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Gröbner-Shirshov Basis for Complex Reflection Group

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Abstract

The aim of this paper is to obtain a (non-commutative) Gröbner-Shirshov basis for the braid group associated with the complex reflection group G_{24}. This gives us an opportunity to get normal forms of the elements of group G_{24}, which represent a new and effective algorithm to solve the word problem over it.

Keywords: *Braid group; Gröbner-Shirshov basis; Presentation; Word problem.*
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1. Introduction

Presentations arise in various areas of mathematics such as knot theory, topology, and geometry. Another motivation for studying presentations is the advent of softwares for symbolic computations like *GAP* (Groups, Algorithms and Programming). Providing algorithms to compute presentations of given groups (semigroups) is a great help for the developers of these softwares. *GAP* provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the *GAP* language as well as large data libraries of algebraic objects. In *GAP*, although there exist some accepted package programs that shows the solvability of the word problem of some groups types, there is no any general algorithm to solve word problem for arbitray group (monoid) presentations. So, in the present paper, it is worth to calculate Gröbner-Shirshov basis of the braid group associated with the complex reflection group G_{24} and then study solvability of the word problem of that group.

In [10], Buchberger introduced the Gröbner basis theory for commutative algebras and that study provides a solution to the reduction problem for commutative algebras. Later, in [4] Bergman generalized the Gröbner basis theory to associative algebras by proving the *Diamond Lemma*. This theory was developed and advanced for Lie algebras by Shirshov in [22]. The importance of this theory is the so-called *Composition Lemma* which characterizes the leading terms of elements in the given ideal. Then, in 1976, Bokut noticed that Shirshov's method works for associative algebras as well ([6]). Thus, for this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the *Gröbner-Shirshov basis* theory in litarature. There are many valuable and important studies on this subject related to the groups (see, for instance, [7, 11]). We may refer the papers [2, 3, 8, 9, 14, 15, 16, 17, 18] for some other recent studies over Gröbner-Shirshov bases.

The method of Gröbner-Shirshov bases which is the main theme of this paper gives a new algorithm to get normal forms of elements of groups, monoids, semigroups and so a new algorithm for solving the word problem in studied algebraic structures. By considering this fact, our aim in this paper is to find Gröbner-Shirshov basis of the braid group associated with the complex reflection group G_{24} and thus solve the word problem over it.

The word problem is one and the most popular of the algorithmic problems. These kinds of problems have played an important role in group theory since the work of M. Dehn in early 1900's. These problems are called *decision problems* which ask for a yes or no answer to a specific question. It is well known that the word problem for finitely presented groups is not solvable in general; that is, given any two words obtained by generators of the group, there may be no algorithm to decide whether these words represent the same element in this group ([1]).

Shephard and Todd (1954) classified all finite complex reflection groups in [19]. Later Cohen (1976) gave a more systematic description for these groups in terms of root systems, vector graphs and root graphs [12]. Recently, in [13], Howlett and Shi defined a simple root system (B,w) for such these groups which is analogous to the corresponding concept for a Coxeter group. In [20, 21], Shi obtained representatives of all the equivalence classes of simple root systems for the complex reflection groups G_{12}, G_{24}, G_{25} and G_{26}. Then, again in the same paper, the author defined representatives of all the congruence classes of presentations for these groups by generators and relations.

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Throughout this paper, by considering the lengths of any two words, we will use the deg-lex ordering if the lengths of these words are different or lexicographically ordering if otherwise. Additionally the notations $(i)\wedge(j)$ and $(i)\vee(j)$ will denote the intersection and inclusion compositions of relations (i) and (j), respectively.

2. Gröbner-Shirshov Bases and Composition-Diamond Lemma

Let K be a field and $K\langle X\rangle$ be the free associative algebra over K generated by X. Denote X^* the free monoid generated by X, where the empty word is the identity denoted by 1. For a word $w\in X^*$, we denote the length of w by $|w|$. Suppose that X^* is a well ordered set. Then every nonzero polynomial $f\in K\langle X\rangle$ has the leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1, then f is called monic.
Let f and g be two monic polynomials in $K\langle X\rangle$. We then have two compositions as follows:

