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Kompleks düzlemin dairesel bölgesindeki lineer diferansiyel denklemlerin çözümleri için bir polinom yaklaşımı

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Anahtar Kelimeler:
Taylor sıralama yöntemi; Polinom yaklaşımı; Kompleks diferansiyel denklemler.

ÖZET

Bu makalede, dairesel bölgelerde yüksek mertebeden lineer kompleks diferansiyel denklemlerin çözümü için bir polinom yaklaşımı verilmektedir. Kullanılan bu sıralama yöntemi esas olarak denklemdeki bilinmeyen fonksiyon ve türev ifadelerinin kesilmiş Taylor seri temsillerinin matris gösterimlerine dayanır ki bunlar verilen bölgede tanımlanan sıralama noktalarını içerir. Yöntemin özelliklerini göstermek için karışık koşullu bazı sayısal örnekler verilmiştir.

A Polynomial Approximation for Solutions of Linear Differential Equations in Circular Domains of the Complex Plane

Keywords:
Taylor Collocation method; Polynomial approximation; Complex differential equations.

ABSTRACT

In this paper we give a polynomial approach to the solution of higher order linear complex differential equations in the circular domains. The used collocation method is essentially based on the matrix representations of the truncated Taylor series of the expressions in equation and their derivatives, which consist of collocation points defined in the given domains. Some numerical examples with the mixed conditions are given to show the properties of the technique.

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1. Introduction

When a mathematical model is formulated for a physical problem it is often represented by complex differential equations that are not solvable exactly by analytic techniques. Therefore one must resort to approximation and numerical methods. For example, the vibrations of a one-mass system with two degrees of freedom are mostly described using differential equation with a complex dependent variable. The differential equation is usually linear. The solution of the differential equation clarifies the linear phenomena which occur in the system. The study of these systems is of interest to several fields of statistical mechanics, physics, electronics and engineering. Examples of such applications include rotor dynamics, particle beams in high energy accelerators, plasma physics, kinetic theory, etc. [1, and references therein]. The various methods for solving differential equations with complex dependent variable are introduced by Cveticanin and in the references of the papers [2, 3].

In recent years, the studies on complex differential equations, i.e. a geometric approach based on meromorphic function in arbitrary domains [4], a topological description of solutions of some complex differential equations with multi-valued coefficients [5], the zero distribution [6] and growth estimates [7] of linear complex differential equations, the rational and polynomial approximations of analytic functions in the complex plane [8, 9], have been developed very rapidly and intensively.

On the other hand, some Taylor and Chebyshev (matrix and Collocation) methods to solve linear differential, integral, integro-differential, difference and integro-difference equations have been presented in many papers by Sezer et. al. [10-16].

Our purpose in this study is to develop and to apply the mentioned methods above to the linear complex differential equation

$$\sum_{k=0}^{m} P_k(z) f^{(k)}(z) = g(z), \quad f^{(0)} = f\ , \qquad (1)$$

which is a generalized case of the complex differential equations given in [6,7,17-19], with the mixed conditions

$$\sum_{k=0}^{m-1} \sum_{r=0}^{R} c_{jk} f^{(k)}(\zeta_r) = \lambda_j, \ j = 0,\ 1,\ \ldots,\ m-1, \qquad (2)$$

and to find the solution in terms of the Taylor polynomial of starting point $z = z_0$,

$$f(z) \cong \sum_{n=0}^{N} f_n (z - z_0)^n, \quad f_n = \frac{f^{(n)}(z_0)}{n!};\ z, z_0 \in D, \quad N \ge m, \qquad (3)$$

Here $P_k(z)$ and $g(z)$ are analytical functions in the circular domain

$$D = \left\{ z \in \mathbb{C},\ |z - z_0| \le \rho, \quad \rho \in \mathbb{R}^+ \right\},$$

c_{jk} and λ_j are appropriate complex coefficients, $\zeta_r \in D$ and f_n, $n = 0,1,\ldots,N$, are the Taylor coefficients to be determined.

Besides, the collocation points to be used in the solution method are defined by

$$z_{pq} = z_0 + \frac{\rho p}{N} e^{i\frac{\theta q}{N}}, \quad 0 < \theta \le 2\pi,\ 0 < \rho < \infty, \quad p,q = 0,1,\ldots,N\ . \qquad (4)$$

Note that $z_{0q} = z_0$ for $q = 0,1,\ldots,N;\ z_{p0} = z_{pN} = z_0 + \frac{\rho}{N} p$ for $p = 1,2,\ldots,N$ and $\theta = 2\pi$.

