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Explicit Solutions of a Class of (3+1)-Dimensional Nonlinear Model

Yongyi Gu[a,b*]

[a]*School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, China*
[b]*Big data and Educational Statistics Application Laboratory, Guangdong University of Finance and Economics, Guangzhou 510320, China*
[*]*Corresponding author*

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Abstract

In this article, we employ Lie group analysis to obtain symmetry reduction of a class of (3+1)-dimensional nonlinear model. This nonlinear model plays a critical role in the study of nonlinear sciences. By the $\exp(-\varphi(z))$-expansion method, we construct explicit solutions for the proposed equation. Four types of explicit solutions are obtained, which are hyperbolic, exponential, trigonometric and rational function solutions.

1. Introduction

Consider the following (3+1)-dimensional nonlinear differential equation (NLDE):

$$u_t + b_1 u^2 u_x + b_2 u_{xxx} + b_3 u_{xyy} + b_4 u_{xss} + b_5 u u_x = 0. \tag{1.1}$$

where b_i $(i = 1, 2, \cdots, 5)$ are arbitrary constants.
It is know that many famous NLDEs are the special cases of Eq.(1.1). For example, if $b_1 = b_3 = b_4 = 0$, then Eq.(1.1) is the Korteweg-de Vries (KdV) equation [1, 2]. If $b_1 = b_4 = 0$, then Eq.(1.1) is the Zakharov-Kuznetsov (ZK) equation [3]. If $b_3 = b_4 = b_5 = 0$, then Eq.(1.1) is the modified KdV equation [4]. If $b_3 = b_4 = 0$, then Eq.(1.1) is the Gardner equation [5]. If $b_4 = b_5 = 0$, then Eq.(1.1) is the modified ZK equation [6].
Eq.(1.1) is a significant nonlinear model which can be used to depict important phenomena and dynamic processes in physics and engineering. It is an interesting and meaningful subject to find exact solutions of NLDEs. During the past few years, there has been extraordinary progress in constructing explicit solutions of NLDEs, for instance, the sine-cosine method [7], the modified simple equation method [8], the bifurcation method of dynamic systems [9], the enhanced $(\frac{G'}{G})$-expansion method [10], the complex method [11]-[15], the $\exp(-\varphi(z))$-expansion method [16]-[18], and the Lie group method [19]-[21] and so on. More related works are in Ref. [22]-[25].
The paper is organized as follows: The algorithm of the $\exp(-\varphi(z))$-expansion method have been introduced in Section 2. Symmetry reduction of the mentioned (3+1)-dimensional NLDE are obtained in Section 3. By the proposed method, we gain explicit solutions of this kind of (3+1)-dimensional NLDE in Section 4. In Section 5, some computer simulations will be given to illustrate our results, and conclusions are presented in the last Section.

2. Algorithm of the $\exp(-\varphi(z))$-expansion method

We consider a nonlinear PDE as follows:

$$F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0, \tag{2.1}$$

where F is a polynomial of an unknown function $u(x, y, t)$ and its derivatives, and it contains highest order derivatives and nonlinear terms are involved.

Email addresses: gdguyongyi@163.com, 0000-0002-6651-1714 (Y. Gu)

Step 1. Substitute traveling wave transformation

$$u(x,y,t) = w(z), \quad z = kx + ly + rt, \tag{2.2}$$

into Eq.(2.1) to convert it to the ODE,

$$P(w, w', w'', w''', \cdots) = 0, \tag{2.3}$$

where P is a polynomial of w and its derivatives, while $' := \frac{d}{dz}$.
Step 2. Suppose that Eq.(2.3) has the exact solutions as follows:

$$w(z) = \sum_{j=0}^{n} B_j (\exp(-\varphi(z)))^j, \tag{2.4}$$

where B_j, $(0 \le j \le n)$ are constants to be determined latter, such that $B_n \neq 0$ and $\varphi = \varphi(z)$ satisfies the ODE as below:

$$\varphi'(z) = \gamma + \exp(-\varphi(z)) + \mu \exp(\varphi(z)). \tag{2.5}$$

Eq.(2.5) has the solutions as follows:
When $\gamma^2 - 4\mu > 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2 - 4\mu)}\tanh(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a)) - \gamma}{2\mu}\right), \tag{2.6}$$

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2 - 4\mu)}\coth(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a)) - \gamma}{2\mu}\right). \tag{2.7}$$

When $\gamma^2 - 4\mu < 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\mu - \gamma^2)}\tan(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z+a)) - \gamma}{2\mu}\right), \tag{2.8}$$

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\mu - \gamma^2)}\cot(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z+a)) - \gamma}{2\mu}\right). \tag{2.9}$$

