

Hermite-Hadamard Type Inequalities for Convex Stochastic Processes on n-Coordinates

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ABSTRACT. The main subject of this study is initially to consider convex stochastic processes on n-dimensional interval. Besides, Hermite-Hadamard type inequalities are obtained for these processes.

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1. INTRODUCTION

Convex functions are important and provide a base to build literature of mathematical inequalities. A function $f : I \rightarrow \mathbb{R}$, where I is an interval in \mathbb{R} is called convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. A classical inequality for convex functions is Hadamard inequality, this is given as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}.$$

where $f : I \rightarrow \mathbb{R}$ is a convex function for all $a, b \in I$, $a < b$.

In many areas of analysis, application of the Hadamard inequality appears for different classes of functions. Some useful mappings connected to this inequality are also defined by many authors. In recent years, the concept of convexity has been extended and generalized in various directions. In this regards, very novel and innovative techniques are used by different authors [8].

For $n \geq 2$, let a_i, b_i , ($i = 1, 2, \dots, n$) be real numbers such that $a_i < b_i$ for $i = 1, 2, \dots, n$. Ellahi et al [1] consider n-dimensional interval $\Delta^n = \prod_{i=1}^n [a_i, b_i] \subseteq [0, \infty)^n$. The definition of convex functions on n-coordinates is given as follows:

Let $x = (x_1, x_2, \dots, x_n) \in \Delta^n$. A mapping $f : \Delta^n \rightarrow [0, \infty)$ is called convex on n-coordinates if the functions $f_{x_n}^i$ where $f_{x_n}^i(t) = f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$, are convex on $[a_i, b_i]$ for $i = 1, 2, \dots, n$.

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Recall that a mapping $f : \mathcal{A}^n \rightarrow [0, \infty)$ is convex on \mathcal{A}^n if for $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathcal{A}^n$ and $\alpha \in [0, 1]$ the following inequality holds

$$f(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) \leq \alpha f(\mathbf{x}) + (1 - \alpha) f(\mathbf{y}).$$

It is interesting to note that if $f : \mathcal{A}^n \rightarrow [0, \infty)$ is convex function on n -coordinates, then $f_{x_n}^i : [a_i, b_i] \rightarrow [0, \infty)$ is convex on $[a_i, b_i]$ for each $i = 1, 2, \dots, n$. From Hermite-Hadamard inequality, we have

$$f_{x_n}^i \left(\frac{a_i + b_i}{2} \right) \leq \frac{1}{b_i - a_i} \int_{a_i}^{b_i} f_{x_n}^i(x_i), \quad i = 1, 2, \dots, n.$$

This gives us

$$\sum_{k=1}^n f_{x_n}^k \frac{a_k + b_k}{2} \leq \sum_{k=1}^n \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f_{x_n}^k(x_k) dx_k.$$

Nowadays, there are many similar results belong to stochastic processes. Then, let us see some studies in the literature [5–7]. Firstly, convex stochastic processes were proposed and some properties were given for classical convex stochastic processes by Nikodem in [3]. Stochastic convexity and its applications were defined by Shaked et al in [7]. Jensen-convex, λ -convex stochastic processes were introduced by Skowronski in [8]. The classical Hermite-Hadamard inequality to convex stochastic processes was extended by Kotrys in [2] as follows:

$$X\left(\frac{u + v}{2}, \cdot\right) \leq \frac{1}{v - u} \int_u^v X(t, \cdot) dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}.$$

In recent years, Set et al in [7] established the following similar inequality of Hadamard’s type for coordinated convex stochastic processes on a rectangle from the plane \mathbb{R}^2 :