- If w is a word such that $w=\overline{f}b=a\overline{g}$ for some $a,b\in X^*$ with $|\overline{f}|+|\overline{g}|>|w|$, then the polynomial $(f,g)_w=fb-ag$ is called the *intersection composition* of f and g with respect to w. The word w is called an *ambiguity* of intersection.
- If $w=\overline{f}=a\overline{g}b$ for some $a,b\in X^*$, then the polynomial $(f,g)_w=f-agb$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

If g is monic, $\overline{f}=a\overline{g}b$ and α is the coefficient of the leading term $\overline{f}$, then transformation $f\mapsto f-\alpha agb$ is called elimination (ELW) of the leading word of g in f.
Let $S\subseteq K\langle X\rangle$ with each $s\in S$ is monic. Then the composition $(f,g)_w$ is called trivial modulo (S,w) if $(f,g)_w=\sum\alpha_i a_i s_i b_i$, where each $\alpha_i\in K, a_i,b_i\in X^*, s_i\in S$ and $a_i\overline{s_i}b_i<w$. If this is the case, then we write $(f,g)_w\equiv 0\ mod(S,w)$.
We call the set S endowed with the well ordering $<$ a *Gröbner-Shirshov basis* for $K\langle X\mid S\rangle$ if any composition $(f,g)_w$ of polynomials in S is trivial modulo S and corresponding w.
The following lemma was proved by Shirshov [22] for free Lie algebras with deg-lex ordering.

Lemma 2.1 (Composition-Diamond Lemma). *Let K be a field, $A=K\langle X\mid S\rangle = K\langle X\rangle/Id(S)$ and $<$ a monomial ordering on X^*, where $Id(S)$ is the ideal of $K\langle X\rangle$ generated by S.Then the following statements are equivalent:*

1. *S is a Gröbner-Shirshov basis.*
2. *$f\in Id(S)\Rightarrow \overline{f}=a\overline{s}b$ for some $s\in S$ and $a,b\in X^*$.*
3. *$Irr(S)=\{u\in X^*\mid u\neq a\overline{s}b, s\in S, a,b\in X^*\}$ is a basis for the algebra $A=K\langle X\mid S\rangle$.*

If a subset S of $K\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process many times (maybe infinitely), we eventually obtain a Gröbner-Shirshov basis S^{comp}. We should note that such a process is called the *Shirshov algorithm*.

3. The Main Result

In the study [5], the authors first defined the presentations for the braid groups associated with the complex reflection groups G_{24} and G_{27} and they used VKCURVE that is a GAP package implementing Van Kampen's method to obtain these presentations. Moreover, they added some conjectures for the cases of G_{29}, G_{31}, G_{33} and G_{34}.

Proposition 3.1. *[5] The braid group associated with the complex reflection group G_{24} admits the presentation*

$$\langle s,t,u; stst=tsts, susu=usus, tutu=utut, stustus=tustust=ustustu\rangle . \tag{3.1}$$

Actually, as noted in the same reference, one can obtain some other presentations that present the same group G_{24} as in the following. First consider presentation (3.1), and then replace t by $usts^{-1}u^{-1}$. Thus we get the presentation

$$\langle s,t,u; sts=tst, tut=utu, susu=usus, sustustus=ustustust\rangle .$$

Additionally, again in presentation (3.1), if we replace t by $susts^{-1}u^{-1}s^{-1}$, then we have the following presentation

$$\langle s,t,u; sts=tst, tutu=utut, susu=usus, sutsuts=usutsut\rangle , \tag{3.2}$$

as a new presentation of G_{24}. We now give a monoid presentation of the group G_{24} given in (3.2) as follows:

$$< s,t,u; sts=tst, tutu=utut, susu=usus, sutsuts=usutsut, ss^{-1}=s^{-1}s=1, tt^{-1}=t^{-1}t=1, uu^{-1}=u^{-1}u=1 > . \tag{3.3}$$

Let us order the generators as $u>s>t$. We also note that the presentation of the group G_{24}, given in (3.3), will be used in our result.
The main result of this paper is the following.

