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MODULES WITH FINITELY MANY SUBMODULES

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Abstract. We characterize ring extensions $R \subset S$ having FCP (FIP), where S is the idealization of some R-module. As a by-product we exhibit characterizations of the modules that have finitely many submodules. Our tools are minimal ring morphisms, while Artinian conditions on rings are ubiquitous.

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1. Introduction and notation

All rings R considered are commutative, nonzero and unital; all morphisms of rings are unital. Let $R \subseteq S$ be a (ring) extension. The set of all R-subalgebras of S is denoted by $[R, S]$. The extension $R \subseteq S$ is said to have FIP (for the "finitely many intermediate algebras property") if $[R, S]$ is finite. A *chain* of R-subalgebras of S is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. We say that the extension $R \subseteq S$ has FCP (for the "finite chain property") if each chain of R-subalgebras of S is finite. It is clear that each extension that satisfies FIP must also satisfy FCP. If the extension $R \subseteq S$ has FIP (FCP), we will sometimes say that $R \subseteq S$ is an FIP (FCP) extension. Our main tool are the minimal (ring) extensions, a concept introduced by Ferrand-Olivier [10]. Recall that an extension $R \subset S$ is called *minimal* if $[R, S] = \{R, S\}$. The key connection between the above ideas is that if $R \subseteq S$ has FCP, then any maximal (necessarily finite) chain $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$, of R-subalgebras of S, with *length* $n < \infty$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$. The *length* of $[R, S]$, denoted by $\ell[R, S]$, is the supremum of the lengths of chains of R-subalgebras of S. In particular, if $\ell[R, S] = r$, for some integer r, there exists a maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_{r-1} \subset R_r = S$ of R-subalgebras of S with length r. Against the general trend, we characterized arbitrary FCP and FIP extensions in [8], a joint paper by D. E. Dobbs and ourselves whereas most of papers on the subject are concerned with extensions of integral domains. Note that other papers by D. E. Dobbs [6], and D. E. Dobbs with P.-J. Cahen, T. G. Lucas [5], J. Shapiro [9], B. Mullins and ourselves [7] also went against the same trend. It is worth noticing here that FCP extensions of integral domains are (ignoring fields) extensions of overrings as a quick look at [5, Theorems 4.1,4.4] shows because FCP extensions are composites of finitely many minimal extensions.

The seminal work on FIP and FCP by R. Gilmer is settled for R-subalgebras of K (also called overrings of R), where R is a domain and K its quotient field. In particular, [12, Theorem 2.14] shows that $R \subseteq S$ has FCP for each overring S of R only if R/C is an Artinian ring, where $C = (R : \overline{R})$ is the conductor of R in its integral closure. This necessary Artinian condition is not surprisingly present in all our results.

This paper is concerned with R-modules M over a ring R and ring extensions $R \subseteq R(+)M$, where $R(+)M$ is the idealization of M. The main results are as follows. Proposition 2.2 shows that $R \subseteq R(+)M$ has FCP if and only if the length of the R-module M is finite, while Proposition 2.4 says that $R \subseteq R(+)M$ has FIP if and only if M has finitely many R-submodules. This leads us to characterize R-modules having finitely many R-submodules in Corollary 2.7. An R-module M, with $C := (0 : M)$, has finitely many submodules if and only if the three following conditions are satisfied: M is finitely generated, R/C has finitely many ideals and M_P is cyclic for any prime ideal P of R containing C such that R/P is infinite. Then Theorem 2.13 gives a structure theorem for these modules that are faithful.

Let R be a ring. As usual, $\mathrm{Spec}(R)$ (resp. $\mathrm{Max}(R)$) denotes the set of all prime ideals (resp. maximal ideals) of R. If I is an ideal of R, we set $\mathrm{V}_R(I) := \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$. If $R \subseteq S$ is a ring extension and $P \in \mathrm{Spec}(R)$, then S_P is the localization $S_{R \setminus P}$ and $(R : S)$ is the conductor of $R \subseteq S$. If E is an R-module, $\mathrm{L}_R(E)$ is its length. We will shorten finitely generated module to f.g. module. Recall that a *special principal ideal ring* (SPIR) is a principal ideal ring R with a unique nonzero prime ideal $M = Rt$, such that M is nilpotent of index $p > 0$. Hence a SPIR is not a field. Each nonzero element of a SPIR is of the form ut^k for some unit u and some *unique* integer $k < p$. Finally, as usual, $\subset$ denotes proper inclusion and $|X|$ denotes the cardinality of a set X.

There are four types of minimal extension, but we only need ramified minimal extensions.

