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Convergence of SP-iteration for generalized nonexpansive mapping in Hadamard spaces

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Abstract

In this paper, we study the convergence of SP-iteration scheme for a class of mappings satisfying the condition (C) and prove Δ-convergence as well as strong convergence theorems in Hadamard spaces. Our results generalize and improve several relevant results of the existing literature.

Keywords: Hadamard spaces, Δ-convergence.

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1. Introduction

Approximation of fixed points remains a widely used technique to prove the existence of solutions of ordinary as well as partial differential equations. In recent years, a multitude of iterative procedures has been developed and utilized to approximate the fixed points of various classes of mappings. Indeed, the Mann and Ishikawa iteration procedures are two basic iteration schemes which now form the foundation of iterative fixed point theory.

In an attempt to construct a convergent sequence of iterates involving a nonexpansive mapping, Mann [15] defined an iteration method as (for any $x_1 \in K$)

(1.1) $$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n \in \mathbb{N}$$

where $\alpha_n \in (0,1)$.

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In 1974, with a view to approximate the fixed point of pseudo-contractive mappings in Hilbert spaces, Ishikawa [11] introduced a new iteration procedure as (for $x_1 \in K$)

$$(1.2)\qquad \begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, & n \in \mathbb{N} \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \in (0,1)$.

Iterative techniques for approximating fixed points have been investigated by various authors (e.g., [12, 17, 18, 22, 24, 25, 26, 27]) using the Mann iteration scheme or Ishikawa iteration scheme. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an up-to date account of literature on this topic, we refer the readers to Berinde [2].

As a genuine extension of Mann and Ishikawa iteration schemes, Xu and Noor [28] introduced a three step iteration scheme as (for $x_1 \in K$)

$$(1.3)\qquad \begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ z_n = (1-\beta_n)x_n + \beta_n T y_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n, \end{cases}$$

where $\{\alpha_n\},\ \{\beta_n\}$ and $\{\gamma_n\} \in (0,1)$.
Thianwan [23] introduced the following two-step iteration scheme as (for $x_1 \in K$)

$$(1.4)\qquad \begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \end{cases}$$

where $\{\alpha_n\},\ \{\beta_n\} \in (0,1)$.
Recently, Phuengrattana and Suantai [16] defined the SP-iteration as (for $x_1 \in K$)

$$(1.5)\qquad \begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ z_n = (1-\beta_n)y_n + \beta_n T y_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n T z_n, \end{cases}$$

where $\{\alpha_n\},\ \{\beta_n\}$ and $\gamma_n \in (0,1)$.

In [16], Phuengrattana and Suantai showed that the rate of convergence of the Mann, Ishikawa, Xu and Noor and SP-iteration are equivalent for nonexpansive mapping and SP-iteration converges better than the other schemes for the class of continuous and nondecreasing functions. On the other hand, in 2008, Suzuki [21] introduced a new class of mappings which is larger than the class of nonexpansive mappings and name the defining condition as condition (C) (sometimes also referred as generalized nonexpansive mapping) and prove some existence and convergence theorems.

In this paper, we prove Δ as well as strong convergence theorems under SP-iteration in Hadamard spaces for generalized nonexpansive mappings. In process, several relevant results contained in Xu and Noor [28], Phuengrattana and Suantai [16] and Şahin and Başarır [19] are generalized and improved.

2. Basic definitions and relevant results

To make our presentation self contained, we collect some basic definitions and needed results. We begin with a metric space (X, d) wherein a geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset R$ to X such that $c(0) = x,\ c(r) = y$ and $d(c(t),\ c(s)) = |s - t|$ for all $s,\ t \in [0,\ r]$. In particular, the mapping c is an isometry and $d(x,\ y) = r$. The image of c is called a geodesic segment joining x and y which is denoted by $[x,\ y]$ whenever such a segment exists uniquely. For any $x,\ y \in X$, we denote the point $z \in [x,\ y]$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \le \alpha \le 1$ if $d(x,\ z) = \alpha d(x,\ y)$ and

$d(z,\ y) = (1-\alpha)d(x,\ y)$. The space $(X,\ d)$ is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,\ y \in X$. A subset C of X is called convex if C contains every geodesic segment joining any two points in C.

