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SPANNING SIMPLICIAL COMPLEXES OF n-CYCLIC GRAPHS WITH A COMMON VERTEX

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Abstract. In this paper, we characterize some algebraic and combinatorial properties of spanning simplicial complex $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ of the n-cyclic graphs $G_{l_1,l_2,\cdots,l_n}$ with a common vertex. We show that $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ is pure simplicial complex of dimension $\sum\limits_{i=1}^{n} l_i - n - 1$. We determine the Stanley-Reisner ideal $I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$ of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ and its primary decomposition. Under the condition that the length of each cyclic graph is t, we also give a formula for f-vector of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ and consequently a formula for Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{l_1,l_2,\cdots,l_n})]$, where k is a field.

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1. Introduction

The note of spanning simplicial complex $\Delta_s(G)$ on edge set E of a graph $G = G(V,E)$ was introduced in [1], the set of its facets is exactly the edge set $s(G)$ of all possible spanning trees of G, i.e.

$$\Delta_s(G) = \langle\, F_i \mid F_i \in s(G)\,\rangle.$$

Note that for a graph G, the problem of finding $s(G)$ is not always easy to handle. Anwar, Raza and Kashif [1] proved some algebraic and combinatorial properties of spanning simplicial complex of the uni-cyclic graph U_n (i.e., if the vertex set of U_n is $V = \{x_1,\ldots,x_n\}$, then the edge set of U_n is $E = \{x_ix_{i+1} \mid i = 1,\ldots,n,$ and $x_{n+1} = x_1\}$). Zhu et al. [5] discussed some properties of the spanning simplicial complexes of the n-cyclic graphs with a common edge. In this paper,

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our goal is to characterize some algebraic and combinatorial properties of spanning simplicial complexes of some n-cyclic graphs $G_{l_1, l_2, \cdots, l_n}$ with a common vertex, which is obtained by joining n disjoint cycles $G_{l_1}, \dots, G_{l_n}$ of length $l_1, \dots, l_n$ at a common vertex. For $n = 2$ and $l_1 = l_2 = 3$, the graph of $G_{3,3}$ is shown in Figure 1 of Section 2.

We give a brief overview of this paper. In Section 2, we recall some definitions and results from commutative algebra and algebraic combinatorics. In Section 3, we determine the Stanley-Reisner ideal $I_{\Delta_s(G_{l_1, l_2, \cdots, l_n})}$ of $\Delta_s(G_{l_1, l_2, \cdots, l_n})$ and its primary decomposition in Theorem 3.2. In Section 4, under the assumption that the length of every cycle is t, we give a formula for f-vector of $\Delta_s(G_{l_1, l_2, \cdots, l_n})$ and consequently a formula for Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{l_1, l_2, \cdots, l_n})]$.

2. Preliminaries

We firstly recall some definitions and basic facts about graph and simplicial complex in order to make this paper self-contained.

Definition 2.1. A spanning tree of a simple connected finite graph $G = G(V, E)$ is a subgraph of G, which is a tree and contains all vertices of G. We denote the collection of all edge sets of the spanning trees of G by $s(G)$, i.e.

$s(G) = \{E(T_i) \subset E \mid T_i \text{ is a spanning tree of } G\}$ (See [3] for more details).

It is well known that for any simple connected finite graph, spanning trees always exist. One can find a spanning tree systematically by the cutting-down method, which says that a spanning tree is obtained by removing one edge from each cycle appearing in the graph. For example, for the following graph G, we obtain that

$$\begin{aligned} s(G) \quad = \quad & \{\{e_2, e_3, e_5, e_6\}, \{e_2, e_3, e_4, e_6\}, \{e_2, e_3, e_4, e_5\}, \{e_1, e_3, e_5, e_6\}, \{e_1, e_3, \\ & e_4, e_6\}, \{e_1, e_3, e_4, e_5\}, \{e_1, e_2, e_5, e_6\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_2, e_4, e_5\}\}. \end{aligned}$$

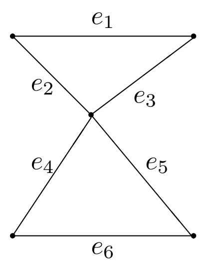

Figure 1. 2-cyclic graph with a common vertex

Definition 2.2. A simplicial complex Δ on a set of vertices $[n] = \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ such that

(1) $\{i\} \in \Delta$ for each $i \in [n]$;

(2) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F and denoted by $dim\, F$. The faces of dimension 0 and 1 are called vertices and edges, respectively, and $dim\, \emptyset = -1$.

