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Spectral Properties of Schrodinger Operator with a General Boundary Conditions on Finite Time Scale

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ABSTRACT

In this paper we consider the operator L generated in $L_\nabla^2[a,b]$ by the boundary problem

$-[y^\Delta(t)]^\nabla + [\lambda + q(t)]^2 y(t) = 0, \qquad t \in [a,b],$

$y(a) - k y^\Delta(a) = 0, \quad y(b) + K y^\Delta(b) = 0$

where $q(t)$ is partial continuous, $q(t) \geq 0, k \geq 0, K \geq 0$. In this paper, spectral properties of Schrodinger problem on finite time scale is examined and the formula of convergent expansion is obtained which is form of series in terms of the eigenfunctions in $L_\nabla^2[a,b]$ space.

Keywords: *Time scale, delta and nabla derivatives, Schrödinger operator, eigenvalue, eigenfunction.*

1. INTRODUCTION

The first articles on eigenvalues problems for linear Δ-differential equations on time scales have been investigated in [2] and [7].

Guseinov [8] investigated eigenfunction expansions for the simple Sturm-Liouville eigenvalue problem

$$-y^{\Delta\nabla}(t) = \lambda y(t), \quad t \in (a,b) \tag{1}$$

$$y(a) = y(b) = 0 \tag{2}$$

where a and b are some fixed points in a time scale T with $a < b$ and such that the time scale interval (a,b) is not empty.

In that paper [8], existence of the eigenvalues and eigenfunctions for problem (1), (2) is proved and mean square convergent and uniformly convergent expansions in eigenfunctions are established.

Huseynov and Bairamov in [1] have extended the results of [8] to more general following eigenvalue problem

$$-[p(t)y^\Delta(t)]^\nabla + q(t)y(t) = \lambda y(t), \quad t \in (a,b],$$
$$y(a) - h y^\Delta(a) = 0, \qquad y(b) + H y^\Delta(b) = 0$$

Let us consider the operator L generated in

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$$L^2_\nabla[a,b] := \left\{ y:[a,b] \to \mathbb{R} \left| \int_a^b y^2(t)\nabla \mathrm{t} < \infty \right. \right\}$$

by the eigenvalue problem

$$-[y^\Delta(t)]^\nabla + [\lambda + q(t)]^2 y(t) = 0,\ t \in [a,b] \tag{3}$$

and the boundary condition

$$y(a) - ky^\Delta(a) = 0, \quad y(b) + Ky^\Delta(b) = 0 \tag{4}$$

We will assume that the following two conditions are satisfied.

(C1) $q(t)$ is piecewise continuous on $[a,b]$, k and K are real numbers.

(C2) $q(t) \geq 0$ for $t \in [a,b]$ and $k \geq 0, K \geq 0$.

In this paper, the Hilbert-Schmidt theorem on self-adjoint completely continuous operators is applied to show that the eigenvalue problem (*3*), (*4*) has a system of eigenfunctions that forms an orthonormal basis for an appropriate Hilbert space. Moreover, uniformly convergent expansions in eigenfunctions are obtained when the expanded functions satisfy some smoothness conditions.

Let T be a time scale and $a,b \in T$ be fixed points with $a < b$ such that the time scale interval

$$(a,b) = \{t \in T: a < t < b\}$$

is not empty. For standard notions and notations connected to time scales calculus we refer to [5, 6].

2. L^2- CONVERGENT EXPANSION

Denote by H Hilbert space of all real $\nabla-$measurable functions $y:(a,b) \to \mathbb{R}$ such that $y(b) = 0$ in the case b is left-scattered and $K = 0$, and that

$$\int_a^b y^2(t)\nabla t < \infty,$$

with the inner product

$$\langle y,z\rangle = \int_a^b y(t)z(t)\nabla t$$

and the norm

$$\|y\| = \sqrt{\langle y,y\rangle} = \left\{\int_a^b y^2(t)\nabla t\right\}^{1/2}.$$

Next denote by D the set of all functions $y \in H$ satisfying the following three conditions

(i) y is continuous on $[a,\sigma(b)]$, where σ denotes the forward jump operator.

(ii) $y^\Delta(t)$ is defined for $t \in [a,b]$
$y(a) - ky^\Delta(a) = 0, \quad y(b) + Ky^\Delta(b) = 0$

(iii) $y^\Delta(t)$ is ∇-differentiable on $[a,b]$ and $[y^\Delta(t)]^\nabla \in H$.