When $\gamma^2 - 4\mu > 0$, $\gamma \neq 0$, $\mu = 0$,

$$\varphi(z) = -\ln\left(\frac{\gamma}{\exp(\gamma(z+a)) - 1}\right). \tag{2.10}$$

When $\gamma^2 - 4\mu = 0$, $\gamma \neq 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(-\frac{2(\gamma(z+a)+2)}{\gamma^2(z+a)}\right). \tag{2.11}$$

When $\gamma^2 - 4\mu = 0$, $\gamma = 0$, $\mu = 0$,

$$\varphi(z) = \ln(z+a). \tag{2.12}$$

Where a is an arbitrary constant and $B_n \neq 0, \gamma, \mu$ are constants in Eq.(2.6)-Eq.(2.12). We determine the positive integer n through considering the homogeneous balance between highest order derivatives and nonlinear terms of Eq.(2.3).

Step 3. Inserting Eq.(2.4) into Eq.(2.3) and then considering the function $\exp(-\varphi(z))$ yields a polynomial of $\exp(-\varphi(z))$. Let the coefficients of same power about $\exp(-\varphi(z))$ equal to zero, then we get a set of algebraic equations. We solve the algebraic equations to obtain the values of $B_n \neq 0, \gamma, \mu$ and then we put these values into Eq.(2.4) along with Eq.(2.6)-Eq.(2.12) to finish the determination of the solutions for the given PDE.

3. Symmetry reduction

With the aim of obtaining the symmetry $\sigma = \sigma(x,y,s,t,u)$ of Eq.(1.1), we let

$$\sigma = au_x + bu_y + cu_s + du_t + eu + f, \tag{3.1}$$

where u is the solution of Eq.(1.1), a,b,c,d,e,f are unknown functions of real variables x,y,s,t. By the Lie group method [19, 20], σ satisfies

$$\sigma_t + b_1\sigma^2 u_x + b_1 u^2\sigma_x + b_2\sigma_{xxx} + b_3\sigma_{xyy} + b_4\sigma_{xss} + b_5\sigma u_x + b_5 u\sigma_x = 0. \tag{3.2}$$

Putting Eq.(3.1) into Eq.(3.2), we obtain a new differential equation, where

$$b_2 u_{xxx} = -b_1 u^2 u_x - b_3 u_{xyy} - b_4 u_{xss} - b_5 u u_x - u_t. \tag{3.3}$$

By Eq.(3.1), Eq.(3.2) and Eq.(3.3), we get

$$a = a_5, b = (a_2 s + a_3), c = (a_4 - \frac{b_4}{b_3}a_2 y), d = a_1, e = 0, f = 0, \tag{3.4}$$

where $a_i (i = 1,2,\cdots,5)$ are real constants. Inserting Eqs.(3.4) into Eq.(3.1), we obtain the symmetry of Eq.(1.1)

$$\sigma = a_5 u_x + (a_2 s + a_3)u_y + (a_4 - \frac{b_4}{b_3}a_2 y)u_s + a_1 u_t.$$

To solve the above characteristic equation of σ

$$\frac{dx}{a_5} = \frac{dy}{a_2 s + a_3} = \frac{ds}{a_4 - \frac{b_4}{b_3}a_2 y} = \frac{dt}{a_1} = \frac{du}{0},$$

we get symmetry reduced equations.
Setting $a_1 = a_3 = a_4 = a_5 = 0$, $a_2 = 1$, we obtain one similarity solution of Eq.(1.1)

$$u = \phi(\xi,\eta), \tag{3.5}$$

in which $\eta = \frac{y^2}{2b_3} + \frac{s^2}{2b_4}$, $\xi = x + t$. Substituting Eq.(3.5) into Eq.(1.1), we get one symmetry reduced equation of Eq.(1.1), which is

$$\phi_\xi + b_1\phi^2\phi_\xi + (b_2 + b_3)\phi_{\xi\xi\xi} + 2\phi_{\xi\eta\eta} + b_5\phi\phi_\xi = 0.$$

Setting $a_1 = a_2 = 0$, $a_3 = a_4 = a_5 = 1$, solving $\sigma = 0$, we obtain the other similarity solution of Eq.(1.1)

$$u = \phi(\xi,\eta), \tag{3.6}$$

in which $\eta = s$, $\xi = x + y$. Substituting Eq.(3.6) into Eq.(1.1), we get the other symmetry reduced equation of Eq.(1.1), which is

$$b_1\phi^2\phi_\xi + (b_2 + b_3)\phi_{\xi\xi\xi} + b_4\phi_{\xi\eta\eta} + b_5\phi\phi_\xi = 0. \tag{3.7}$$