The maximum faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ, which is denoted by $dim\, \Delta$, is the maximal dimension of its facets, i.e.

$$dim\, \Delta = \max\{dim\ \ F \mid F \text{ is a facet of } \Delta\}.$$

We denote the simplicial complex Δ with facets $\{F_1, \ldots, F_q\}$ by

$$\Delta = \langle F_1, \ldots, F_q \rangle.$$

Definition 2.3. A simplicial complex Δ is pure if all of its facets have the same dimension.

Definition 2.4. Given a simplicial complex Δ of dimension d, we define its f-vector to be the $(d+1)$-tuple $f = (f_0, f_1, \ldots, f_d)$, where f_i is the number of i-dimensional faces of Δ.

Definition 2.5. For a simple connected finite graph $G = G(V, E)$ with $s(G) = \{E_1, \ldots, E_s\}$, we define a simplicial complex $\Delta_s(G)$ on E such that facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, called the spanning simplicial complex of $G(V, E)$. In other words,

$$\Delta_s(G) = \langle E_1, \ldots, E_s \rangle.$$

As the number of elements of both E_i and E_j are $|E| - m$, where m denotes the number of cycles in G, we have that $E_i \nsubseteq E_j$ for $i \neq j$.

For example, the spanning simplicial complex of the graph G with edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ in Figure 1 is given by

$$\begin{aligned}\Delta_s(G) \quad = \quad & \langle \{e_2, e_3, e_5, e_6\}, \{e_2, e_3, e_4, e_6\}, \{e_2, e_3, e_4, e_5\}, \{e_1, e_3, e_5, e_6\}, \{e_1, e_3, \\ & e_4, e_6\}, \{e_1, e_3, e_4, e_5\}, \{e_1, e_2, e_5, e_6\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_2, e_4, e_5\} \rangle.\end{aligned}$$

Definition 2.6. An n-cyclic graph $G_{l_1, l_2, \cdots, l_n}$ with a common vertex is a graph which is obtained by joining n disjoint cycles $G_{l_1}, G_{l_2}, \ldots, G_{l_n}$ at a common vertex, where G_{l_i} denotes the cycle of length l_i and $l_i \geq 3$ for each $i \in \{1, 2, \ldots, n\}$.

Remark 2.7. *It is easy to see $G_{l_1, l_2, \cdots, l_n}$ has $\sum\limits_{i=1}^{n} l_i - n + 1$ vertices and $\sum\limits_{i=1}^{n} l_i$ edges.*

3. Primary decomposition of $I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$

In this section, we will determine the Stanley-Reisner ideal $I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$ of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ and its primary decomposition.

We label the edge set of $G_{l_1,l_2,\cdots,l_n}$ such that $\{e_{i1}, e_{i2}, \ldots, e_{i\,l_i}\}$ is the edge set of the cycle G_{l_i} for $1 \leq i \leq n$. First, we have the following proposition.

Proposition 3.1. *$\Delta_s(G_{l_1,l_2,\cdots,l_n})$ is a pure simplicial complex of dimension $\sum_{i=1}^{n} l_i - n - 1$.*

Proof. Let $E = \{e_{11}, \ldots, e_{1\,l_1}, e_{21}, \ldots, e_{2\,l_2}, \ldots, e_{n1}, \ldots, e_{n\,l_n}\}$ be the edge set of the n-cyclic graph $G_{l_1,l_2,\cdots,l_n}$. As $G_{l_1,l_2,\cdots,l_n}$ contains exactly n cycles of length l_1, l_2, $\ldots$, l_n, by the cutting-down method, its spanning trees are obtained by removing one edge from each cycle G_{l_i}, $1 \leq i \leq n$. Hence, the subset $E(T_i) \subset E$ is in $s(G_{l_1,l_2,\cdots,l_n})$ if and only if $E(T_i) = E\backslash\{e_{1\,i_1}, \ldots, e_{n\,i_n}\}$ for some $i_j \in \{1, \ldots, l_j\}$, where j runs from 1 to n, i.e.

$$s(G_{l_1,l_2,\cdots,l_n}) = \{E\backslash\{e_{1\,i_1}, \ldots, e_{n\,i_n}\} \mid i_j \in \{1, \ldots, l_j\} \text{ and } j \in \{1, \ldots, n\}\}.$$

It is easily seen that each spanning tree of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ has $\sum_{i=1}^{n}(l_i - 1) = \sum_{i=1}^{n} l_i - n$ edges, thus the result follows. $\square$