Obviously D is a linear subset dense in K. Now we define the operator $L: D \subset H \to H$ as follows. The domain of definition of L is D and we put

$$(Ly)(t) = -[y^\Delta(t)]^\nabla + [\lambda + q(t)]^2 y(t),$$
$$t \in [a,b] \text{for } y \in D$$

Definition 1: $\lambda \in \mathbb{C}$ is called an eigenvalue of problem (3)-(4) if there exists a nonidentically zero function $y \in D$ such that

$$-[y^\Delta(t)]^\nabla + [\lambda + q(t)]^2 y(t) = 0,\ t \in (a,b]$$

The function y is called an eigenfunction of problem (*3*)-(*4*), corresponding to the eigenvalue λ. We see that the eigenvalue problem (*3*)-(*4*) is equivalent to the equation

$$Ly - \lambda y = -[y^\Delta(t)]^\nabla + [\lambda^2 + \lambda(2q(t) - 1) + q^2(t)]y(t),\ \ y \in D,\ \ y \neq 0 \tag{5}$$

Theorem 1: Under the condition (C1) we have, for all $y,z \in D$,

$$\langle Ly,z\rangle = \langle y,Lz\rangle \tag{6}$$

Proof: We have for all $y,z \in D$

$$\langle Ly,z\rangle = \int_a^b \{-[y^\Delta(t)]^\nabla + [\lambda + q(t)]^2 y(t)\} z(t)\nabla t$$

$$= -y^\Delta(t)z(t)\Big|_a^b + \int_a^b y^\Delta(t)z^\Delta(t)\Delta t + \int_a^b [\lambda + q(t)]^2 y(t)z(t)\nabla t$$

$$= y^\Delta(t)z(t)\Big|_a^b + y(t)z^\Delta(t)\Big|_a^b - \int_a^b y(t)[z^\Delta(t)]^\nabla \nabla t + \int_a^b [\lambda + q(t)]^2 y(t)z(t)\nabla t$$

$$= \int_a^b y(t)\{-[z^\Delta(t)]^\nabla + [\lambda + q(t)]^2 z(t)\}\nabla t$$

$$= \langle y, Lz\rangle$$

where we have used the boundary conditions (*4*) for functions $y,z \in D$. (*6*) shows that the operator L is symmetric (self-adjoint).

Theorem 2: Under the conditions (C1) and (C2) we have, for all $y \in D$,

$\langle Ly,y\rangle > 0$.

Proof: We have for all $y \in D$

$$\langle Ly,y\rangle = \int_a^b \{-[y^\Delta(t)]^\nabla + [\lambda + q(t)]^2 y(t)\} y(t)\nabla t$$

$$= -y^\Delta(t)y(t)|_a^b + \int_a^b [y^\Delta(t)]^2\Delta t + \int_a^b [\lambda + q(t)]^2 y^2(t)\nabla t$$

$$= k[y^\Delta(a)]^2 + K[y^\Delta(b)]^2 + \int_a^b [y^\Delta(t)]^2\Delta t + \int_a^b [\lambda + q(t)]^2 y^2(t)\nabla t \qquad (7)$$

(7) shows that it is positive $\langle Ly,y\rangle > 0$ for all $y \in D$, $y \neq 0$. Therefore all eigenvalues of the operator L are real, positive and any two eigenfunctions corresponding to the distinct eigenvalues are orthogonal.
Now we would like to show that the existence of eigenvalues for problem (*3*)-(*4*).

Theorem 3:

$$kerL = \{y \in D: Ly = 0\} = \{0\}$$

Proof: If $y \in D$ and $Ly = 0$, then from (7) we have by the condition (C2) that $y^\Delta(t) = 0$ for $t \in (a,b]$. Hence $y(t)$ is constant on $[a,b]$. Then using boundary conditions (*4*) we get that $y(t) \equiv 0$.

It follows that the inverse operator L^{-1} exists.