Theorem 3.2. *A Gröbner-Shirshov basis of the braid group associated with the complex reflection group G_{24} consists of the following relations:*

$$
\begin{array}{ll}
(1) & sts = tst, \\
(2) & utut = tutu, \\
(3) & usus = susu, \\
(4) & usutsut = sutsuts, \\
(5) & st^n st = tst^2 s^{n-1}, \\
(6) & ut^n utu = tutuut^{n-1}, \\
(7) & us^n usu = susuus^{n-1}, \\
(8) & usut^{n'} st = susuts^{n'}, \\
(9) & usutst^{n'} utu = sutsutsut^{n'}, \\
(10) & us^n utsut^m st^{k'} = susuutst^{n-2}ut(ts^{m-1})^{k'}, \\
(11-a) & ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d usu = tutuut^{k-1}(A_1'B_1'A_2'B_2'\cdots A_N'B_N')^{\lambda} sus^d, \\
(11-b) & ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda} t^d utu = tutuut^{k-1}sus^{m'}(B_1'A_1'B_2'A_2'\cdots B_N'A_N')^{\lambda} tut^d, \\
(11-c) & ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d utsut^h st^p = tutuut^{k-1}(A_1'B_1'A_2'B_2'\cdots A_N'B_N')^{\lambda} sus^{d-1}tsut(ts^{h-1})^p, \\
(11-c)^* & ut^k ut(A_1B_1A_2B_2\cdots A_N)^{\lambda} s^d utsut^h st^p = tutuut^{k-1}(A_1'B_1'A_2'B_2'\cdots A_N')^{\lambda} sus^{d-1}tsut(ts^{h-1})^p, \\
(12-a) & usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d usu = sutsutsut^{n'}(A_1'B_1'A_2'B_2'\cdots A_N'B_N')^{\lambda} sus^d, \\
(12-b) & usuts(t^{n'}ut)^c(A_1B_1A_2B_2\cdots A_N)^{\lambda} t^d utu = sutsut(sut^{n'})^c(A_1'B_1'A_2'B_2'\cdots A_N')^{\lambda} tut^d, \\
(12-c) & usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d utsut^h st^p = sutsutsut^{n'}(A_1'B_1'A_2'B_2'\cdots A_N'B_N')^{\lambda} sus^{d-1}tsut(ts^{h-1})^p, \\
(12-c)^* & usutst^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda} s^d utsut^h st^p = sutsutsut^{n'}(A_1'B_1'A_2'B_2'\cdots A_N')^{\lambda} sus^{d-1}tsut(ts^{h-1})^p, \\
(13-a) & us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d usu = susuus^{k-1}tut^{n'}(A_1'B_1'A_2'B_2'\cdots A_N'B_N')^{\lambda} sus^d, \\
(13-b) & us^k us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda} t^d utu = susuus^{k-1}(B_1'A_1'B_2'A_2'\cdots B_N'A_N')^{\lambda} tut^d, \\
(13-c) & us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d utsut^h st^p = susuus^{k-1}tut^{n'}(A_1'B_1'A_2'B_2'\cdots A_N'B_N')^{\lambda} sus^{d-1}tsut(ts^{h-1})^p, \\
(13-c)^* & us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda} s^d utsut^h st^p = susuus^{k-1}tut^{n'}(A_1'B_1'A_2'B_2'\cdots A_N')^{\lambda} sus^{d-1}tsut(ts^{h-1})^p, \\
(14) & ss^{-1} = 1, \qquad (15)\, s^{-1}s = 1, \\
(16) & tt^{-1} = 1, \qquad (17)\, t^{-1}t = 1, \\
(18) & uu^{-1} = 1, \qquad (19)\, u^{-1}u = 1,
\end{array}
$$

where $k,m,n \geq 2$; $m',n',d,h \geq 1$; $k',p,c = \{0,1\}$; $A_i = s^{a_i}us$; $A_i' = sus^{a_i}$; $B_i = t^{b_i}ut$; $B_i' = tut^{b_i}$ $(1 \leq i \leq N)$ *and* $\lambda = \{0,1\}$.

Proof. We need to prove that all compositions among relations $(1)-(19)$ are trivial. We classified the relations (No-a, No-b, No-c) in theorem by regarding the following cases.

- *For the cases* No-a: Since the left hand side of relations ended by the sub word $s^d usu$ $(d \geq 1)$, the sub word B_N must be placed before it.
- *For the cases* No-b: Since the left hand side of relations ended by the sub word $t^d utu$ $(d \geq 1)$, the sub word A_N must be placed before it.
- *For the cases* No-c: Since the left hand side of relations ended by the sub word $s^d utsut^h st^p$ $(d,h \geq 1,\ p = 0,1)$, there are two possibilities either A_N or B_N must be placed before it.