Let $E = \{e_{11}, \ldots, e_{1\,l_1}, e_{21}, \ldots, e_{2\,l_2}, \ldots, e_{n1}, \ldots, e_{n\,l_n}\}$ be the edge set of the n-cyclic graph $G_{l_1,l_2,\cdots,l_n}$, and let $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ be the spanning simplicial complex of $G_{l_1,l_2,\cdots,l_n}$. We can assume that $S = k[x_{11}, \ldots, x_{1\,l_1}, x_{21}, \ldots, x_{2\,l_2}, \ldots, x_{n1}, \ldots, x_{n\,l_n}]$ is a polynomial ring in $\sum_{i=1}^{n} l_i$ variables over a field k, $I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$ is the Stanley-Reisner ideal of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$, which is a squarefree monomial ideal. The standard graded algebra $k[\Delta_s(G_{l_1,l_2,\cdots,l_n})] = S/I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$ is called the Stanley-Reisner ring of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$. We can give a primary decomposition of ideal $I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$, Hilbert series and h-vector of $k[\Delta_s(G_{l_1,l_2,\cdots,l_n})]$. We refer readers to [2] and [4] for detailed information about the Stanley-Reisner ideal, primary decomposition and Hilbert series.

Now, we give a primary decomposition of the Stanley-Reisner ideal $I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$ of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$.

Theorem 3.2. *Let $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ be the spanning simplicial complex of the n-cyclic graph $G_{l_1,l_2,\cdots,l_n}$. Then the Stanley-Reisner ideal $I_{\Delta_s(G_{l_1,l_2,\cdots,l_n})}$ of*

$\Delta_s(G_{l_1, l_2, \cdots, l_n})$ *is given by*

$$\begin{aligned} I_{\Delta_s(G_{l_1, l_2, \cdots, l_n})} &= \bigcap_{\substack{i_j \in \{1,2,\ldots,l_j\} \\ j \in \{1,2,\ldots,n\}}} (x_{1i_1}, x_{2i_2}, \ldots, x_{ni_n}) \\ &= (x_{11}x_{12}\cdots x_{1l_1}, x_{21}x_{22}\cdots x_{2l_2}, \ldots, x_{n1}x_{n2}\cdots x_{nl_n}). \end{aligned}$$

Proof. As each of facets of $\Delta_s(G_{l_1, l_2, \cdots, l_n})$ is obtained by removing exactly one edge from each cycle G_{l_i}, $1 \leq i \leq n$. From [4, Proposition 5.3.10], we get that

$$\begin{aligned} I_{\Delta_s(G_{l_1, l_2, \cdots, l_n})} &= \bigcap_{\substack{i_j \in \{1,2,\ldots,l_j\} \\ j \in \{1,2,\ldots,n\}}} (x_{1i_1}, x_{2i_2}, \ldots, x_{ni_n}) \\ &= (x_{11}x_{12}\cdots x_{1l_1}, x_{21}x_{22}\cdots x_{2l_2}, \ldots, x_{n1}x_{n2}\cdots x_{nl_n}). \end{aligned}$$

□

As corollaries, we obtain the following two results.

Corollary 3.3. *Let $\Delta_s(G_{l_1, l_2, \cdots, l_n})$ be the spanning simplicial complex of the n-cyclic graph $G_{l_1, l_2, \cdots, l_n}$. Then the Stanley-Reisner ring $S/I_{\Delta_s(G_{l_1, l_2, \cdots, l_n})}$ is Gorenstein.*

Proof. By the above theorem, we have that

$$I_{\Delta_s(G_{l_1, l_2, \cdots, l_n})} = (x_{11}x_{12}\cdots x_{1l_1}, x_{21}x_{22}\cdots x_{2l_2}, \ldots, x_{n1}x_{n2}\cdots x_{nl_n}).$$

It is clear that $x_{11}x_{12}\cdots x_{1l_1}, x_{21}x_{22}\cdots x_{2l_2}, \ldots, x_{n1}x_{n2}\cdots x_{nl_n}$ is a regular sequence in any order. As S is Gorenstein, the Stanley-Reisner ring $S/I_{\Delta_s(G_{l_1, l_2, \cdots, l_n})}$ is Gorenstein by [2, Proposition 3.1.19]. □

Corollary 3.4. *Let $\Delta_s(G_{l_1, l_2, \cdots, l_n})$ be the spanning simplicial complex of the n-cyclic graph $G_{l_1, l_2, \cdots, l_n}$. Then the Stanley-Reisner ideal $I_{\Delta_s(G_{l_1, l_2, \cdots, l_n})}$ is unmixed of height n.*

4. The computation of f-vector of $\Delta_s(G_{l_1, l_2, \cdots, l_n})$

In this section, we will give a formula for f-vector of $\Delta_s(G_{l_1, l_2, \cdots, l_n})$ and consequently a formula for Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{l_1, l_2, \cdots, l_n})]$ under the assumption that the length of every cycle G_{l_i} is t for $1 \leq i \leq n$. But before this we need the following proposition, its proof can be seen in Proposition 2.2 of [1].