Theorem 1.1. [10, Théorème 2.2], [16, Theorem 3.3] *Let $R \subset T$ be a ring extension and $M := (R : T)$. Then $R \subset T$ is a* **ramified** *minimal extension if and only if $M \in \mathrm{Max}(R)$ and there exists $M' \in \mathrm{Max}(T)$ such that ${M'}^2 \subseteq M \subset M'$, $[T/M :$*

$R/M] = 2$ (resp. $\mathrm{L}_R(M'/M) = 1$), and the natural map $R/M \to T/M'$ is an isomorphism.

Definition 1.2. An integral extension $f : R \hookrightarrow S$ is termed *subintegral* if all its residual extensions are isomorphisms and ${}^a f$ is bijective [18].

A minimal morphism is ramified if and only if it is subintegral.

According to J. A. Huckaba and I. J. Papick [14], an extension $R \subseteq S$ is termed a Δ_0-*extension* provided each R-submodule of S containing R is an element of $[R, S]$. We recall here for later use an unpublished result of the Gilbert's dissertation.

Proposition 1.3. [11, Proposition 4.12] *Let $R \subseteq S$ be a ring extension with conductor I and such that $S = R + Rt$ for some $t \in S$. Then the R-modules R/I and S/R are isomorphic. Moreover, each of the R-modules between R and S is a ring (and so there is a bijection from $[R, S]$ to the set of ideals of R/I).*

We will use the following result. If $R_1, \dots, R_n$ are finitely many rings, the ring $R_1 \times \cdots \times R_n$ localized at the prime ideal $P_1 \times R_2 \times \cdots \times R_n$ is isomorphic to $(R_1)_{P_1}$ for $P_1 \in \mathrm{Spec}(R_1)$. This rule works for any prime ideal of the product.

Rings which have finitely many ideals are characterized by D. D. Anderson and S. Chun [1], a result that will be often used.

Proposition 1.4. [1, Corollary 2.4] *A commutative ring R has only finitely many ideals if and only if R is a finite direct product of finite local rings, SPIRs, and fields, and these are the localizations of R at its maximal ideals.*

Note that if (R, M) is a local Artinian ring, then R is finite if and only if R/M is finite, since $M^n = 0$ for some integer n. If (R, M) is an Artinian local ring, we denote by $n(R)$ the nilpotency index of M.

From now on, a ring R with finitely many ideals is termed an FMIR.

2. Idealizations which are FCP or FIP extensions

Let M be an R-module. We consider the ring extension $R \subseteq R(+)M$, where $R(+)M$ is the idealization of M in R.

Recall that $R(+)M := \{(r, m) \mid (r, m) \in R \times M\}$ is a commutative ring whose operations are defined as follows:

$(r, m) + (s, n) = (r + s, m + n)$ and $(r, m)(s, n) = (rs, rn + sm)$

Then $(1, 0)$ is the unit of $R(+)M$, and $R \subseteq R(+)M$ is a ring morphism defining $R(+)M$ as an R-module, so that we can identify any $r \in R$ with $(r, 0)$. The following lemma will be useful for all this section.

Lemma 2.1. *Let M be an R-module, then $R \subseteq R(+)M$ is a subintegral extension with conductor* $(0 : M)$.

Proof. If $(r, m) \in R(+)M$, then $(r, m)^2 = 2r(r, m) - r^2(1, 0)$ shows that $R(+)M$ is integral over R. Moreover, by [13, Theorem 25.1(3)], $\mathrm{Spec}(R(+)M) = \{P(+)M \mid P \in \mathrm{Spec}(R)\}$ implies that $R \subseteq R(+)M$ is subintegral.

Set $S := R(+)M$ and let $x \in (R : S)$. Then, we have $(x, 0)(0, m) = (0, xm) \in R$ for any $m \in M$, so that $x \in (0 : M)$. Conversely, any $x \in (0 : M)$ gives $x(r, m) = (xr, 0) \in R$ for any $(r, m) \in R(+)M$, which implies $x \in (R : S)$. So, we get $(R : S) = (0 : M)$. □

Proposition 2.2. *Let M be an R-module, then $R \subseteq R(+)M$ has FCP if and only if $\mathrm{L}_R(M) < \infty$ and, if and only if $R/(0 : M)$ is Artinian and M is f.g. over R.*

Proof. Set $S := R(+)M$. Since $R \subseteq S$ is integral, $R \subseteq S$ has FCP if and only if $\mathrm{L}_R(S/R) < \infty$ by [8, Theorem 4.2]. By the same reference, this condition is equivalent to $R/(0 : M) \cong R/(R : S)$ is Artinian and $R \subseteq S$ is module finite. Finally, note that $S/R \cong M$; and that S is f.g over R if (and only if) S/R is f.g. over R. □

For a submodule N of an R-module M, we denote by $[\![N, M]\!]$ the set of all submodules of M containing N and set $[\![M]\!] := [\![0, M]\!]$. Recall that M is called *uniserial* if $[\![M]\!]$ is linearly ordered.