Let $k,m,n \geq 2$; $m',n',d,h \geq 1$; $k',p,c = \{0,1\}$; $A_i = s^{a_i}us$; $A_i' = sus^{a_i}$; $B_i = t^{b_i}ut$; $B_i' = tut^{b_i}$ $(1 \leq i \leq N)$ and $\lambda = \{0,1\}$. Let us, firstly, consider the intersection compositions of the relation (1) with relations $(1)-(19)$ and start with listing all intersections compositions. Actually we have the following ambiguities w:

$(1)\wedge(1) : w = ststs,$ $\qquad (1)\wedge(5) : w = stst^n st,$
$(5)\wedge(1) : w = st^n sts,$ $\qquad (8)\wedge(1) : w = usut^{n'} sts,$
$(1)\wedge(14) : w = stss^{-1},$ $\qquad (10)\wedge(1) : w = us^n utsut^m sts,$
$(11-c)\wedge(1) : w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d utsut^h sts,$
$(11-c)^*\wedge(1) : w = ut^k ut(A_1B_1A_2B_2\cdots A_N)^{\lambda} s^d utsut^h sts,$
$(12-c)\wedge(1) : w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d utsut^h sts,$
$(12-c)^*\wedge(1) : w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda} s^d utsut^h sts,$
$(13-c)\wedge(1) : w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda} s^d utsut^h sts,$
$(13-c)^*\wedge(1) : w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda} s^d utsut^h sts,$
$(15)\wedge(1) : w = s^{-1}sts.$

It is seen that these compositions are trivial. Let us check one of them as an example:

$$\begin{aligned}(5)\wedge(1):w &= st^nsts,\\ (f,g)_w &= (st^nst-tstts^{n-1})s-st^n(sts-tst)\\ &= st^nsts-tstts^{n-1}s-st^nsts+st^ntst\\ &= st^{n+1}st-tstts^{n-1}s \quad (\text{by } (5))\\ &\equiv tstts^n-tstts^n\equiv 0.\end{aligned}$$

Let us proceed with intersection compositions of (2) with the relations $(2)-(19)$. The ambiguities are the following:

$(2)\wedge(2): w=ututut,\qquad (4)\wedge(2): w=usutsutut,$
$(2)\wedge(6): w=utut^nutu,\qquad (6)\wedge(2): w=ut^nutut,$
$(6)\wedge(2): w=ut^nututut,\qquad (9)\wedge(2): w=usutst^{n'}utut,$
$(9)\wedge(2): w=usutst^{n'}ututut,\qquad (2)\wedge(16): w=ututt^{-1},$
$(2)\wedge(11-a): w=utut^kut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dusu,$
$(2)\wedge(11-b): w=utut^kuts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^dutu,$
$(2)\wedge(11-c): w=utut^kut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dutsut^hst^p,$
$(2)\wedge(11-c)^*: w=utut^kut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^dutsut^hst^p,$
$(11-a)\wedge(2): w=ut^kut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dusutut,$
$(11-b)\wedge(2): w=ut^kuts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^dutut,$
$(11-b)\wedge(2): w=ut^kuts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)t^dututut,$
$(12-a)\wedge(2): w=usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dusutut,$
$(12-b)\wedge(2): w=usuts(t^{n'}ut)^c(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^dutut,$
$(12-b)\wedge(2): w=usuts(t^{n'}ut)^c(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^dututut,$
$(13-a)\wedge(2): w=us^kust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dusutut,$
$(13-b)\wedge(2): w=us^kus(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^dutut,$
$(13-b)\wedge(2): w=us^kus(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^dututut,$
$(19)\wedge(2): w=u^{-1}utut.$

These intersection compositions are trivial. Let us check two of them.