Proposition 4.1. *For a simplicial complex Δ on $[n]$ of dimension d, if $f_t = \binom{n}{t+1}$ for some $t \leq d$, then $f_i = \binom{n}{i+1}$ for all $0 \leq i < t$.*

Now, under the assumption that the length of every cycle G_{l_i} is t for $1 \le i \le n$, we give the formula to compute the f-vector of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$.

Theorem 4.2. *Let $l_i = t$ for any $1 \le i \le n$. Then the f-vector of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ is given by $f = (f_0, f_1, \ldots, f_d)$, where $d = n(t-1) - 1$ and*

$$f_j = \sum_{i=0}^{k} (-1)^i \binom{n}{i} \binom{nt - it}{j - it + 1},$$

where $0 \le j \le d$, $k = [\frac{j+1}{t}]$ and symbol $[a], a \in Q$ denotes the maximum integer not exceeding a.

Proof. As $l_i = t$ for any $1 \le i \le n$, we can set $E = \{e_{11}, \ldots, e_{1\,t}, e_{21}, \ldots, e_{2\,t}, \ldots, e_{n1}, \ldots, e_{n\,t}\}$ be the edge set of the n-cyclic graph $G_{l_1,l_2,\cdots,l_n}$. By the definition of f-vector of $\Delta_s(G_{l_1,l_2,\cdots,l_n})$, f_j is the number of all those subsets of the edge set E of the graph $G_{l_1,l_2,\ldots,l_n}$ with $j+1$ elements, that do not contain these cycles $\{e_{i_1 1}, \ldots, e_{i_1 t} \,|\, 1 \le i_1 \le n\}$, $\{e_{i_1 1}, \ldots, e_{i_1 t}, e_{i_2 1}, \ldots, e_{i_2 t} \,|\, 1 \le i_1 < i_2 \le n\}, \ldots,$ $\{e_{i_1 1}, \ldots, e_{i_1 t}, \ldots, e_{i_k 1}, \ldots, e_{i_k t} \,|\, 1 \le i_1 < \cdots < i_k \le n\}$.

By Remark 2.7, $G_{l_1,l_2,\cdots,l_n}$ has nt edges, thus there are $\binom{n}{k}\binom{nt-kt}{j-kt+1}$ subsets of E with $j+1$ elements, which contain the edge set $\{e_{i_1 1}, e_{i_1 2}, \ldots, e_{i_1 t}, \ldots, e_{i_k 1}, \ldots, e_{i_k t} \,|\, 1 \le i_1 < \cdots < i_k \le n\}$. Similarly, there are $\binom{n}{k-1}[\binom{nt-(k-1)t}{j-(k-1)t+1} - \binom{n-(k-1)}{1}\binom{nt-kt}{j-kt+1}]$ $= \binom{n}{k-1}\binom{nt-kt+t}{j-kt+t+1} - \binom{n}{k-1}\binom{n-k+1}{1}\binom{nt-kt}{j-kt+1} = \sum_{i=0}^{1}(-1)^i\binom{n}{k-1}\binom{n-k+1}{i}\binom{nt-(k-1+i)t}{j-(k-1+i)t+1}$ subsets of E with $j+1$ elements, containing the edge set $\{e_{i_1 1}, \ldots, e_{i_1 t}, \ldots, e_{i_{k-1} 1}, \ldots, e_{i_{k-1} t} \,|\, 1 \le i_1 < \cdots < i_{k-1} \le n\}$. By analogy, there are

$$\begin{aligned}
& \binom{n}{k-2}\{\binom{nt-(k-2)t}{j-(k-2)t+1} - \binom{n-(k-2)}{1}[\binom{nt-(k-1)t}{j-(k-1)t+1} \\
- & \binom{n-(k-1)}{1}\binom{nt-kt}{j-kt+1}] - \binom{n-(k-2)}{2}\binom{nt-kt}{j-kt+1}\} \\
= & \binom{n}{k-2}\binom{nt-kt+2t}{j-kt+2t+1} - \binom{n}{k-2}\binom{n-k+2}{1}\binom{nt-kt+t}{j-kt+t+1} \\
+ & \binom{n}{k-2}[\binom{n-k+2}{1}\binom{n-k+1}{1} - \binom{n-k+2}{2}]\binom{nt-kt}{j-kt+1} \\
= & \binom{n}{k-2}\binom{nt-kt+2t}{j-kt+2t+1} - \binom{n}{k-2}\binom{n-k+2}{1}\binom{nt-kt+t}{j-kt+t+1} \\
+ & \binom{n}{k-2}\binom{n-k+2}{2}\binom{nt-kt}{j-kt+1} \\
= & \sum_{i=0}^{2}(-1)^i\binom{n}{k-2}\binom{n-k+2}{i}\binom{nt-(k-2+i)t}{j-(k-2+i)t+1}
\end{aligned}$$