A geodesic triangle $\triangle(x_1,\ x_2,\ x_3)$ in a geodesic metric space $(X,\ d)$ is consisted of three points of X (as the vertices of $\triangle$) and a geodesic segment between each pair of points (as the edges of $\triangle$). A comparison triangle for $\triangle(x_1,\ x_2,\ x_3)$ in $(X,\ d)$ is a triangle $\overline{\triangle}(x_1,\ x_2,\ x_3) := \triangle(\overline{x}_1,\ \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i,\ \overline{x}_j) = d(x_i,\ x_j)$ for $i,\ j \in \{1,2,3\}$. A point $\overline{x} \in [\overline{x}_1,\ \overline{x}_2]$ is said to be comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\overline{x}_1, \overline{x})$. Comparison points on $[\overline{x}_2, \overline{x}_3]$ and $[\overline{x}_3, \overline{x}_1]$ are defined in same way.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom (CAT(0) inequality):

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,\ y \in \triangle$ and all comparison points $\overline{x},\ \overline{y}\ \in\ \overline{\triangle}$,

$$d(x,\ y) \leq d_{\mathbb{R}^2}(\overline{x},\ \overline{y}).$$

If $x,\ y_1$ and y_2 are points of CAT(0) space and y_0 is the midpoint of the segment $[y_1,\ y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2\ \leq\ \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

The above inequality is known as (CN) inequality and was given by Bruhat and Tits [5]. A geodesic space is a CAT(0) space if and only if it satisfies (CN) inequality. The following classes of subsets are examples of CAT(0) spaces:

(i)Any convex subset of a Euclidean space $\mathbb{R}^n$, when endowed with the induced metric is a CAT(0) space.

(ii) Every pre -Hilbert space is a CAT(0) space.

(iii)If a normed real vector space X is CAT(0) space, then it is a pre-Hilbert space.

(iv) If X_1 and X_2 are CAT(0) spaces, then $X_1 \times X_2$ is also a CAT(0) space.

A complete CAT(0) space is called Hadamard space. For further details on these spaces, one can be referred to [3, 4, 5, 6].

Now, we collect some basic geometric properties which will be utilized throughout the subsequent discussion. Let X be Hadamard space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$ set:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x\ \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, x_n) = r(\{x_n\})\}.$$

It is well known for a Hadamard space that $A(\{x_n\})$ consists of exactly one point (see Proposition 5 of [8]).

In 2008, Kirk and Panyanak [13] gave a concept of convergence in CAT(0) spaces which is an analogue of weak convergence in Banach spaces and restriction of Lim's concepts of convergence [14] to CAT(0) spaces.

2.1. Definition. ([13]) A sequence $\{x_n\}$ in X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of u_n for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and read as x is the Δ-limit of $\{x_n\}$.

Notice that given $\{x_n\} \subset X$ such that x_n Δ-converges to x and given $y \in X$ with $y \neq x$, by uniqueness of asymptotic center we have,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property. Now, we collect some basic facts about CAT(0) spaces which will be used throughout the text.

2.2. Lemma. ([13]) *Every bounded sequence in a Hadamard space admits a Δ-convergent subsequence.*

2.3. Lemma. ([7]) *If C is closed convex subset of a Hadamard space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

2.4. Lemma. ([9]) *Let $(X,\ d)$ be a CAT(0) space. For $x,\ y \in X$ and $t \in [0,1]$, there exists a unique $z \in [x,\ y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1-t)d(x, y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point z of the above lemma.

2.5. Lemma. ([9]) *For $x, y, z \in X$ and $t \in [0,1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

2.6. Lemma. (([9]) *Let X be a CAT (0) space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Now, we give the definition of condition (C) in CAT(0) spaces.