$$\begin{aligned}(9)\wedge(2):w &= usutst^{n'}utut,\\ (f,g)_w &= (usutst^{n'}utu-sutsutsut^{n'})t-usutst^{n'}(utut-tutu)\\ &= usutst^{n'}utut-sutsutsut^{n'}t-usutst^{n'}utut+usutst^{n'}tutu\\ &= usutst^{n'}tutu-sutsutsut^{n'}t \quad (\text{by } (9))\\ &\equiv sutsutsut^{n'+1}-sutsutsut^{n'+1}\equiv 0.\end{aligned}$$

$$\begin{aligned}(13-b)\wedge(2)\ &:\ w=us^kus(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^dututut,\\ \underline{for\ \lambda=1}\ &\\ (f,g)_w &= (us^kusB_1A_1B_2A_2\cdots B_NA_Nt^dutu-susuus^{k-1}B_1'A_1'B_2'A_2'\cdots B_N'A_N'tut^d)tut\\ &- us^kusB_1A_1B_2A_2\cdots B_NA_Nt^dut(utut-tutu)\\ &= us^kusB_1A_1B_2A_2\cdots B_NA_Nt^dututut-susuus^{k-1}B_1'A_1'B_2'A_2'\cdots B_N'A_N'tut^dtut\\ &- us^kusB_1A_1B_2A_2\cdots B_NA_Nt^dututut+us^kusB_1A_1B_2A_2\cdots B_NA_Nt^duttutu\\ &= us^kusB_1A_1B_2A_2\cdots B_NA_Nt^duttutu-susuus^{k-1}B_1'A_1'B_2'A_2'\cdots B_N'A_N'tut^dtut \quad (\text{by } (6))\\ &\equiv us^kusB_1A_1B_2A_2\cdots B_NA_Nt^dtutuut-susuus^{k-1}B_1'A_1'B_2'A_2'\cdots B_N'A_N'tut^{d+1}ut\ (\text{by}(13-b))\\ &\equiv susuus^{k-1}B_1'A_1'B_2'A_2'\cdots B_N'A_N'tut^{d+1}ut-susuus^{k-1}B_1'A_1'B_2'A_2'\cdots B_N'A_N'tut^{d+1}ut\equiv 0.\end{aligned}$$

Our next compositions will be (3) with $(3)-(19)$. The ambiguties of these intersection composition are the following:

$(3)\wedge(3): w=ususus,\qquad (3)\wedge(4): w=ususutsut,$
$(3)\wedge(7): w=usus^nusu,\qquad (3)\wedge(8): w=ususut^{n'}st,$
$(3)\wedge(9): w=ususutst^{n'}utu,\qquad (3)\wedge(10): w=usus^nutsut^mst^p,$
$(3)\wedge(14): w=ususs^{-1},$
$(3)\wedge(12-a): w=ususutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dusu,$
$(3)\wedge(12-b): w=ususuts(t^{n'}ut)^c(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^dutu,$
$(3)\wedge(12-c): w=ususutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dutsut^hst^p,$
$(3)\wedge(12-c)^*: w=ususutst^{n'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^dutsut^hst^p,$
$(3)\wedge(13-a): w=usus^kust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^dusu,$
$(3)\wedge(13-b): w=usus^kus(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^dutu,$

$(3)\wedge(13-c): w = usus^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d utsut^h st^p,$
$(3)\wedge(13-c)^*: w = usus^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^d utsut^h st^p,$
$(6)\wedge(3): w = ut^n utusus, \qquad (7)\wedge(3): w = us^n usus,$
$(7)\wedge(3): w = us^n ususus, \qquad (9)\wedge(3): w = usutst^{n'} utusus,$
$(11-a)\wedge(3): w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus,$
$(11-a)\wedge(3): w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususus,$
$(11-b)\wedge(3): w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusus,$
$(12-a)\wedge(3): w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus,$
$(12-a)\wedge(3): w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususus,$
$(12-b)\wedge(3): w = usuts(t^{n'} ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^d utusus,$
$(13-a)\wedge(3): w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus,$
$(13-a)\wedge(3): w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususus,$
$(13-b)\wedge(3): w = us^k us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusus,$
$(19)\wedge(3): w = u^{-1}usus.$

These composition are trivial. Let us check one of them as follows.

$$\begin{aligned}
(7)\wedge(3): w &= us^n usus,\\
(f,g)_w &= (us^n usu - susuus^{n-1})s - us^n(usus - susu)\\
&= us^n usus - susuus^{n-1}s - us^n usus + us^n susu\\
&= us^n susu - susuus^{n-1}s\\
&\equiv us^{n+1}usu - susuus^n \quad \text{(by (7))}\\
&\equiv susuus^n - susuus^n \equiv 0.
\end{aligned}$$

Now we proceed with intersection composition of (4) with $(4)-(19)$. The ambiguities are as follows.