The technique presented in the paper is a formal method. For this reason, existence and uniqueness of the solution of the problem (1)-(2) is beyond of the paper, and convergence of the method is not analyzed.

2. Fundamental matrix relations

We first consider the solution *f*(*z*) and its derivative $f^{(k)}(z)$ in the form

$$f^{(k)}(z) \cong \sum_{n=0}^{N} f_n^{(k)} (z - z_0)^n. \qquad (5)$$

The relation between the Taylor coefficients is

$$f_n^{(k+1)} = (n+1) f_{n+1}^{(k)}, \quad n,\ k = 0,\ 1,\ 2,\ \ldots \qquad (6)$$

Then we convert the expression (5) to the matrix form

$$f^{(k)}(z) \cong \mathbf{Z}(z)\mathbf{F}^{(k)} \qquad (7)$$

where

$$\mathbf{Z}(z) = \begin{bmatrix} 1 & (z-z_0) & (z-z_0)^2 & \ldots & (z-z_0)^N \end{bmatrix}$$

$$\mathbf{F}^{(k)} = \begin{bmatrix} f_0^{(k)} & f_1^{(k)} & \cdots & f_N^{(k)} \end{bmatrix}^T.$$

Note that $\mathbf{F}^{(0)} = \mathbf{F} = \begin{bmatrix} f_0 & f_1 & \cdots & f_N \end{bmatrix}^T$.

In addition, from the recurrence relation (6) it is obtained the matrix relation [15].

$$\mathbf{F}^{(k)} = \mathbf{M}^k \mathbf{F} \tag{8}$$

where $\mathbf{M}^0$ is a unit matrix and

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Substituting (8) into (7), we have the relation

$$f^{(k)}(z) \cong \mathbf{Z}(z)\mathbf{M}^k\mathbf{F}. \tag{9}$$

For the collocation points $z = z_{pq},\ p,q = 0,1,\ldots,N,$ the matrix relation (9) becomes

$$f^{(k)}(z_{pq}) \cong \mathbf{Z}(z_{pq})\mathbf{M}^k\mathbf{F} \tag{10}$$

where

$$\mathbf{Z}(z_{pq}) = \begin{bmatrix} 1 & (z_{pq}-z_0) & (z_{pq}-z_0)^2 & \ldots & (z_{pq}-z_0)^N \end{bmatrix}.$$

For $p = 0,1,\ldots,N$, we can write the relation (10) in the form

$$\mathbf{F}_q^{(k)} \cong \mathbf{Z}_q\mathbf{M}^k\mathbf{F},\quad q = 0,1,\ldots,N, \tag{11}$$

where

$$\mathbf{F}_q^{(k)} = \begin{bmatrix} f^{(k)}(z_{0q}) & f^{(k)}(z_{1q}) & \cdots & f^{(k)}(z_{Nq}) \end{bmatrix}^T$$

and

$$\mathbf{Z}_q = \begin{pmatrix} \mathbf{Z}(z_{0q}) \\ \mathbf{Z}(z_{1q}) \\ \vdots \\ \mathbf{Z}(z_{Nq}) \end{pmatrix} = \begin{pmatrix} 1 & (z_{0q}-z_0) & (z_{0q}-z_0)^2 & \ldots & (z_{0q}-z_0)^N \\ 1 & (z_{1q}-z_0) & (z_{1q}-z_0)^2 & \ldots & (z_{1q}-z_0)^N \\ & \cdots & \cdots & \cdots & \cdots \\ 1 & (z_{Nq}-z_0) & (z_{Nq}-z_0)^2 & \ldots & (z_{Nq}-z_0)^N \end{pmatrix}.$$

On the other hand, substituting the collocation points $z = z_{pq}$ defined by (4) into Eq. (1) we have

$$\sum_{k=0}^{m} P_k(z_{pq}) f^{(k)}(z_{pq}) = g(z_{pq}),\quad p,q = 0,1,\ldots,N. \tag{12}$$