Proposition 2.3. (Dobbs) *Let M be an R-module, then $R \subseteq R(+)M$ is a Δ_0-extension because $[R, R(+)M] = \{R(+)N \mid N \in [\![M]\!]\}$.*

Proof. The equality $[R, R(+)M] = \{R(+)N \mid N \in [\![M]\!]\}$ was proved by D. E. Dobbs in [6, Remark 2.9] using the bijection $[\![M]\!] \to [R, R(+)M]$, $N \mapsto R(+)N$. □

We say that an R-module M is an FMS module if M has finitely many R-submodules. An FMS R-module M is Noetherian and Artinian and $R/(0 : M)$ is a Noetherian and Artinian ring. We denote by $\nu_R(M)$ (or $\nu(M)$) the number of submodules of an FMS R-module M. Hence, $\nu(R)$ is the number of ideals of an FMIR R.

Proposition 2.4. *Let M be an R-module, then $R \subseteq R(+)M$ has FIP if and only if M is an FMS module. In this case, $|[R, R(+)M]| = \nu(M)$.*

Proof. Set $S := R(+)M$. By Proposition 2.3, it follows that $R \subseteq S$ has FIP if and only if M is an FMS module. In this case, $|[R, R(+)M]| = \nu(M)$. □

We now intend to characterize FMS modules by using the previous proposition.

Theorem 2.5. *An R-module M over a quasi-local ring (R, P) is an FMS module if and only if the next conditions* (1) *and* (2) *hold with $C := (0 : M)$:*

(1) *M is finitely generated, and cyclic when $|R/P| = \infty$.*
(2) *R/C is an FMIR.*

If M is an FMS R-module, (R, P) is quasi-local, $|R/P| = \infty$, and $M = Re$ for some $e \in M$, then M is uniserial, $[\![M]\!] = \{P^j e \mid j = 0, \ldots, m\}$, with $m := n(R/C) = \nu(R/C) - 1$ and $|[R, R(+)M]| = m + 1$.

Assume in addition that $P = (0 : M)$ and $|R/P| = \infty$. Then $R \subseteq R(+)M$ has FIP if and only if M is simple, if and only if $R \subseteq R(+)M$ is minimal ramified.

Proof. Note that R-submodules and R/C-submodules of M coincide.

Assume that M is an FMS module. We first prove (1). Then Proposition 2.4 shows that $R \subseteq R(+)M$ has FIP, whence has FCP. We deduce from Proposition 2.2 that M is f.g. and $(R/C, P/C)$ is local Artinian. Assume that $|R/P| = \infty$. Denote by $Re_1, \ldots, Re_n$, with $e_i \in M$, the finitely many cyclic submodules of M. Then for any $m \in M$, there is some i such that $Rm = Re_i$, so that $M = \cup_{i=1}^n Re_i$. We can then suppose that $M = \cup_{i=1}^p Rf_i$, where $f_i \in \{e_1, \ldots, e_n\}$ and the Rf_i are incomparable. If $p = 1$, then M is cyclic. The case $p = 2$ cannot happen because a group cannot be the union of two proper incomparable subgroups. We now show that $p > 2$ leads to a contradiction. Let $\mathcal{F}$ be a(n infinite) set of representatives of the non-zero elements of R/P. Then, each $\alpha \in \mathcal{F}$ is a unit of R. For each $\alpha \in \mathcal{F}$, set $m_\alpha := f_1 + \alpha f_2$. Obviously $m_\alpha \notin Rf_1 \cup Rf_2$. It follows that $m_\alpha \in Rf_i$, for some $i \neq 1, 2$. Let $\alpha, \beta \in \mathcal{F},\ \alpha \neq \beta$. We claim that m_α and m_β are not in the same Rf_i. Deny, then $m_\alpha - m_\beta = (\alpha - \beta) f_2 \in Rf_i$ and $\alpha - \beta$ is a unit implies $f_2 \in Rf_i$, a contradiction. Therefore, M is cyclic and (1) is proved.

To prove (2), we consider two cases. If $|R/P| < \infty$, then $|R/C| < \infty$ (see the remark after Proposition 1.4), so that R/C is an FMIR.

Assume that $|R/P| = \infty$. It follows from (1) that $M = Re$ for some $e \in M$, so that $C = (0 : e)$. Set $R' := R/C,\ P' := P/C$ and $I_N := (N :_R e)$ for $N \in [\![M]\!]$. Then, $I_N \in [\![C, R]\!]$ and is such that $N = I_N e$. Conversely, $I \in [\![C, R]\!]$ is such that $I = I_{Ie}$ with $Ie \in [\![M]\!]$, since $C \subseteq I$. We define a preserving order bijective map $\psi : [\![C, R]\!] \to [\![M]\!]$ by $I \mapsto Ie$. It follows that R' is an FMIR (either a field or a SPIR) and $\nu(M) = \nu(R/C)$. Then, (2) is proved.