2.7. Definition. ([21]) A mapping T defined on a subset K of a CAT(0) space X is said to satisfy condition (C) if (for all $\mathrm{x, y} \in K$)

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

It is straightforward to notice that every nonexpansive mapping satisfies condition (C). If a mapping T satisfies condition (C) and has a fixed point, then T remains a quasinonexpansive mapping. But the converse of above statements need not be true in general. The following examples demonstrate such facts.

2.8. Example. ([21]) Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0, & \text{when } x \neq 3 \\ 1, & \text{when } x = 3. \end{cases}$$

Then T satisfies condition (C) but T is not a nonexpansive mapping.

2.9. Example. ([21]) Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0, & \text{when } x \neq 3 \\ 2, & \text{when } x = 3. \end{cases}$$

Then $F(T) \neq \emptyset$ and T is a quasinonexpansive mapping but does not satisfy condition (C).

Also, the following theorem is quite interesting.

2.10. Theorem. ([21]) *Let T be a mapping on a closed subset K of a Banach space X. Assume that T satisfies condition (C). Then $F(T)$ is closed. Moreover, if X is strictly convex and K is convex, then $F(T)$ is also convex.*

The following result is crucial and will be used repeatedly.

2.11. Lemma. ([21]) *Let K be a subset of a CAT(0) space X and $T : K \to K$ be a mapping which satisfies condition (C), then for all $x, y \in K$ the following holds:*

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y).$$

Now, we write the iteration scheme of Thianwan [23] in CAT(0) space as (for $x_1 \in K$)

$$\begin{cases} y_n = (1-\gamma_n)x_n \oplus \gamma_n T x_n \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T y_n, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \in (0,1)$, while the SP-iteration as (for $x_1 \in K$)

$$\begin{cases} y_n = (1-\gamma_n)x_n \oplus \gamma_n T x_n \\ z_n = (1-\beta_n)y_n \oplus \beta_n T y_n, \\ x_{n+1} = (1-\alpha_n)z_n \oplus \alpha_n T z_n. \end{cases} \tag{2.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \in (0,1)$.

In this paper, we study the convergence behaviour of SP-iteration scheme (2.2) for generalized nonexpansive mappings in Hadamard spaces which generalize several relevant existing results in literature.

3. Main results

We begin with the following auxiliary lemmas.

3.1. Lemma. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ be generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. If $\{x_n\}$ is described by (2.2), then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. Since,

$$\frac{1}{2} d(p, Tp) = 0 \leq d(x_n, p),$$

which due to condition (C) gives rise $d(Tx_n, Tp) \leq d(x_n, p)$.

Similarly, we have $d(Ty_n, Tp) \leq d(y_n, p)$ and $d(Tz_n, Tp) \leq d(z_n, p)$. By Equation (2.2) and owing to Lemma 2.5, we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T x_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, Tp) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)y_n \oplus \gamma_n T y_n, p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, Tp) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(y_n, p) \\ &= d(y_n, p). \end{aligned} \tag{3.2}$$

In view of equations (3.1) and (3.2), we get

$$d(z_n, p) \leq d(x_n, p). \tag{3.3}$$

Now,

$$d(x_{n+1},p) = d((1-\alpha_n)z_n \oplus \alpha_n Tz_n, p)$$
$$\leq (1-\alpha_n)d(z_n,p) + \alpha_n d(Tz_n, Tp) \tag{3.4}$$
$$\leq (1-\alpha_n)d(z_n,p) + \alpha_n d(z_n,p)$$
$$= d(z_n,p).$$

On combining (3.3) and (3.4), we get

$$d(x_{n+1},p) \leq d(x_n,p) \tag{3.5}$$

which shows that $\{d(x_n,p)\}$ is a decreasing sequence of non-negative reals. Thus in all, sequence $\{d(x_n,p)\}$ is bounded below and decreasing and hence remains convergent. □