Theorem 4: The Green function $G(t,s)$ of (*3*)-(*4*) is defined as

$$G(t,s) = \begin{cases} G_1(t,s), & Im\lambda \leq 0 \\ G_2(t,s), & Im\lambda \geq 0 \end{cases} \qquad (8)$$

Furthermore the Green function is symmetric that is

$$G(t,s) = G(s,t)$$

for *s,t.* Where $G_1(t,s)$ on the plane $Im\lambda \leq 0$ is defined as

$$G_1(t,s) = -\frac{1}{w_1}\begin{cases} u_1(t)v_1(s), & t \leq s \\ u_1(s)v_1(t), & t \geq s \end{cases}$$

and $G_2(t,s)$ on the plane $Im\lambda \geq 0$ is defined as

$$G_2(t,s) = -\frac{1}{w_2}\begin{cases} u_2(t)v_2(s), & t \le s \\ u_2(s)v_2(t), & t \ge s \end{cases}$$

In here, $u_1(t)$ and $u_2(t)$ are the solution of (3) satisfying boundary conditions

$u_1(a) = k, \quad u_1^{\Delta}(a) = 1$

$v_1(b) = K, \quad v_1^{\Delta}(b) = -1$

and

$$u_2(a) = k, \quad u_2^{\Delta}(a) = 1$$
$$v_2(b) = K, \quad v_2^{\Delta}(b) = -1$$

respectively, w_1 and w_2 are Wronskian of the solution u and v which are defined as

$$w_1 = w_t(u_1, v_1) = u_1(t)v_1^{\Delta}(t) - u_1^{\Delta}(t)v_1(t)$$

and

$$w_2 = w_t(u_2, v_2) = u_2(t)v_2^{\Delta}(t) - u_2^{\Delta}(t)v_2(t)$$

Note that $w_1 \neq 0$ and $w_2 \neq 0$.

Then,

$$(L^{-1}f)(t) = \int_a^b G(t,s)f(s)\nabla s, \quad \forall f \in H \tag{9}$$

for any $f \in H$ [3, 4].

The equations (8) and (9) imply that L^{-1} is completely continuous (or compact) self-adjoint and lineer operator in the Hilbert space H.

The eigenvalue problem (5) is equivalent (note that $\lambda = 0$ is not an eigenvalue of L) to the eigenvalue problem

$Bg = \mu, \quad g \in H, \quad g \neq 0$

where

$$B = L^{-1} \text{ and } \mu = \frac{1}{\lambda}.$$

In other words, if λ is an eigenvalue and $y \in D$ is a corresponding eigenfunction for L, then $\mu = \lambda^{-1}$ is an eigenvalue for B with the same corresponding eigenfunction y conversely, if $\mu \neq 0$ is an eigenvalue and $g \in H$ is corresponding eigenfunction for B, then $g \in D$ and $\lambda = \mu^{-1}$ is an eigenvalue for L with the same eigenfunction g.

Next we use the following well-known Hilbert-Schmidt theorem. For every completely continuous self-adjoint linear operator B in a Hilbert H there exists an orthonormal system $\{\varphi_k\}$ of eigenvectors corresponding to eigenvalues $\{\mu_k\}(\mu_k \neq 0)$ such that element $f \in H$ can be written uniquely in the form

$$f = \sum_k c_k \varphi_k + \psi$$

where $\psi \in kerB$, that is, $B\psi = 0$. Moreover,

$$Bf = \sum_k \mu_k c_k \varphi_k$$

and if the system $\{\varphi_k\}$ is infinite, then

$\lim \quad \mu_k = 0 \quad (k \to \infty)$.

As a corollary of the Hilbert-Schmidt theorem we have if B is a completely continuous self-adjoint linear operator in a Hilbert space H and if $kerB = \{0\}$, then the eigenvectors of B form an orthogonal basis of H.

Applying the corollary of the Hilbert-Schmidt theorem to the operator $B = L^{-1}$ and using the above described connection between the eigenvalues and eigenfunctions of L and the eigenvalues and eigenfunctions of B we use the following result in [1].

Theorem 5: Under the conditions (C1) and (C2), for the eigenvalue problem (3)-(4) there exists an orthonormal system $\{\varphi_k\}$ of eigenfunctions corresponding to eigenvalues $\{\lambda_k\}$. Each eigenvalue λ_k is positive and simple. The system $\{\varphi_k\}$ forms an orthonormal basis for the Hilbert space H. Therefore the number of the eigenvalues is equal to $N = dimH$. Any function $f \in H$ can be expanded in eigenfunctions φ_k in the form

$$f(t) = \sum_{k=1}^{N} c_k \varphi_k(t) \tag{10}$$

wherec_k are the Fourier coefficients of f defined by

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t \tag{11}$$

In the case $N = \infty$ the sum in (10) becomes an infinite series and it converges to the function f in metric of the space H, that is, in mean square metric

$$\lim_{n\to\infty} \int_a^b \left[f(t) - \sum_{k=1}^{n} c_k \varphi_k(t) \right]^2 \nabla t = 0 \tag{12}$$