Now, assume that (1) and (2) hold. There is no harm to suppose that $C = 0$ and that R is an FMIR, so that (R, P) is local Artinian. If $|R/P| < \infty$, we get that $|M| < \infty$ and then M is an FMS module. Assume that $|R/P| = \infty$, and that $M = Re$ is cyclic. The assertion is clear if $M = 0$. Assume $M \neq 0$. If $P = 0$, then

M is a one-dimensional vector space over the field R, so that $\nu(M) = 2 = \nu(R)$. If $P \neq 0$, consider $S := R(+)M = R + Rf$, where $f = (0, e)$. From Proposition 1.3 we deduce that $|[\![R, S]\!]| < \infty$, since R is an FMIR and also that there are bijective maps $[\![R]\!] \to [R, S]$ and $[\![R, S]\!] \to [\![M]\!]$. In fact $[\![R, S]\!] = \{R(+)N \mid N \in [\![M]\!]\}$. By Proposition 2.3, M is an FMS module.

Assume that M is an FMS R-module, (R, P) is quasi-local, $|R/P| = \infty$, and $M = Re$ for some $e \in M$. If R' is a SPIR, there is some $x \in P$, whose class $\bar{x} \in R'$ is such that $P' = R'\bar{x}$, $\bar{x}^m = 0$ and $\bar{x}^{m-1} \neq 0$, for $m := n(R') > 1$. It follows that $[\![C, R]\!] = \{P^j + C \mid j \in \{0, \dots, m\}\}$ and $[\![M]\!] = \{P^j e \mid j \in \{0, \dots, m\}\}$ (to see this, use the above bijection ψ). If R' is a field, then $P = C$ gives $m = 1$. In both cases, M is uniserial, $m := n(R/C) = \nu(R/C) - 1$ and $|[R, R(+)M]| = m + 1$.

To end, assume that (R, P) is quasi-local with $|R/P| = \infty$. Let M be a simple R-module, with $P = (0 : M)$. Then $[R, R(+)M] = \{R, R(+)M\}$ by Proposition 2.3. It follows that $R \subsetneq R(+)M$ has FIP and is a minimal ramified extension since minimal subintegral. The converse is obvious. □

Example 2.6. *We give this example due to the referee showing that the condition* $|R/P| = \infty$ *in Theorem 2.5 is necessary in order to have* M *a simple module when* M *is an FMS module. Let* R *be a finite field, and let* $M := R \bigoplus R$. *Then,* $R \subseteq R(+)M$ *has FIP since* M *has only finitely many submodules and* $(0 : M) = \{0\} = P$, *but* M *is not a simple* R*-module.*

Corollary 2.7. *Let* M *be an* R*-module and* $C := (0 : M)$. *Then* M *is an FMS module if and only if the two following conditions hold:*

(1) M *is f.g. and* M_P *is cyclic over* R_P *for all* $P \in \mathrm{V}(C)$ *such that* $|R/P| = \infty$.

(2) R/C *is an FMIR.*

In case (1), (2) *both hold, set* $\{P_1, \dots, P_n\} = \mathrm{V}(C)$ *and suppose that each* $|R/P_i| = \infty$. *Then, for each* i, *there exist some* $e_i \in M$, *such that* $M_{P_i} = R_{P_i}(e_i/1)$ *and,* M *is generated by the* $e_1, \dots, e_n$.

Proof. If M is an FMS module, Proposition 2.4 shows that $R \subsetneq R(+)M$ has FIP, and then has FCP. Hence, M is f.g. and R/C is Artinian by Proposition 2.2. Let $P \in \mathrm{V}(C)$, then M_P is an FMS R_P-module, so that we can use Theorem 2.5. It follows that $R_P/C_P \cong (R/C)_P$ is an FMIR, and so is R/C, since $|\mathrm{V}(C)| < \infty$, which gives (2). Moreover, for $P \in \mathrm{V}(C)$ with $|R/P| = \infty$, Theorem 2.5 gives that M_P is cyclic and (1) holds.

Conversely, if (1) and (2) hold, they also hold for each M_P, where $P \in \mathrm{V}(C)$. Theorem 2.5 gives that M_P is an FMS module for any $P \in \mathrm{V}(C)$. To show that M

is an FMS module, there is no harm to suppose that $C = 0$, so that R is Artinian, with $\mathrm{Max}(R) = \{P_1, \dots, P_n\}$. Now if N is a submodule of M, it is well known that $N = \cap_{i=1}^n \varphi_i^{-1}(N_{P_i})$, where $\varphi_i : M \to M_{P_i}$ is the natural map and thus M is an FMS module.