3.2. Lemma. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. If $\{x_n\}$ is described by (2.2), then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. From Lemma 3.1, $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$. Let us write,

$$\lim_{n\to\infty} d(x_n,p) = c. \tag{3.6}$$

In view of Equations (3.4) and (3.6), we have

$$\liminf_{n\to\infty} d(z_n,p) \geq c$$

while in view of Equations (3.3) and (3.6), we also have

$$\limsup_{n\to\infty} d(z_n,p) \leq c$$

so that

$$\lim_{n\to\infty} d(z_n,p) = c. \tag{3.7}$$

Also, owing to Equations (3.2) and (3.7), we get

$$\liminf_{n\to\infty} d(y_n,p) \geq c$$

while Equation (3.1) implies that

$$\limsup_{n\to\infty} d(y_n,p) \leq c,$$

so that

$$\lim_{n\to\infty} d(y_n,p) = c. \tag{3.8}$$

Now, in view of Lemma 2.6, we can have

$$\begin{aligned} d(y_n,p)^2 &= d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p)^2 \\ &\leq (1-\gamma_n)d(x_n,p)^2 + \gamma_n d(Tx_n,Tp)^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \\ &\leq (1-\gamma_n)d(x_n,p)^2 + \gamma_n d(x_n,p)^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \\ &\leq d(x_n,p) - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \end{aligned}$$

implying thereby

$$\gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \leq d(x_n,p)^2 - d(y_n,p)^2$$

so that

$$d(x_n - Tx_n)^2 \leq \frac{1}{\gamma_n(1-\gamma_n)}\Big\{d(x_n,p)^2 - d(y_n,p)^2\Big\}.$$

As $\{\gamma_n\} \in [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, therefore

$$d(x_n, Tx_n)^2 \leq \frac{1}{\epsilon^2}\Big\{d(x_n,p)^2 - d(y_n,p)^2\Big\}.$$

Now, in view of equations (3.7) and (3.8), $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. This concludes the proof. $\square$

Now, we prove the following $\Delta-$ convergence theorem for SP-iteration scheme.

3.3. Theorem. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let α_n and β_n be two sequences in $[0,1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. If $\{x_n\}$ is described by (2.2), then the sequence x_n Δ-converges to a fixed point of T.*

Proof. In view of Lemma 3.1, $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$ so that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let $W_\omega(\{x_n\}) =: \cup A(\{u_n\})$, where union is taken over all subsequence $\{u_n\}$ of $\{x_n\}$. In order to show the Δ-convergence of $\{x_n\}$ to a fixed point of T, firstly we show that $W_\omega(\{x_n\}) \subset F(T)$ and thereafter prove that $W_\omega(\{x_n\})$ is a singleton set. To show $W_\omega(\{x_n\}) \subset F(T)$, let $y \in W_\omega(\{x_n\})$. Then, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $A(\{y_n\}) = y$. By Lemmas 2.2 and 2.3, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n z_n = z$ and $z \in K$. Since $\lim_{n\to\infty} d(z_n, Tz_n) = 0$ and T satisfies condition (C), therefore by Lemma 2.11, we have

$$d(z_n, Tz) \leq 3d(z_n, Tz_n) + d(z_n, z).$$

By taking lim sup of both the sides, we have

$$\limsup_{n\to\infty} d(z_n, Tz) \leq \limsup_{n\to\infty}\{3d(z_n, Tz_n) + d(z_n, z)\}$$

$$\leq \limsup_{n\to\infty} d(z_n, z).$$

As $\Delta - \lim_n z_n = z$, by Opial's property, we have

$$\limsup_{n\to\infty} d(z_n, z) \leq \limsup_{n\to\infty} d(z_n, Tz).$$

Hence $Tz = z$, i.e. $z \in F(T)$. Now, we assert that $z = y$. If not, by Lemma 3.1, $\lim_n d(x_n, z)$ exists and owing to the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, y)\\
&\leq \limsup_{n\to\infty} d(y_n, y)\\
&< \limsup_{n\to\infty} d(y_n, z)\\
&= \limsup_{n\to\infty} d(x_n, z)\\
&= \limsup_{n\to\infty} d(z_n, z),
\end{aligned}$$

which is a contradiction so that $y = z$. To show that $W_\omega(\{(x_n\})$ is a singleton, let $\{y_n\}$ be a subsequence of $\{x_n\}$. In view of Lemmas 2.2 and 2.3, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n z_n = z$. Let $A(\{y_n\}) = y$ and $A(\{x_n\}) = x$. Earlier, we have shown that $y = z$, therefore it is enough to show $z = x$. If $z \neq x$, by Lemma 3.2