By using the expressions (10), (11) and (12), we obtain the fundamental matrix equation

$$\sum_{k=0}^{m}\sum_{q=0}^{N} \mathbf{P}_{kq}\mathbf{Z}_q\mathbf{M}^k\mathbf{F} = \sum_{q=0}^{N}\mathbf{G}_q \tag{13}$$

where

$$\mathbf{G}_q = \begin{pmatrix} g(z_{0q}) \\ g(z_{1q}) \\ \vdots \\ g(z_{Nq}) \end{pmatrix},\quad \mathbf{P}_{kq} = \begin{pmatrix} P_k(z_{0q}) & 0 & \cdots & 0 \\ 0 & P_k(z_{1q}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(z_{Nq}) \end{pmatrix}.$$

Note that, as we have $(N+1)^2$ collocation points, $(N+1)^2$ equations for the Taylor coefficients are obtained. But in formula (13) we leave only $(N+1)$ equations summing $\mathbf{G}_q$ from $q = 0$ up to $q = N$.

Besides, we can obtain the corresponding matrix form of conditions (2) as follows. By means of the relation (9) we have the matrix equation

$$\left\{\sum_{k=0}^{m-1}\sum_{r=0}^{R} c_{jk}\mathbf{Z}(\zeta_r)\mathbf{M}^k\right\}\mathbf{F} = \lambda_j,\quad j = 0,\ 1,\ \ldots,\ m-1 \tag{14}$$

where $\zeta_r \in D,\quad c_{jk} \in \square$, and

$$\mathbf{Z}(\zeta_r) = \begin{bmatrix} 1 & (\zeta_r - z_0) & (\zeta_r - z_0)^2 & \ldots & (\zeta_r - z_0)^N \end{bmatrix}.$$

Briefly, the matrix equation (14) is

$$\mathbf{U}_j\mathbf{F} = \lambda_j,\quad j = 0,\ 1,\ \ldots,\ m-1, \tag{15}$$

where

$$\mathbf{U}_j = \sum_{k=0}^{m-1}\sum_{r=0}^{R} c_{jk}\mathbf{Z}(\zeta_r)\mathbf{M}^k \equiv \begin{bmatrix} u_{j0} & u_{j1} & \cdots & u_{jN} \end{bmatrix}.$$

3. Method of Solution

Let us now consider the fundamental matrix equation (13) corresponding to Eq. (1). We can write Eq. (13) in the form

$$\mathbf{WF=G} \tag{16}$$

where

$$\mathbf{W} = [w_{in}] = \sum_{k=0}^{m}\sum_{q=0}^{N}\mathbf{P}_{kq}\mathbf{Z}_q\mathbf{M}^k,\quad i,n = 0,1,\ldots,N$$

and

$$\mathbf{G} = \sum_{q=0}^{N}\mathbf{G}_q \equiv [g_0 \quad g_1 \quad \cdots \quad g_N]^T.$$

The augmented matrix of Eq. (16) becomes

$$\left[\mathbf{W};\mathbf{G}\right]=\left[w_{in};g_n\right] \tag{17}$$

and also the augmented matrix of Eq. (15) corresponding to conditions (2) can be written in the form

$$\left[\mathbf{U}_j;\lambda_j\right]\equiv\left[u_{j0}\quad u_{j1}\quad\cdots\quad u_{jN};\lambda_j\right] \tag{18}$$

Replacing the last m rows of the augmented matrix (17) by the m rows of matrix (18), we get the new augmented matrix $[\mathbf{W}^*;\mathbf{G}^*]$. If $\det\mathbf{W}^*\neq 0$, we can write

$$\mathbf{F}=\left(\mathbf{W}^*\right)^{-1}\mathbf{G}^* \tag{19}$$

and the array **F** of the unknown Taylor coefficients f_n is uniquely determined. Thus the m*th*-order linear complex differential equation (1) under the conditions (2) has a solution in the form (3).

Besides, we can easily check the accuracy of the solutions as follows. Since the Taylor polynomial (3) is an approximate solution of Eq. (1), when the solution $f(z)$ and its derivatives are substituted in Eq. (1), the resulting equation must be satisfied approximately, i.e., for $z=z_r$, $r=0,1,2,\ldots$, k_r positive integer.

$$E(z_r)=\left|\sum_{k=0}^{m}P_k(z_r)\left(\sum_{n=0}^{N}f_n^{(k)}(z_r-z_0)^n\right)-g(z_r)\right|\leq 10^{-k_r} \tag{20}$$

If k-1 exact decimal digits are required for the solution, then the truncation limit N is increased until

$$\max_r E(z_r)\leq 10^{-k}.$$

4. Illustrative Examples

Taking $z_0=0, \rho=1, \theta=2\pi$ for the collocation points defined in (4), the following examples were solved.