Now, assume that (1) and (2) hold and that $|R/P| = \infty$ for any $P \in \mathrm{V}(C) = \{P_1, \dots, P_n\}$. For each $j = 1, \dots, n$, there is some $e_j \in M$ such that $M_{P_j} = R_{P_j}(e_j/1)$. Set $M' := Re_1 + \cdots + Re_n$. It is easy to show that $M'_{P_j} = M_{P_j}$ for $j = 1, \dots, n$. Observe that $\mathrm{V}(C) = \mathrm{Supp}(M)$, because M is f.g. ([2, Proposition 17, ch. II, p.133]). Now let $P \in \mathrm{Max}(R) \setminus \mathrm{V}(C)$. We get that $M'_P \subseteq M_P = 0$ and then $M' = M$. □

Let N be a submodule of an R-module M. By Proposition 2.3, $R(+)N$ is an R-subalgebra of $R(+)M$ and then $R(+)M$ is an $(R(+)N)$-algebra. Even if $R \subseteq R(+)M$ does not have FCP (resp. FIP), it may be that $R(+)N \subseteq R(+)M$ has FCP (resp. FIP).

Any $(R(+)N)$-subalgebra of $R(+)M$ is an R-subalgebra of $R(+)M$, and then is of the form $R(+)N'$, for some $N' \in [\![N, M]\!]$ since $R(+)N \subseteq R(+)N'$. Conversely, for any R-subalgebra N' of M containing N, $R(+)N'$ is an $(R(+)N)$-subalgebra of $R(+)M$. In particular, $R(+)N \subseteq R(+)M$ is a minimal extension if and only if M/N is a simple module.

Proposition 2.8. *Let N be a submodule of an R-module M. Then:*

(1) *$R(+)N \subseteq R(+)M$ is a Δ_0-extension.*
(2) *$R(+)N \subseteq R(+)M$ has FCP if and only if $\mathrm{L}_R(M/N) < \infty$. In this case, $\ell[R(+)N, R(+)M] = \mathrm{L}_R(M/N)$.*
(3) *$R(+)N \subseteq R(+)M$ has FIP if and only if M/N is an FMS module. In this case, $|[R(+)N, R(+)M]| = \nu(M/N)$.*

Proof. (1) By Proposition 2.3, $R \subseteq R(+)M$ is a Δ_0-extension. Since an $(R(+)N)$-submodule S of $R(+)M$ containing R is also an R-submodule of $R(+)M$, we get that S is a ring, so that $R(+)N \subseteq R(+)M$ is a Δ_0-extension.

(2) By Lemma 2.1, $R \subseteq R(+)M$ is integral and so is $R(+)N \subseteq R(+)M$. Therefore, the following conditions are equivalent:

- $R(+)N \subseteq R(+)M$ has FCP
- there exists a finite chain of minimal finite extensions going from $R(+)N$ to $R(+)M$ ([8, Theorem 4.2(2)])
- there is a finite maximal chain of R-submodules of M going from N to M
- $\mathrm{L}_R(M/N) < \infty$.

In this case, $\ell[R(+)N, R(+)M] = \mathrm{L}_R(M/N)$, the supremum of the lengths of chains of submodules of M containing N.

(3) The following conditions are equivalent:

- $R(+)N \subseteq R(+)M$ has FIP
- there are finitely many $(R(+)N)$-subalgebras of $R(+)M$
- there are finitely many R-subalgebras of $R(+)M$ containing $R(+)N$
- there are finitely many R-submodules of M containing N
- M/N is an FMS module.

In this case, $|[R(+)N, R(+)M]|$ is also the number of R-submodules of M containing N, which is also $\nu(M/N)$. □

We consider now the special case where M is an ideal I of R.

Proposition 2.9. *Let I be an ideal of a ring R, $S := R(+)R$ and $T := R(+)I$. Then:*

1. *$R \subseteq S$ has FCP if and only if $\mathrm{L}_R(R) < \infty$ if and only if R is Artinian. In this case, $\ell[R, R(+)R] = \mathrm{L}_R(R)$.*
2. *$R \subseteq T$ has FCP if and only if $\mathrm{L}_R(I) < \infty$ if and only if I is finitely generated and $R/(0 : I)$ is Artinian. In this case, $\ell[R, R(+)I] = \mathrm{L}_R(I)$.*
3. *$R \subseteq S$ has FIP if and only if R is an FMIR. In this case, $|[R, R(+)R]| = \nu(R)$.*
4. *$R \subseteq T$ has FIP if and only if $[\![I]\!]$ is finite. In this case, $|[R, R(+)I]| = \nu(I)$.*

Proof. Propositions 2.2 and 2.8 with M equal to R or I give most of the results because taking $N = 0$ gives $R(+)0 \cong R$. □

Proposition 2.10. *Any f.g. module over a ring R is an FMS module if and only if R is a finite ring.*