subsets of E with $j+1$ elements, containing the edge set $\{e_{i_1 1}, e_{i_1 2}, \ldots, e_{i_1 t}, \ldots, e_{i_{k-2}1}, e_{i_{k-2}2}, \ldots, e_{i_{k-2}t} \,|\, 1 \leq i_1 < \cdots < i_{k-2} \leq n\}$ and so on. Therefore, the number of all subsets of E, which have $j+1$ elements and contain the edge set $\{e_{i_1 1}, \ldots, e_{i_1 t}, \ldots, e_{i_{k-m},1}, \ldots, e_{i_{k-m},t} \,|\, 1 \leq i_1 < \cdots < i_m \leq n\}$, is

$$\sum_{i=0}^{m}(-1)^i \binom{n}{k-m}\binom{n-k+m}{i}\binom{nt-(k-m+i)t}{j-(k-m+i)t+1}.$$

Therefore, by inclusion exclusion principle, we have

$$\begin{aligned}
f_j &= \binom{nt}{j+1} - \binom{n}{k}\binom{nt-kt}{j-kt+1} \\
&- \sum_{i=0}^{1}(-1)^i \binom{n}{k-1}\binom{n-k+1}{i}\binom{nt-(k-1+i)t}{j-(k-1+i)t+1} \\
&- \sum_{i=0}^{2}(-1)^i \binom{n}{k-2}\binom{n-k+2}{i}\binom{nt-(k-2+i)t}{j-(k-2+i)t+1} - \cdots \\
&- \sum_{i=0}^{m}(-1)^i \binom{n}{k-m}\binom{n-k+m}{i}\binom{nt-(k-m+i)t}{j-(k-m+i)t+1} - \cdots \\
&- \sum_{i=0}^{k-1}(-1)^i \binom{n}{1}\binom{n-1}{i}\binom{nt-(1+i)t}{j-(1+i)t+1} \\
&= \binom{nt}{j+1} - [\sum_{j=1}^{k}(-1)^{k-j}\binom{n}{j}\binom{n-j}{k-j}]\binom{nt-kt}{j-kt+1} \\
&- [\sum_{j=1}^{k-1}(-1)^{k-1-j}\binom{n}{j}\binom{n-j}{k-1-j}]\binom{nt-(k-1)t}{j-(k-1)t+1} - \cdots \\
&- [\sum_{j=1}^{2}(-1)^{2-j}\binom{n}{j}\binom{n-j}{2-j}]\binom{nt-2t}{j-2t+1} - \binom{n}{1}\binom{nt-t}{j-t+1} \\
&= \sum_{i=0}^{k}(-1)^i \binom{n}{i}\binom{nt-it}{j-it+1},
\end{aligned}$$

where the last equality holds by combinatorial formula $\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{j}\binom{n-j}{k-j} = 0$. □

We can now give a formula for Hilbert series of $k[\Delta_s(G_{l_1,l_2,\cdots,l_n})]$ under the condition that the length of every cycle G_{l_i} is t for $1 \leq i \leq n$.

Theorem 4.3. *Let $\Delta_s(G_{l_1,l_2,\cdots,l_n})$ be the spanning simplicial complex of the n-cyclic graph $G_{l_1,l_2,\cdots,l_n}$, where $l_i = t$ for every $1 \leq i \leq n$. Then Hilbert series of* *the Stanley-Reisner ring* $k[\Delta_s(G_{l_1,l_2,\cdots,l_n})]$ *is given by*

$$H(k[\Delta_s(G_{l_1,l_2,\cdots,l_n})],z) = 1+\sum_{i=0}^{d}\sum_{l=0}^{k}(-1)^l\binom{n}{l}\binom{nt-lt}{j-lt+1}.$$

Proof. From [4, Corollary 5.4.5], we know that if Δ is a simplicial complex and $f(\Delta)=(f_0,\ldots,f_d)$ is its f-vector, then the Hilbert series of the Stanley-Reisner ring $k[\Delta]$ is given by

$$H(k[\Delta],z)=\sum_{i=-1}^{d}\frac{f_i z^{i+1}}{(1-z)^{i+1}}, \qquad d = dim\,\Delta.$$

The desired formula follows from the theorem above at once. □

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