$$\begin{aligned} & X\left(\left(\frac{u_1 + u_2}{2}, \frac{v_1 + v_2}{2}\right), \cdot\right) \\ & \leq \frac{1}{2(u_2 - u_1)} \int_{u_1}^{u_2} X\left(\left(t, \frac{v_1 + v_2}{2}\right), \cdot\right) dt + \frac{1}{2(v_2 - v_1)} \int_{v_1}^{v_2} X\left(\left(\frac{u_1 + u_2}{2}, s\right), \cdot\right) ds \\ & \leq \frac{1}{(u_2 - u_1)(v_2 - v_1)} \int_{u_1}^{u_2} \int_{v_1}^{v_2} X((t, s), \cdot) ds dt \\ & \leq \frac{1}{4(u_2 - u_1)} \int_{u_1}^{u_2} (X((t, v_1), \cdot) + X((t, v_2), \cdot)) dt + \frac{1}{4(v_2 - v_1)} \int_{v_1}^{v_2} (X((u_1, s), \cdot) + X((u_2, s), \cdot)) ds \\ & \leq \frac{X((u_1, v_1), \cdot) + X((u_1, v_2), \cdot) + X((u_2, v_1), \cdot) + X((u_2, v_2), \cdot)}{4}. \end{aligned}$$

Also, the Hermite-Hadamard type inequalities for harmonically convex stochastic processes on the coordinates were obtained by Okur et al in [8].

The authors’ findings led to our motivation to build our work. The main subject of this paper is to obtain some generalized Hermite-Hadamard type inequalities for convex stochastic processes on the coordinates.

2. PRELIMINARIES

There are well-known definitions and properties of stochastic processes, the continuity in probability and mean square continuity the definition of mean square integral of a stochastic processes, some fundamentals about Hermite Hadamard Inequality for stochastic processes in the literature (see, [2–7]).

Let us give some basic definitions and notions about continuity concepts and differentiability for stochastic processes, and a mean-square integral of a stochastic process.

Definition 2.1 ([7]). Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space. A function $\xi : \Omega \rightarrow \mathbb{R}$ is called a random variable if it is \mathfrak{F} -measurable. $X : I \times \Omega \rightarrow \mathbb{R}$, where $I \in \mathfrak{F}, I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Definition 2.2 ([7]). Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space and $I \in \mathfrak{F}, I \subset \mathbb{R}$ be an interval. The stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is called almost everywhere

(i) convex if

$$X(\lambda t + (1 - \lambda)s, \cdot) \leq \lambda X(t, \cdot) + (1 - \lambda)X(s, \cdot)$$

for all $t, s \in I$ and $\lambda \in [0, 1]$,

(ii) λ -convex if

$$X(\lambda t + (1 - \lambda)s, \cdot) \leq \lambda X(t, \cdot) + (1 - \lambda)X(s, \cdot)$$

for all $t, s \in I$ and λ is a fixed number in $(0, 1)$.

(iii) Jensen-convex if

$$X\left(\frac{t + s}{2}, \cdot\right) \leq \frac{X(t, \cdot) + X(s, \cdot)}{2}$$

for all $t, s \in I$.

Definition 2.3 ([7]). Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space and $I \in \mathfrak{I}, I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is called

(i) continuous in probability on I if for all $t_0 \in I$ if

$$P - \lim_{t \rightarrow t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes limit in probability,

(ii) mean-square continuous on I if for all $t_0 \in I$ if

$$\lim_{t \rightarrow t_0} E[X(t, \cdot) - X(t_0, \cdot)]^2 = 0,$$

where $E[\xi(t, \cdot)]$ denotes expectation value of the random variable $X(t, \cdot)$,

(iii) increasing (decreasing) if for all $t, s \in I$ such that $t < s$ if

$$X(t, \cdot) \leq X(s, \cdot) \quad (X(t, \cdot) \geq X(s, \cdot)),$$

(iv) monotonic if it is increasing or decreasing,

(v) mean-square differentiable at a point if $t \in I$ if there is a random variable $X'(t, \cdot) : I \times \Omega \rightarrow \mathbb{R}$ such that

$$X'(t, \cdot) = P - \lim_{t \rightarrow t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

We say that a stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I .