Proof. If R is finite, then $[\![M]\!]$ is finite for any f.g. R-module M. Conversely, let R be a ring such that any f.g. R-module is an FMS module. Set $S := R[X,Y]/(X^2, XY, Y^2) = R[x,y]$, where x and y are respectively the classes of X and Y in S. Then S is an R-module with basis $\{1, x, y\}$. For each $\alpha \in R$, set $S_\alpha := R(x+\alpha y)$, which is an R-submodule of S. If $\alpha, \beta \in R$, $\alpha \neq \beta$, then $S_\alpha \neq S_\beta$. Therefore, $|R| = \infty$ gives a contradiction and R is a finite ring. □

Remark 2.11. *If N is a submodule of an R-module M, Proposition 2.2 shows that $R \subseteq R(+)M$ has FCP if and only if $R \subseteq R(+)N$ and $R \subseteq R(+)(M/N)$ have FCP. This property does not hold for FIP. It is enough to consider a 2-dimensional vector space M over an infinite field, and a 1-dimensional subspace N because N and M/N are FMS modules, while M is not.*

Example 2.12. *In the following examples, we mix properties of this section and* [17, Section 3].

(1) Let k be a field, $n > 1$ an integer, E an n-dimensional k-vector space with basis $\{e_1, \dots, e_n\}$ and set $R := k^n$. We can equip E with the structure of an R-module by the following law: for $(a_1, \dots, a_n) \in R$ and $x = \sum_{i=1}^n x_i e_i,\ x_i \in k$, we set $(a_1, \dots, a_n)x := \sum_{i=1}^n a_i x_i e_i$. Then E is generated over R by $\{e_1, \dots, e_n\}$ and is faithful, while R is an FMIR. Finally, the prime (maximal) ideals of R are the ideals $P_i := \{(a_1, \dots, a_n) \in R \mid a_i = 0\}$ for $i = 1, \dots, n$, so that $R_{P_i} \cong k$. The canonical base $\{\varepsilon_1, \dots, \varepsilon_n\}$ of R over k is such that each $\varepsilon_i \notin P_i$. We have $\varepsilon_i e_j = 0$ for each $i, j \in \{1, \dots, n\}$ such that $i \neq j$, so that $e_j/1 = 0$ in R_{P_i} for $j \neq i$. It follows that $E_{P_i} = \sum_{j=1}^n R_{P_i}(e_j/1) = R_{P_i}(e_i/1)$ is cyclic over $R_{P_i} \cong k$. Then, whatever $|k|$ may be, Corollary 2.7 gives that E is an FMS R-module. But, as soon as $|k| = \infty$ and $n \geq 2$, E is infinite. Since $E_{P_i} \cong k(e_i/1)$ is one-dimensional over k, E_{P_i} has only two R_{P_i}-submodules. Set $F := \prod_{i=1}^n E_{P_i}$ and consider the canonical injective morphism of R-modules $\varphi : E \to F$ and the projections $\varphi_i : F \to E_{P_i}$. Any R-submodule N of F is of the form $N' := \prod_{i=1}^n N_i$, where $N_i = \varphi_i(N)$, because $N \subseteq N' \subseteq \sum_{i=1}^n \varepsilon_i N$. Now φ is a k-isomorphism because $\mathrm{Dim}_k(E) = \mathrm{Dim}_k(F)$, whence an R-isomorphism. It follows that $\nu_R(E) = 2^n$.

By Proposition 2.4, $k^n \subseteq k^n(+)E$ has FIP, and $k \subseteq k^n$ has FIP by [4, Proposition 3, p. 29] (another proof follows from [7, Theorem III.5]). But, always in view of Proposition 2.4, if $|k| = \infty$ and $n \geq 2$, then $k \subseteq k(+)E$ has not FIP, so that $k \subseteq k^n(+)E$ has not FIP.

(1') We keep the context of (1). Set $\mathcal{R} := \prod_{i=1}^n (k/(0 : e_i))$. Since $(0 : e_i) = 0$ for each i, we get $\mathcal{R} = k^n$. Then $k \subset \mathcal{R}$ has FIP while $k \subseteq k(+)E$ has not FIP.

(2) Let k be an infinite field, $n > 1$ an integer and E an n-dimensional vector space over k. Let $u \in \mathrm{End}(E)$ with minimal polynomial X^n. Then, $u^n = 0$ and $u^{n-1}(e_1) \neq 0$ for some $e_1 \in E$. If $e_i := u^{i-1}(e_1)$ for any $i \in \{1, \dots, n\}$, an easy induction shows that $\{e_1, \dots, e_n\}$ is a basis of E over k. Set $R := k[u]$, then E is a faithful R-module with scalar multiplication defined by $P(u) \cdot x := P(u)(x)$, for $P(X) \in k[X]$ and $x \in E$. Since $R \cong k[X]/(X^n)$ is a SPIR and $E = R \cdot e_1$ because $e_i = u^{i-1} \cdot e_1$ for each i, then by Theorem 2.5, E is an FMS R-module and $R \subseteq R(+)E$ has FIP by Proposition 2.4.