Note that since

$$\int_a^b \left[f(t) - \sum_{k=1}^{n} c_k \varphi_k(t) \right]^2 \nabla t = \int_a^b f^2(t)\nabla t - \sum_{k=1}^{n} c_k^2$$

we get from (12) the Parseval equality

$$\int_a^b f^2(t)\nabla t = \sum_{k=1}^{N} c_k^2 \tag{13}$$

3. UNIFORMLY CONVERGENT EXPANSION

In this section, if the condition (C1) and (C2) are satisfied, we prove the following result.

Theorem 6: Let $f:[a,b] \to \mathbb{R}$ be a function such that it has a Δ −derivative $f^{\Delta}(t)$ everywhere on $[a,b]$, except at a finite number of points $t_1, t_2, \dots, t_m$ belonging to (a,b), the Δ-derivative being continuous everywhere except at these points, at which f^{Δ} has finite limits from the left and right. Besides assume that f satisfies the boundary conditions

$$f(a) - kf^{\Delta}(a) = 0 \quad , \quad f(b) + Kf^{\Delta}(b) = 0$$

Then the series

$$\sum_{k=1}^{\infty} c_k \varphi_k(t) \tag{14}$$

where

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t \tag{15}$$

Converges uniformly on $[a,b]$ to the function *f*.

Proof: Let the function f is Δ-differentiable everywhere on $[a,b]$ and that f^{Δ} is continuous on $[a,b]$. Consider the functional

$$J(y) = k[y^{\Delta}(a)]^2 + K[y^{\Delta}(b)]^2 + \int_a^b [y^{\Delta}(t)]^2 \Delta t + \int_a^b [q^2(t) + 2\lambda q(t)] y^2(t) \nabla t$$

so that we have $J(y) \geq 0$. Substituting in the functional $J(y)$

$$y = f(t) - \sum_{k=1}^{n} c_k \varphi_k(t)$$

where c_k are defined by (15), we obtain

$$\begin{aligned}
J\left(f - \sum_{k=1}^{n} c_k\varphi_k(t)\right) &= k\left[f^{\Delta}(a) - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}(t)\right]^2 + K\left[f^{\Delta}(b) - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}(b)\right]^2 \\
&+ \int_a^b \left[f^{\Delta} - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}(b)\right]^2 \Delta t + \int_a^b [q^2(t) + 2\lambda q(t)]\left[f + \sum_{k=1}^{n} c_k\,\varphi_k\right]^2 \nabla t \\
&= k\left(f^{\Delta}(a)\right)^2 + K\left(f^{\Delta}(b)\right)^2 - 2\sum_{k=1}^{n} c_k[kf^{\Delta}(a)\varphi_k^{\Delta}(a) + Kf^{\Delta}(b)\varphi_k^{\Delta}(b)] \\
&+ \sum_{k,l=1}^{n} c_kc_l[k\varphi_k^{\Delta}(a)\varphi_l^{\Delta}(a) + K\varphi_k^{\Delta}(b)\varphi_l^{\Delta}(b)] + \int_a^b (f^{\Delta})^2\Delta t \int_a^b [q^2(t) + 2\lambda q(t)]f^2\Delta t \\
&-2\sum_{k=1}^{n} c_k\left(\int_a^b f^{\Delta}\,\varphi_k^{\Delta}(t)\Delta t + \int_a^b (q^2 - 2\lambda q)f\varphi_k\nabla t\right) \\
&+ \sum_{k,l=1}^{n} c_kc_l\left(\int_a^b \varphi_k^{\Delta}\varphi_l^{\Delta}\Delta t + \int_a^b (q^2 + 2\lambda q)\,\varphi_k\varphi_l\nabla t\right)
\end{aligned} \tag{16}$$

where δ_{kl} is the Kronecker symbol and where we have used the boundary conditions (*4*).

$$\varphi_k(a) - k\varphi_k^{\Delta}(a) = 0, \quad \varphi_k(b) + K\varphi_k^{\Delta}(b) = 0 \tag{17}$$

and the equation

$$-\left[\varphi_k^{\Delta}(t)\right]^{\nabla} + \left(q^2 + 2\lambda q(t)\right)\varphi_k(t) = \lambda_k\varphi_k(t).$$