Definition 2.4 ([7]). Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space and $I \in \mathfrak{I}, I \subset \mathbb{R}$ be an interval and $X : I \times \Omega \rightarrow \mathbb{R}$ be a stochastic process with $E[X(t, \cdot)^2] \leq \infty$ for all $t \in I$. Let $[0, t] \subset I, 0 = t_0 < t_1 \dots t_n = t$ be a partition of $[0, t]$ and $\Theta_k \in [t_{k-1}, t_k]$ arbitrary for $k = 1, \dots, n$. A random variable $\eta : \Omega \times \Omega \rightarrow \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[0, t]$ if the following identity holds:

$$\lim_{n \rightarrow \infty} E\left(\sum_{k=1}^n X(\Theta_k, \cdot) \cdot (t_k - t_{k-1}) - \eta(t, \cdot)\right)^2 = 0.$$

Then we can write almost everywhere

$$\int_0^t X(u, \cdot) du = \eta(t, \cdot).$$

The mean-square integral operator is increasing on $[0, t]$ almost everywhere, that is,

$$X(t, \cdot) \leq Y(t, \cdot) \Rightarrow \int_0^t X(t, \cdot) dt \leq \int_0^t Y(t, \cdot) dt.$$

Let us consider a two-dimensional interval $\Delta := [u_1, u_2] \times [v_1, v_2]$ in \mathbb{R}^2 with $u_1 < u_2, v_1 < v_2$ and the above mentioned mean-square integrability is used throughout this paper. A convex stochastic process is defined on Δ as follows:

Definition 2.5 ([7]). A stochastic process $X : \Delta \times \Omega \rightarrow \mathbb{R}$ is said to be convex on Δ if the following inequality holds almost everywhere

$$X((\lambda t_1 + (1 - \lambda)t_2, \lambda s_1 + (1 - \lambda)s_2), \cdot) \leq \lambda X((t_1, s_1), \cdot) + (1 - \lambda)X((t_2, s_2), \cdot)$$

for all $(t_1, s_1), (t_2, s_2) \in \Delta$ and $\lambda \in [0, 1]$. If the above inequality is reversed then X is said to be concave on Δ .

Definition 2.6 ([6]). A stochastic process $X : \Delta \times \Omega \rightarrow \mathbb{R}$ is said to be a convex on the coordinates on Δ if the partial mappings

$$\begin{aligned} X_s : [u_1, u_2] \times \Omega &\rightarrow \mathbb{R}, X_s(u, \cdot) := X((u, s), \cdot) \\ X_t : [v_1, v_2] \times \Omega &\rightarrow \mathbb{R}, X_t(v, \cdot) := X((t, v), \cdot) \end{aligned}$$

defined are convex almost everywhere for all $t \in [u_1, u_2]$ and $s \in [v_1, v_2]$.

Now we give a formal definition of coordinated convex stochastic processes:

Definition 2.7 ([6]). A stochastic process $X : \Delta \times \Omega \rightarrow \mathbb{R}$ is said to be a coordinated convex stochastic process on Δ almost everywhere, if

$$\begin{aligned} X((\lambda_1 t_1 + (1 - \lambda_1)t_2, \lambda_2 s_1 + (1 - \lambda_2)s_2), \cdot) &\leq \lambda_1 \lambda_2 X((t_1, s_1), \cdot) + \lambda_1 (1 - \lambda_2) X((t_1, s_2), \cdot) \\ &+ (1 - \lambda_1) \lambda_2 X((t_2, s_1), \cdot) + (1 - \lambda_1)(1 - \lambda_2) X((t_2, s_2), \cdot) \end{aligned}$$

for all $(t_1, s_1), (t_2, s_2) \in \Delta$ and $\lambda_1, \lambda_2 \in [0, 1]$.

3. MAIN RESULTS

The main goal of this section is to present Hermite-Hadamard type inequalities for convex stochastic processes on n-coordinates.

By using Definition 2.5, we generalize it for n-coordinates. For $n \geq 2$, let $u_i, v_i, (i = 1, 2, \dots, n)$ be real numbers such that $u_i < v_i$ for $i = 1, 2, \dots, n$. We consider n-dimensional interval $\Delta^n = \prod_{i=1}^n [u_i, v_i] \subseteq [0, \infty)^n$.