(2') Let R be a ring, $n > 1$ an integer and $I_1, \dots, I_n$ ideals of R distinct from R, but not necessarily distinct, such that $\cap_{j=1}^n I_j = 0$. Such a family $\{I_1, \dots, I_n\}$ of ideals of R is called a *separating family*, a reference to Algebraic Geometry where a finite family of morphisms $\{f_j : M \to M_j \mid j = 1, \dots, n\}$ of R-modules is called separating if $\cap_{j=1}^n \ker f_j = 0$. In [17, Section 3], we study the ring extension $R \subseteq \prod_{j=1}^n (R/I_j) =: \mathcal{R}$ associated to a separating family.

We keep the context of (2). Since $u^n = 0$, $u^{n-1}(e_1) \neq 0$ and $e_j = u^{j-1}(e_1)$ for any $j \in \{1, \dots, n\}$, a short calculation gives $I_j := (0 :_R e_j) = Ru^{n-j+1}$. Then, $\cap_{j=1}^n I_j = 0$ because $I_1 = Ru^n = 0$ and $\{I_1, \dots, I_n\}$ is a separating family such that $I_j \subset I_{j+1}$ for each $j \in \{1, \dots, n-1\}$. Moreover, $R/I_j = R/Ru^{n-j+1} \cong k[X]/(X^{n-j+1})$. Set $M := Ru$, $\mathcal{R} := \prod_{i=1}^n (R/(0 : e_i))$ and $J_j := \cap_{k=1, k\neq j}^n I_k$. Then, $J_1 = I_2 \cong (X^{n-1})/(X^n)$ and $J_j = 0$ for each $j > 1$. Apply [17, Corollary 3.10]. We have $\sum_{j=1}^n J_j = I_2$, giving that $R/\sum_{j=1}^n J_j = R/I_2 \cong k[X]/(X^{n-1})$ is a SPIR and $|R/M| = \infty$, because $R/M \cong k$. Since $I_1 + J_1 = I_2 \cong (X^{n-1})/(X^n)$ and $I_j + J_j = I_j \cong (X^{n-j+1})/(X^n)$ for each $j > 1$, it is enough to take $n > 3$ to get that $R \subset \mathcal{R}$ has not FIP.

(3) Let $M = \sum_{i=1}^n Re_i$ be a faithful Artinian R-module and set $\mathcal{R} := \prod_{i=1}^n (R/(0 : e_i))$. Since M is faithful, we have $(0 : M) = 0$. Then, R is an Artinian ring in view of [15, Theorem 2, page 180] because M is a finitely generated Artinian module, and $R \subseteq R(+)M$ has FCP by Proposition 2.2. Since $(0 : M) = \cap_{i=1}^n (0 : e_i) = 0$, the family $\{(0 : e_i)\}_{i=1,\dots,n}$ is separating and $R \subseteq \mathcal{R}$ has FCP by [17, Proposition 3.1].

Examples (1') and (2') show that for a finitely generated R-module $M = \sum_{i=1}^n Re_i$ such that $\{(0 : e_1), \dots, (0 : e_n)\}$ is a separating family, we may have only one of the two extensions $R \subseteq R(+)M$ and $R \subseteq \prod_{i=1}^n (R/(0 : e_i))$ which has FIP, and not the other one.

(4) Let k be an infinite field, $n > 1$ an integer and E an n-dimensional vector space over k. Let $u \in \mathrm{End}(E)$ with minimal polynomial $\pi_u(X) := \prod_{i=1}^s P_i^{\alpha_i}(X)$, with each $P_i(X) \in k[X]$ of degree 1, $P_i(X) \neq P_j(X)$ for $i \neq j$, and such that $n = \sum_{i=1}^s \alpha_i$. For each i, set $E_i := \ker(P_i^{\alpha_i}(u))$. The "Lemme des noyaux" [4, Proposition 3, ch. VII, p. 30] gives that $E = \bigoplus_{i=1}^s E_i$ $(*)$, with $\alpha_i = \dim_k(E_i)$. If $R := k[u]$, then E is a faithful R-module for the scalar multiplication defined by $P(u) \cdot x := P(u)(x)$, for $P(X) \in k[X]$ and $x \in E$. Since $R \cong k[X]/(\pi_u(X))$ is an Artinian FMIR, to conclude that E is an FMS module over R by applying Corollary 2.7, we need only to show that E_M is cyclic for each $M \in \mathrm{Max}(R) = \{M_1, \dots, M_s\}$ where $M_i := P_i(u)R$. We next prove that $E_{M_i} \cong (E_i)_{M_i}$ as R_{M_i}-modules. Let $x \in E_j$ for some $j \neq i$, then $P_j^{\alpha_j}(u)(x) = 0$ and $P_j^{\alpha_j}(u)$ is a unit in R_{M_i} since $P_j(X) \notin (P_i(X))$. It follows that $x/1 = 0$ in E_{M_i}, so that $E_{M_i} \cong (E_i)_{M_i}$ by $(*)$. Now, we are reduced to (2) with $P_i^{\alpha_i}(u) = 0$ in $(E_i)_{M_i}$, so that each $(E_i)_{M_i}$ is cyclic over R_{M_i} and Corollary 2.7 holds.