Example 1: Let us first consider the problem

$$f''(z)+z^2f'(z)-zf(z)=3z^3-z+6,\quad f(0)=1,\quad f'(0)=2,\quad |z|\leq 1.$$

Choosing $N = 2$, we have the augmented matrix

$$[\mathbf{W}^*;\mathbf{G}^*]=\begin{pmatrix}0 & 0 & 6 & ; & 18\\ 1 & 0 & 0 & ; & 1\\ 0 & 1 & 0 & ; & 2\end{pmatrix}$$

so that $\mathbf{F}=\begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$. This yields the exact solution $f(z)=3z^2+2z+1$.

Table 1. Error analysis of Example 2

z	$\lvert f_N(z)-f(z)\rvert$		$E(z)$	
	N = 10	N = 12	N = 10	N = 12
0	0	0	4.13E-14	4.61E-13
0.1	1.93E-14	2.66E-19	6.97E-13	7.38E-12
0.2	1.20E-13	3.33E-15	2.50E-13	1.37E-11
0.3	3.00E-13	2.11E-14	9.35E-12	9.71E-12
0.4	5.68E-13	6.37E-14	1.64E-10	1.03E-12
0.5	1.02E-12	1.38E-13	1.52E-9	1.98E-12
0.6	2.52E-12	2.44E-13	9.41E-9	3.12E-11
0.7	1.06E-11	3.96E-13	4.40E-8	2.29E-10
0.8	5.23E-11	7.26E-13	1.56E-7	1.09E-9
0.9	2.28E-10	1.90E-12	4.38E-7	4.20E-9
1	8.34E-10	6.59E-12	1.02E-6	1.37E-8

Table 2. Comparison the absolute errors of Example 3

z	Sezer and Gülsu's Method		Present Method		
	N = 9	N = 11	N = 9	N = 11	N = 12
0.0 + 0.0i	0	0	0	0	0
0.1 + 0.1i	0.7359E-9	0.7826E-10	2.5887E-12	2.4318E-15	8.2114E-16
0.2 + 0.2i	0.1237E-7	0.1503E-8	1.7511E-12	7.0230E-15	6.2119E-15
0.3 + 0.3i	0.6315E-7	0.8173E-8	2.9659E-11	4.1154E-14	2.5422E-14
0.4 + 0.4i	0.2001E-6	0.2674E-7	1.0743E-10	3.6518E-13	1.0456E-13
0.5 + 0.5i	0.4887E-6	0.6658E-7	7.2451E-10	1.4451E-12	4.0652E-13
0.6 + 0.6i	0.1013E-5	0.1399E-6	9.5332E-9	1.4661E-11	5.0565E-12
0.7 + 0.7i	0.1877E-5	0.2617E-6	5.8629E-8	2.0680E-10	5.7108E-11
0.8 + 0.8i	0.3203E-5	0.4497E-6	2.5433E-7	1.4309E-9	4.1195E-10
0.9 + 0.9i	0.5132E-5	0.7244E-6	8.8455E-7	7.0334E-9	2.2183E-9
1.0 + 1.0i	0.7828E-5	0.1108E-5	2.6282E-6	2.7704E-8	9.6869E-9

Example 2: Consider the third order initial value problem

$$f'''(z) - \tfrac{4}{9} f'(z) + e^{z} f(z) = 1 - \tfrac{5}{9} e^{-z} + e^{z}, \quad f(0) = 2, \quad f'(0) = -1, \quad f''(0) = 1$$

Applying the presented method to the problem, we obtain the absolute errors shown in Table 1 for N = 10 and N = 12. The exact solution of the problem is $f(z) = 1 + e^{-z}$. Table 1 also shows the accuracy of the solution in Eq. (20).

Example 3: Consider the following problem given in Ref. [20]

$$f''(z) + zf'(z) + 2zf(z) = 2z\sin z + z\cos z - \sin z, \quad f(0) = 0, \quad f'(0) = 1, \quad |z| \le 1$$

In Table 2, for different values of N, the results obtained by the present method are compared with the Taylor collocation method proposed in [20] by Sezer and Gülsu. As you can see, for any values of z, the present method gives better results.

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