In the following we give definition of convexity for stochastic processes on n-coordinates:

Definition 3.1. A stochastic process $X : \Delta^n \times \Omega \rightarrow \mathbb{R}$ is called convex on n-coordinates if the stochastic processes

$$X_{t_n}^i(t, \cdot) := X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot)$$

are convex on $[u_i, v_i]$ almost everywhere for $i = 1, 2, \dots, n$.

Definition 3.2. A stochastic process $X : \Delta^n \times \Omega \rightarrow \mathbb{R}$ is said to be convex on Δ^n if the following inequality holds almost everywhere

$$X((\lambda t + (1 - \lambda)s), \cdot) \leq \lambda X(t, \cdot) + (1 - \lambda)X(s, \cdot)$$

for all $t = (t_1, t_2, \dots, t_n), s = (s_1, s_2, \dots, s_n) \in \Delta^n$ and $\lambda \in [0, 1]$. If the above inequality is reversed then X is said to be concave on Δ^n .

Lemma 3.3. Every convex stochastic process $X : \Delta^n \times \Omega \rightarrow \mathbb{R}$ is convex on n-coordinates almost everywhere but converse is not true.

Proof. Let $X : \Delta^n \times \Omega \rightarrow \mathbb{R}$ be convex on Δ^n . Consider $X_{t_n}^i : [u_i, v_i] \times \Omega \rightarrow \mathbb{R}$ defined by

$$X_{t_n}^i(t, \cdot) := X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot), \quad t \in [u_i, v_i].$$

Now for $t, s \in [u_i, v_i]$ and $\lambda \in [0, 1]$ almost everywhere

$$\begin{aligned} X_{t_n}^i((\lambda t + (1 - \lambda)s), \cdot) &:= X((t_1, \dots, t_{i-1}, \lambda t + (1 - \lambda)s, t_{i+1}, \dots, t_n), \cdot) \\ &\leq \lambda X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot) + (1 - \lambda)X((t_1, \dots, t_{i-1}, s, t_{i+1}, \dots, t_n), \cdot) \\ &= \lambda X_{t_n}^i(t, \cdot) + (1 - \lambda)X_{t_n}^i(s, \cdot) \end{aligned}$$

which implies $X_{t_n}^i$ is convex on $[u_i, v_i]$, that is, X is convex on n-coordinates. For converse we give the following counter example:

Example 3.4. Let us consider a stochastic process $X : [0, 1]^n \times \Omega \rightarrow \mathbb{R}$ defined as

$$X((t_1, t_2, \dots, t_n), \cdot) = t_1 t_2 \dots t_n.$$

It is convex on n-coordinates as follows:

$$\begin{aligned} X_{t_n}^i((\lambda t + (1 - \lambda) s), \cdot) &= t_1 t_2 \dots t_{i-1} (\lambda t + (1 - \lambda) s) t_{i+1} \dots t_n \\ &= \lambda (t_1 t_2 \dots t_{i-1} t t_{i+1} \dots t_n) + (1 - \lambda) (t_1 t_2 \dots t_{i-1} s t_{i+1} \dots t_n) \\ &= \lambda X_{t_n}^i(t, \cdot) + (1 - \lambda) X_{t_n}^i(s, \cdot). \end{aligned}$$

But for $t = (1, 1, \dots, 0)$, $s = (0, 1, \dots, 1) \in [0, 1]^n$, we have

$$X((\lambda t + (1 - \lambda) s), \cdot) = X((\lambda, 1, 1, \dots, (1 - \lambda)), \cdot) = \lambda(1 - \lambda)$$

and

$$\lambda X(t, \cdot) + (1 - \lambda) X(s, \cdot) = \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0.$$

This gives

$$X((\lambda t + (1 - \lambda) s), \cdot) > \lambda X(t, \cdot) + (1 - \lambda) X(s, \cdot)$$

for all $\lambda \in [0, 1]$, that is, X is not convex on $[0, 1]^n$.