Theorem 2.13. *A faithful R-module M is an FMS module if and only if the two following conditions are satisfied:*

(1) *R is an FMIR which is a direct product of two rings $R' \times R''$, where $|R'| < \infty$ and $|R''/P| = \infty$ for any $P \in \mathrm{Spec}(R'')$.*
(2) *M is the direct product of a finite R'-module and a rank one projective R''-module.*

Proof. If M is an FMS module, R is an FMIR and M is f.g. over R by Corollary 2.7. Then by Proposition 1.4, $R = \prod_{i=1}^n R_i$, a product of local rings that are either finite, or a SPIR, or a field. Let R' be the ring product of the R_i that are finite and R'' the product of the others. Then $|R'| < \infty$ and a SPIR factor (R_i, P_i) of R'' is such that $|R_i/P_i| = \infty$ because R_i is local Artinian. When R_i is an infinite field, take $P_i = 0$. So, (1) holds with $R = R' \times R''$.

Set $M' := R'M = \{(r', 0)m \mid r' \in R',\ m \in M\}$ and $M'' := R''M = \{(0, r'')m \mid r'' \in R'',\ m \in M\}$. By [3, Remarque 3, ch.II, p.32], we get $M = M' \bigoplus M'' \cong M' \times M''$, $R'M'' = R''M' = 0$ and $(0 :_{R''} M'') = 0$. Clearly, $|M'| < \infty$ since M' is f.g. over the finite ring R'. In the same way, M'' is f.g. over R''. Now an R''-submodule N of M'' gives an R-submodule of M by the one-to-one function $N \mapsto M' \times N$. It follows that M'' is an FMS R''-module. Therefore, we can assume that R is an FMIR with $|R/P| = \infty$ for each $P \in \mathrm{Spec}(R) = \{P_1, \dots, P_n\}$. By Corollary 2.7, M is generated over R by some $e_1, \dots, e_n \in M$ such that $M_{P_i} = R_{P_i}(e_i/1)$ for each i. Actually, $e_i/1$ is free over R_{P_i}: suppose that $(a/t)(e_i/1) = 0$ for $a \in R$ and $t \in R \setminus P_i$. There is some $s_i \in R \setminus P_i$ such that $s_i a e_i = 0$. Moreover, $e_j/1 \in M_{P_i} = R_{P_i}(e_i/1)$ for $j \neq i$ gives that $e_j/1 = (b_j/t_j)(e_i/1)$, for some $b_j \in R$, $t_j \in R \setminus P_i$ for each $j \neq i$. This allows us to pick up some $s_j \in R \setminus P_i$ such that $s_j a e_j = 0$. Setting $s := s_1 \cdots s_n$, we get $sae_k = 0$ for each $k \in \{1, \dots, n\}$. Since M is faithful, $sa = 0$, so that $a/t = 0$. By [2, Théorème 2, ch.II, p.141], M is a rank one projective R-module and (2) follows.

Conversely, assume that (1) and (2) hold and keep the above notation with $R = R' \times R''$, $|R'| < \infty$, $|R''/P| = \infty$ for any $P \in \mathrm{Spec}(R'')$ and $M = M' \times M''$, where M' is a finite R'-module and M'' is a rank one projective R''-module. Then, from [2, Théorème 2, ch. II, p. 141], we deduce that M'' is f.g. over R'', with M''_P cyclic for each maximal ideal P of R''. Since M' is also f.g. over R' because finite, M is f.g. over R. For each $N \in \mathrm{Max}(R)$ such that $|R/N| = \infty$, there exists $P \in \mathrm{Max}(R'')$ such that $N = R' \times P$ and in this case $M_N \cong M''_P$ as R_N-modules. Indeed, consider the R_N-linear isomorphism $u : M_N \cong (M' \times M'')_{R' \times P} \to M''_P$ defined by $u((m', m'')/(s, t)) = m''/t$, using the ring isomorphism $R_N \cong R''_P$. It follows that M_N is cyclic over R_N. By Corollary 2.7, we can conclude that M is an FMS module. $\square$

Remark 2.14. (1) *For the proof of Theorem 2.13, it was convenient to suppose that M is a faithful R-module. However, one should note that Theorem 2.13 can be used to characterize when an arbitrary (not necessarily faithful) module is FMS. In fact, an R-module M is FMS (as an R-module) if and only if M is an FMS module over the ring $R/(0 : M)$.*

(2) *The rings R' and R'' in the statement of Theorem 2.13 are necessarily each FMIRs. In fact, if A and B are rings, then $A \times B$ is an FMIR if and only if both A and B are FMIRs.*

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