$\{d(x_n, z)\}$ is convergent. By uniqueness of asymptotic centers

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x)\\ &\leq \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, z)\\ &= \limsup_{n\to\infty} d(z_n, z)\end{aligned}$$

which is a contradiction so that the conclusion follows. This concludes the proof. □

By setting $\beta_n = 0$ for all $n \in \mathbb{N}$, we can get the following $\Delta-$convergence theorem for Thaiwan iteration scheme (2.1) as a direct consequence of Theorem 3.1.

3.4. Corollary. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. If $\{x_n\}$ is described by (2.1), then the sequence x_n Δ-converges to a fixed point of T.*

3.5. Theorem. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let α_n and β_n be sequences in $[0, 1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. If $\{x_n\}$ is described by (2.2), then $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

Proof. If $\{x_n\}$ converges to a fixed point p of T, then

$$\liminf_{n\to\infty} d(x_n, p) = 0$$

so that

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

For the converse part, let $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. In view of Equation (3.5) for all $p \in F(T)$, we have

$$d(x_{n+1}, p) \leq d(x_n, p)$$

so that

$$\inf_{p\in F(T)} d(x_{n+1}, p) \leq \inf_{p\in F(T)} d(x_n, p),$$

which amounts to say that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T))$$

and hence $\lim_{n\to\infty} d(x_n, F(T))$ exists so that in view of our supposition, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore for any $\epsilon > 0$, there exists a positive integer k such that (for all $n \geq k$)

$$d(x_n, F(T)) < \frac{\epsilon}{4},$$

or

$$\inf\{d(x_k, p) : p \in F(T)\} < \frac{\epsilon}{4}$$

so that there exists a $p \in F(T)$ such that

$$d(x_k, p) < \frac{\epsilon}{2}.$$

Now, for all $m,\ n \geq k$, we have

$$\begin{aligned}d(x_m, x_n) &\leq d(x_m, p) + d(p, x_n)\\ &\leq 2d(x_k, p)\\ &< 2(\tfrac{\epsilon}{2}) = \epsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C and converges to some x in C. As $\lim_{n\to\infty} d(x_n, F(T)) = 0$ which amounts to say that $d(x, F(T)) = 0$. In view of Theorem 2.10, $F(T)$ is closed so that $x \in F(T)$. This completes the proof. □

Again by setting $\beta_n = 0$ (for all $n \in \mathbb{N}$), we get the following corollary.

3.6. Corollary. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. If $\{x_n\}$ is described by (2.1), then the sequence $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

In 1974, Senter and Dotson [20] introduced the condition (I) as follows.

A mapping $T : C \to C$ is said to satisfy the condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) = f(d(x, F(T)))$ for all $x \in C$.

3.7. Theorem. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ a generalized nonexpansive mapping with $F(T) \neq \emptyset$ which satisfies condition (I) . Let α_n and β_n be sequences in $[0, 1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. If $\{x_n\}$ is described by (2.2), then $\{x_n\}$ converges to a fixed point of T.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and let us assume it to be c. If $c = 0$, then there is nothing to prove. If $c > 0$, then as argued in Theorem 3.5, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Owing to condition (I) there exists a nondecreasing function f such that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

so that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since, f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now, in view of Theorem 3.5, we are through. □

Again by choosing $\beta_n = 0$ (for all $n \in \mathbb{N}$), we get the following corollary.

3.8. Corollary. *Let K be a nonempty closed convex subset of a Hadamard space X and $T : K \to K$ a generalized nonexpansive mapping which satisfies Condition (I) wherein $F(T) \neq \emptyset$. Let α_n and β_n be sequences in $[0, 1]$ and $\{\gamma_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. If $\{x_n\}$ is described by (2.1), then $\{x_n\}$ converges to a fixed point of T.*

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References

[1] Agarwal, R. P., O'Regan, D., Sahu, D. R., *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex. Anal. **8** (1), 61-79, 2007.