Therefore we have from (16)

$$J\left(f - \sum_{k=1}^{n} c_k\varphi_k(t)\right) = k[f^{\Delta}(a)]^2 + K[f^{\Delta}(b)]^2 + \int_a^b [f^{\Delta 2} + (q^2 + 2\lambda q)f^2]\Delta t - \sum_{k=1}^{n} \lambda_k c_k^2$$

Since the left-hand side is nonnegative we get the inequality

$$\sum_{k=1}^{\infty} \lambda_k\, c_k^2 \leq k[f^{\Delta}(a)]^2 + K[f^{\Delta}(b)]^2 + \int_a^b [f^{\Delta 2} + (q^2 + 2\lambda q)f^2]\, \Delta t \tag{18}$$

analogous to Bessel's inequality, and the convergence of the series on the left follows. All the terms of this series are nonnegative, since $\lambda_k > 0$. Note that the proof of (18) is entirely unchanged if we assume that the function f satisfies only the conditions stated in the theorem. Indeed, when integrating by parts, it is sufficient to integrate over the intervals on which f^{Δ} is continuous and then add all these integrals. (The integrated terms vanish by (*4*), (17) and the fact that f, φ_k and φ_k^{Δ} are continuous on $[a,b]$).

We now show that the series

$$\sum_{k=1}^{n}|c_k\varphi_k(t)| \qquad (19)$$

is uniformly convergent on the interval $[a,b]$.
Using the integral equation

$$\varphi_k(t)=\lambda_k\int_a^b G(t,s)\varphi_k(s)\nabla s$$

which follows from $\varphi_k=\lambda_k L^{-1}\varphi_k$ by (9), we can rewrite (19) as

$$\sum_{k=1}^{n}\lambda_k|c_k g_k(t)| \qquad (20)$$

where

$$g_k(t)=\int_a^b G\,(t,s)\varphi_k(s)\nabla s$$

can be regarded as the Fourier coefficient of $G(t,s)$ as a function of s.

By using inequality (18), we can write

$$\sum_{k=1}^{\infty}\lambda_k g_k^2\le k[G^{\Delta_s}(t,a)]^2+K[G^{\Delta_s}(t,b)]^2+\int_a^b\left[[G^{\Delta_s}]^2(t,s)+\left(q^2+2\lambda q(s)\right)G^2(t,s)\right]\Delta s \qquad (21)$$

where $G^{\Delta_s}(t,s)$ is the delta derivative of $G(t,s)$ with respect to s. The function appearing under the integral sign is bounded (see (8)) and it follows from (21) that

$$\sum_{k=1}^{\infty}\lambda_k g_k^2\le M$$

Where M is a constant. Now replacing $\lambda_k$by $\sqrt{\lambda_k}$, we apply the Cauch-Schwarz inequality to the segment of series (20),

$$\sum_{k=m}^{m+p}\lambda_k|c_k g_k(t)|=\sum_{k=m}^{m+p}\left|\sqrt{\lambda_k}c_k\right|\left|\sqrt{\lambda_k}g_k(t)\right|$$

$$\le\sqrt{\sum_{k=m}^{m+p}\lambda_k c_k^2}\,.\,\sqrt{\sum_{k=m}^{m+p}\lambda_k q_k^2(t)}$$

$$\le\sqrt{M}\sqrt{\sum_{k=m}^{m+p}\lambda_k c_k^2}$$

And this inequality, together with the convergence of the series with terms $\lambda_k c_k^2$ (see (18)) at once implies that series with terms (20), and hence series (19) is uniformly convergent on the interval $[a,b]$. Denote the sum of series (14) by $f_1(t)$

$$f_1(t)=\sum_{k=1}^{\infty}c_k\varphi_k(t) \qquad (22)$$

Since the series in (22) is convergent uniformly on $[a,b]$, we can multiply both sides of (22) by $\varphi_l(t)$ and then ∇ integrate it term-by-term to get

$$\int_a^b f_1(t)\varphi_l(t)\nabla t=c_l$$

Therefore the Fourier coefficients of f_1 and f are the same. Then the Fourier coefficients of the difference f_1-f are zero and applying the Parseval equality (13) to the function f_1-f we get that $f_1-f=0$, so that the sum of series (14) is equal to $f(t)$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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