Remark 3.5. If $X : \mathcal{A}^n \times \Omega \rightarrow \mathbb{R}$ is convex stochastic process on n-coordinates, then $X_{t_n}^i : [u_i, v_i] \times \Omega \rightarrow \mathbb{R}$ is convex on $[u_i, v_i]$ for each $i = 1, 2, \dots, n$. From Hermite-Hadamard inequality, we have almost everywhere

$$X_{t_n}^i\left(\frac{u_i + v_i}{2}, \cdot\right) \leq \frac{1}{v_i - u_i} \int_{u_i}^{v_i} X_{t_n}^i(t_i, \cdot) dt_i, \quad i = 1, 2, \dots, n.$$

This gives us

$$\sum_{k=1}^n X_{t_n}^k\left(\frac{u_k + v_k}{2}, \cdot\right) \leq \sum_{k=1}^n \frac{1}{v_k - u_k} \int_{u_k}^{v_k} X_{t_n}^k(t_k, \cdot) dt_k. \tag{3.1}$$

Following results comprise Hermite-Hadamard’s inequality for convex stochastic processes of first sense on n-coordinates:

Theorem 3.6. Let $X : \mathcal{A}^n \times \Omega \rightarrow \mathbb{R}$ be convex stochastic process on n-coordinates. Then we have almost everywhere

$$\begin{aligned} &\sum_{k=1}^n \frac{1}{v_k - u_k} \int_{u_k}^{v_k} X_{t_n}^{k+1}\left(\frac{u_{k+1} + v_{k+1}}{2}, \cdot\right) dt_k \\ &\leq \sum_{k=1}^n \frac{1}{(v_k - u_k)(v_{k+1} - u_{k+1})} \int_{u_k}^{v_k} \int_{u_{k+1}}^{v_{k+1}} X(t, \cdot) dt_{k+1} dt_k \\ &\leq \sum_{k=1}^n \frac{1}{2(v_k - u_k)} \int_{u_k}^{v_k} [X_{t_n}^{k+1}(u_{k+1}, \cdot) + X_{t_n}^{k+1}(v_{k+1}, \cdot)] dt_k \end{aligned} \tag{3.2}$$

with $n + 1 \rightarrow 1$. The above inequalities are sharp.

Proof. By applying Hermite-Hadamard’s inequality for convex stochastic process $X_{t_n}^{i+1}$ on interval $[u_{i+1}, v_{i+1}]$ we have almost everywhere

$$\begin{aligned} X_{t_n}^{i+1}\left(\frac{u_{i+1} + v_{i+1}}{2}, \cdot\right) &\leq \frac{1}{v_{i+1} - u_{i+1}} \int_{u_{i+1}}^{v_{i+1}} X_{t_n}^{i+1}(t_{i+1}, \cdot) dt_{i+1} \\ &\leq \frac{X_{t_n}^{i+1}(u_{i+1}, \cdot) + X_{t_n}^{i+1}(v_{i+1}, \cdot)}{2}. \end{aligned}$$

By integrating on $[u_i, v_i]$ we get

$$\frac{1}{v_i - u_i} \int_{u_i}^{v_i} X_{t_n}^{i+1}\left(\frac{u_{i+1} + v_{i+1}}{2}, \cdot\right) dt_i$$

$$\begin{aligned} &\leq \frac{1}{(v_i - u_i)(v_{i+1} - u_{i+1})} \int_{u_i}^{v_i} \int_{u_{i+1}}^{v_{i+1}} X_n^{i+1}(t_{i+1}, \cdot) dt_{i+1} dt_i \\ &\leq \frac{1}{2(v_i - u_i)} \int_{u_i}^{v_i} (X_n^{i+1}(u_{i+1}, \cdot) + X_n^{i+1}(v_{i+1}, \cdot)) dt_i. \end{aligned}$$

Taking summation from 1 to n we get (3.2). If we consider $X((t_1, t_2, \dots, t_n), \cdot) = t_1 t_2 \dots t_n$, then inequalities in (3.2) become equality, which shows these are sharp.