4. Application of the exp$(-\varphi(z))$-expansion method to the nonlinear model

Substitute traveling wave transform

$$\phi(\xi,\eta) = w(z), \quad z = k\xi + l\eta,$$

into Eq.(3.7), and integrate it with respect to z, then

$$(b_2 k^2 + b_3 k^2 + b_4 l^2)w'' + \frac{b_5}{2}w^2 + \frac{b_1}{3}w^3 - \lambda = 0, \tag{4.1}$$

where λ is the integration constant.
Taking the homogeneous balance between w^3 and w'' in Eq.(4.1), we have

$$w(z) = B_0 + B_1 \exp(-\varphi(z)), \tag{4.2}$$

where $B_1 \neq 0$, B_0 are constants.
Substituting w^2, w^3, w'' into Eq.(4.1) and equating the coefficients of $\exp(-\varphi(z))$ to zero, we get

$$B_1 b_4 l^2\mu\gamma + B_1 k^2 b_3\mu\gamma + B_1 k^2 b_2\mu\gamma + \frac{1}{3}b_1 {B_0}^3 + \frac{1}{2}b_5 {B_0}^2 - \lambda = 0,$$

$$B_1 l^2 b_4\gamma^2 + B_1 b_2 k^2\gamma^2 + B_1 b_3 k^2\gamma^2 + 2B_1 b_2 k^2\mu + 2B_1 b_3 k^2\mu + 2B_1 l^2 b_4\mu$$
$$+ {B_0}^2 B_1 b_1 + B_0 B_1 b_5 = 0,$$

$$3B_1 b_4 l^2\gamma + b_1 B_0 {B_1}^2 + \frac{1}{2}b_5 {B_1}^2 + 3B_1 k^2 b_2\gamma + 3B_1 k^2 b_3\gamma = 0,$$

$$2B_1 b_4 l^2 + 2B_1 k^2 b_2 + 2B_1 k^2 b_3 + \frac{1}{3}b_1 {B_1}^3 = 0.$$

Solving the above algebraic equations yields

$$\lambda = -\sqrt{\frac{(4\mu-\gamma^2)^3(b_2k^2+b_3k^2+b_4l^2)^3}{18b_1}},$$

$$B_0 = \frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma - \sqrt{2b_1(4\mu-\gamma^2)\left(b_4l^2+b_3k^2+b_2k^2\right)}}{2b_1},$$

$$B_1 = \sqrt{\frac{-6\left(b_4l^2+b_3k^2+b_2k^2\right)}{b_1}}, \tag{4.3}$$

where γ and μ are arbitrary constants.
We substitute Eqs.(4.3) into Eq.(4.2), then

$$w(z) = \frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma - \sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1} + \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\exp(-\varphi(z)). \tag{4.4}$$

Using Eq.(2.6) to Eq.(2.12) into Eq.(4.4) respectively, we gain traveling wave solutions to the nonlinear model in the following.
When $\gamma^2-4\mu>0$, $\mu\neq 0$,

$$w_1(z) = \frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma - \sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1} - \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(\gamma^2-4\mu)}\tanh(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a))+\gamma},$$

$$w_2(z) = \frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma - \sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1} - \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(\gamma^2-4\mu)}\coth(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a))+\gamma}.$$

When $\gamma^2-4\mu<0$, $\mu\neq 0$,

$$w_3(z) = \frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma - \sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1} + \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(4\mu-\gamma^2)}\tan(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+a))-\gamma},$$

$$w_4(z) = \frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma - \sqrt{2b_1(4\mu-\gamma^2)\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1} + \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{2\mu}{\sqrt{(4\mu-\gamma^2)}\cot(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+a))-\gamma}.$$

When $\gamma^2-4\mu>0$, $\gamma\neq 0$, $\mu=0$,

$$w_5(z) = \frac{\sqrt{-6b_1\left(b_2k^2+b_3k^2+b_4l^2\right)}\gamma - \sqrt{-2b_1\gamma^2\left(b_2k^2+b_3k^2+b_4l^2\right)}}{2b_1} + \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{\gamma}{\exp(\gamma(z+a))-1}.$$

When $\gamma^2-4\mu=0$, $\gamma\neq 0$, $\mu\neq 0$,

$$w_6(z) = \sqrt{\frac{-3\left(b_2k^2+b_3k^2+b_4l^2\right)}{2b_1}}\gamma - \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{\gamma^2(z+a)}{2(\gamma(z+a)+2)}.$$

When $\gamma^2-4\mu=0$, $\gamma=0$, $\mu=0$,

$$w_7(z) = \sqrt{\frac{-6\left(b_2k^2+b_3k^2+b_4l^2\right)}{b_1}}\frac{1}{z+a}.$$

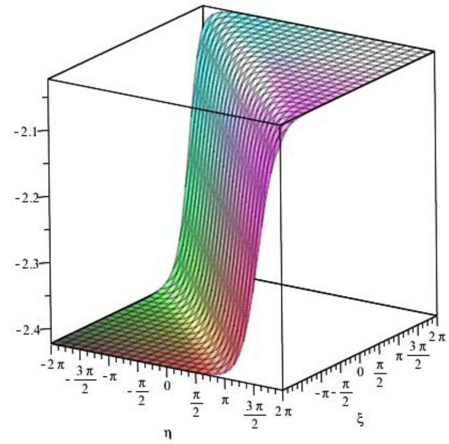

Figure 5.1: 3D profile of $w_1(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 4$, and $\mu = 3$.