[2] Berinde, V., *Iterative Approximation of Fixed Points*, Lecture Notes Math., vol. 1912, Springer, Berlin, 2007.

[3] Bridson, M., Haefliger, A., *Metric Spaces of Nonpositive Curvature*, Springer-Verlag, Berlin, 1999.

[4] Brown, K. S., *Buildings*, Springer-Verlag, New York, 1989.

[5] Bruhat, F., Tits, J., *Groupes reductifs sur un corps local*, I. Donnees radicielles valuees Inst Hautes Etudes Sci Publ Math. **41**, 1972.

[6] Burago, D., Burago, Y., Ivanov, S., *A Course in Metric Geometry*, in: Graduate Studies in Math., vol. 33, Amer. Math. Soc., Providence, RI, 2001.

[7] Dhompongsa, S., Kirk, W. A., Panyanak, B., *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. **8**, 35-45, 2007.
[8] Dhompongsa, S., Kirk, W. A., Sims, B., *Fixed points of uniformly lipschitzian mappings*, Nonlinear Anal. **65**, 762-772, 2006.
[9] Dhompongsa, S., Panyanak, B., *On Δ-convergence theorems in CAT(0) space*, Comput. Math. Appl. **56**, 2572-2579, 2008.
[10] Goebel, K., Reich, S., *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, Inc., New York, 1984.
[11] Ishikawa, S., *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44**, 147-150, 1974.
[12] Ishikawa, S., *Fixed point and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc., **59**, 65-71, 1976.
[13] Kirk, W. A., Panyanak, B., *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68**, 3689-3696, 2008.
[14] Lim, T. C., *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. **60**, 179-182, 1976.
[15] Mann, W. R., *Mean value methods in iterations*, Proc. Amer. Math. Soc., **4**, 506-510, 1953.
[16] Phuengrattana, W., Suantai, S., *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comput. Appl. Math. **235**, 3006-3014, 2011.
[17] Razani, A., Salahifrd, H., *Approximating fixed points of generalized non-expansive mappings*, Bull. Iranian Math. Soc. **37** (1), 235-246, 2011.
[18] Reich, S., *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**, 274-276, 1979.
[19] Şahin, A., Başarır, M., *On the strong and Δ-convergence of SP-iteration on CAT(0) space*, J. Inequal. Appl., **311**, 10pp., 2013.
[20] Senter, H. F., Dotson, W. G., *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44**, 375-380, 1974.
[21] Suzuki, T., *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340**, 1088-1095, 2008.
[22] Tan, K. K., Xu, H. K., *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., **178**, 301-308, 1993.
[23] Thianwan, S., *Common fixed points of new iterations for two asymptotically nonexpansive nonself mappings in a Banach space*, J. Comput. Appl. Math. **224**, 688-695, 2009.
[24] Uddin, I., Imdad, M., *On certain convergence of S-iteration scheme in CAT(0) spaces* Kuwait J. Sci. **42** (2), 93-106, 2015.
[25] Uddin, I., Imdad, M., *Some convergence theorems for a hybrid pair of generalized nonexpansive mappings in CAT (0) spaces*, J. Nonlinear Convex Anal. **16** (3), 447-457, 2015.
[26] Uddin, I., Imdad, M., Ali, J., *Convergence Theorems for a Hybrid Pair of Generalized Nonexpansive Mappings in Banach Spaces*, Bull. Malays. Math. Sci. Soc., **38** (2), 695-705, 2015.
[27] Wangkeeree, R., Preechasilp, P., *Δ-Convergence for Generalized Hybrid Mappings in CAT(0) Spaces*, Bull. Malays. Math. Sci. Soc., **38** (3), 1127-1141, 2015.
[28] Xu, B., Noor, M. A., *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **267**, 444-453, 2002.