Theorem 3.7. *Let $X : \Delta^n \times \Omega \rightarrow \mathbb{R}$ be convex stochastic process on n-coordinates. Then we have almost everywhere*

$$\begin{aligned} &\sum_{k=1}^n \frac{1}{2(v_k - u_k)} \int_{u_k}^{v_k} (X_{u_n}^k(t_i, \cdot) + X_{v_n}^k(t_k, \cdot)) dt_k \\ &\leq \frac{n}{2} [X(u, \cdot) + X(v, \cdot)] + \frac{1}{2} \sum_{k=1}^n [X_{u_n}^k(v_k, \cdot) + X_{v_n}^k(u_k, \cdot)] \end{aligned} \tag{3.3}$$

where $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$. The above equality is sharp.

Proof. Since $X : \Delta^n \times \Omega \rightarrow \mathbb{R}$ is convex stochastic on n-coordinates, there for $X_{t_n}^i : [u_i, v_i] \times \Omega \rightarrow \mathbb{R}$ is convex stochastic process on $[a_i, b_i]$ for each $i = 1, 2, \dots, n$, we have almost everywhere

$$\begin{aligned} \frac{1}{v_i - u_i} \int_{u_i}^{v_i} X_{u_n}^i(t_i, \cdot) dt_i &\leq \frac{X(u, \cdot) + X_{u_n}^i(v_i, \cdot)}{2} \\ \frac{1}{v_i - u_i} \int_{u_i}^{v_i} X_{v_n}^i(t_i, \cdot) dt_i &\leq \frac{X_{v_n}^i(u_i, \cdot) + X(v, \cdot)}{2} \end{aligned}$$

Adding above two inequalities we have

$$\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [X_{u_n}^i(t_i, \cdot) + X_{v_n}^i(t_i, \cdot)] dt_i \leq \frac{X(u, \cdot) + X(v, \cdot) + X_{u_n}^i(v_i, \cdot) + X_{v_n}^i(u_i, \cdot)}{2},$$

where $i = 1, 2, \dots, n$. Adding above n inequalities we get (3.3). If we consider $X((t_1, t_2, \dots, t_n), \cdot) = t_1 t_2 \dots t_n$, then inequality in (3.3) becomes equality, which gives the sharpness of the inequality.

A special case of inequalities (3.1), (3.2) and (3.3) is stated in the following corollary:

Corollary 3.8. *Let $X : \Delta^2 \times \Omega \rightarrow \mathbb{R}$ be convex stochastic process on 2-coordinates. Then we have almost everywhere*

$$\begin{aligned} &X\left(\left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}\right), \cdot\right) \\ &\leq \frac{1}{2} \left[\frac{1}{(v_1 - u_1)} \int_{u_1}^{v_1} X\left(t_1, \frac{u_2 + v_2}{2}, \cdot\right) dt_1 + \frac{1}{(v_2 - u_2)} \int_{u_2}^{v_2} X\left(\frac{u_1 + v_1}{2}, t_2, \cdot\right) dt_2 \right] \\ &\leq \frac{1}{(v_1 - u_1)(v_2 - u_2)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} X((t_1, t_2), \cdot) dt_2 dt_1 \\ &\leq \frac{1}{4(v_1 - u_1)} \int_{u_1}^{v_1} [X((t_1, u_2), \cdot) + X((t_1, v_2), \cdot)] dt_1 \\ &\leq \frac{1}{4(v_1 - u_1)} \int_{u_1}^{v_1} [X((t_1, u_2), \cdot) + X((t_1, v_2), \cdot)] dt_1 \\ &+ \frac{1}{4(v_2 - u_2)} \int_{u_2}^{v_2} [X((u_1, t_2), \cdot) + X((v_1, t_2), \cdot)] dt_2 \\ &\leq \frac{X(u_1, u_2), \cdot + X((u_1, v_2), \cdot) + X((v_1, u_2), \cdot) + X((v_1, v_2), \cdot)}{4}. \end{aligned} \tag{3.4}$$

Proof. By using (3.1), (3.2) and (3.3) for $n = 2$ and after combining them we get (3.4).

4. CONCLUSION

In this paper, we obtain some new Hermite-Hadamard type inequalities for convex stochastic processes on n-coordinates and obtain. As special cases, one can obtain several new and correct versions of the previously known results for various classes of these stochastic processes. Applying some type of inequalities for stochastic processes is another promising direction for future research.

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