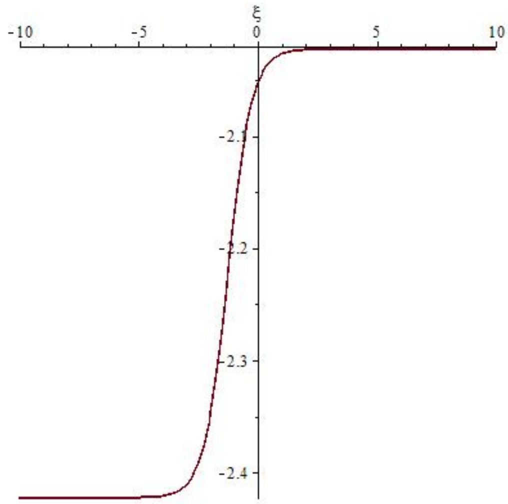

Figure 5.2: 2D profile of $w_1(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 4$, $\mu = 3$ and $\eta = 0$.

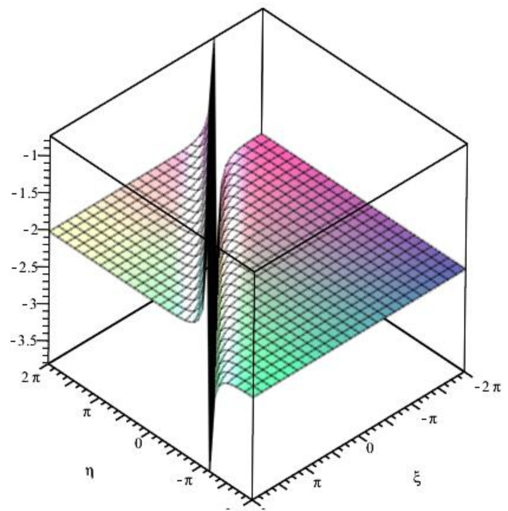

Figure 5.3: 3D profile of $w_2(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 2$, and $\mu = 2$.

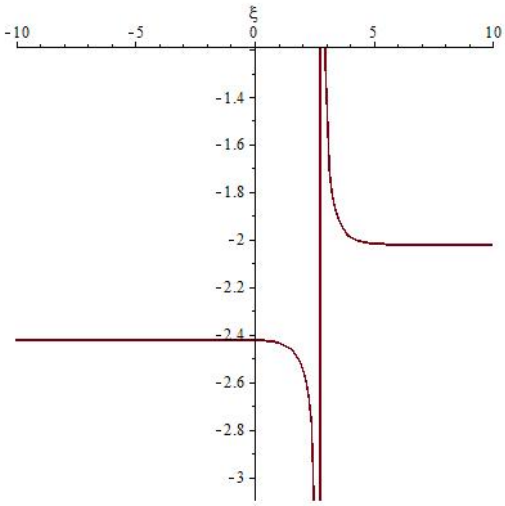

Figure 5.4: 2D profile of $w_2(z)$ for $b_4 = 1$, $b_3 = 1$, $b_2 = -1$, $b_1 = -6$, $k = 1$, $l = 1$, $\gamma = 2$, $\mu = 2$ and $\eta = 0$.

5. Computer simulations

In this section, the computer simulations are given to illustrate our results by the figures.

6. Conclusion

The $\exp(-\varphi(z))$-expansion method allows us to express the explicit solutions of NLDEs as a polynomial of $\exp(-\varphi(z))$, in which $\varphi(z)$ satisfies the ODE (2.5). We can determine the degree of the polynomial via the homogeneous balance and get the coefficients of the polynomial via the simple calculation from the process of this method, and then we obtain the exact solutions.
In this article, symmetry reduction of a class of (3+1)-dimensional nonlinear model are obtained via Lie group analysis. Then, we achieve to reduce the dimension of the NLDEs that is meaningful in engineering and mathematical physics. By the $\exp(-\varphi(z))$-expansion method, we obtain four kinds of explicit solutions. The results demonstrate that the applied method is direct and efficient method, which allow us to do tedious and complicated algebraic calculation.

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