$(4)\wedge(6): w = usutsut^n utu,$
$(4)\wedge(11-a): w = usutsut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usu,$
$(4)\wedge(11-b): w = usutsut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utu,$
$(4)\wedge(11-c): w = usutsut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d utsut^h st^p,$
$(4)\wedge(11-c)^*: w = usutsut^k ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^d utsut^h st^p,$
$(4)\wedge(16): w = usutsutt^{-1}, \qquad (6)\wedge(4): w = ut^n utusutsut,$
$(7)\wedge(4): w = us^n usutsut, \qquad (7)\wedge(4): w = us^n ususutsut,$
$(9)\wedge(4): w = usutst^{n'} utusutsut,$
$(11-a)\wedge(4): w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usutsut,$
$(11-a)\wedge(4): w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)s^d ususutsut,$
$(11-b)\wedge(4): w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusutsut,$
$(12-a)\wedge(4): w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usutsut,$
$(12-a)\wedge(4): w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususutsut,$
$(12-b)\wedge(4): w = usuts(t^{n'} ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^d utusutsut,$
$(13-a)\wedge(4): w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usutsut,$
$(13-a)\wedge(4): w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususutsut,$
$(13-b)\wedge(4): w = us^k us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusutsut,$
$(19)\wedge(4): w = u^{-1}usutsut.$

These intersection compositions are trivial. Let us check one of them as an example:

$$\begin{aligned}
(12-a)\wedge(4): w &= usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususutsut,\\
\underline{for\ \lambda=1} &\\
w &= usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d ususutsut\\
(f,g)_w &= (usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d usu - sutsutsut^{n'} sus^{a_1}tut^{b_1}sus^{a_2}tut^{b_2}\cdots sus^{a_N}tut^{b_N}sus^d)sutsut\\
&- usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d us(usutsut - sutsuts)\\
&= usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d ususutsut - sutsutsut^{n'} sus^{a_1}tut^{b_1}sus^{a_2}tut^{b_2}\cdots sus^{a_N}tut^{b_N}sus^d sutsut\\
&- usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d ususutsut + usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d ussutsuts\\
&= usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d ussutsuts - sutsutsut^{n'} sus^{a_1}tut^{b_1}sus^{a_2}tut^{b_2}\cdots sus^{a_N}tut^{b_N}sus^d sutsut\\
&\equiv usutst^{n'} uts^{a_1}ust^{b_1}uts^{a_2}ust^{b_2}ut\cdots s^{a_N}ust^{b_N}uts^d susuutsut - sutsutsut^{n'} sus^{a_1}tut^{b_1}sus^{a_2}tut^{b_2}\cdots sus^{a_N}tut^{b_N}sus^{d+1}utsut\\
&\equiv sutsutsut^{n'} sus^{a_1}tut^{b_1}sus^{a_2}tut^{b_2}\cdots sus^{a_N}tut^{b_N}sus^{d+1}utsut - sutsutsut^{n'} sus^{a_1}tut^{b_1}sus^{a_2}tut^{b_2}\cdots sus^{a_N}tut^{b_N}sus^{d+1}utsut\\
&\equiv 0.
\end{aligned}$$

Now we consider composition of intersection of (5) with $(5)-(19)$. We have the ambiguties as follows.

$(5)\wedge(5): w=st^nst^mst, \quad (8)\wedge(5): w=usut^{n'}st^mst,$
$(5)\wedge(16): w=st^nstt^{-1}, \quad (10)\wedge(5): w=us^nutsut^mst^kst,$
$(11-c)\wedge(5): w=ut^kut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dutsut^hst^nst,$
$(11-c)^*\wedge(5): w=ut^kut(A_1B_1A_2B_2\cdots A_N)^\lambda s^dutsut^hst^nst,$
$(12-c)\wedge(5): w=usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dutsut^hst^mst,$
$(12-c)^*\wedge(5): w=usutst^{n'}ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^dutsut^hst^mst,$
$(13-c)\wedge(5): w=us^kust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dutsut^hst^mst,$
$(13-c)^*\wedge(5): w=us^kust^{n'}ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^dutsut^hst^mst,$
$(15)\wedge(5): w=s^{-1}st^nst.$

These composition are trivial. Let us check one of them as examples:

$$\begin{aligned}
(8)\wedge(5): w &= usut^{n'}st^mst,\\
(f,g)_w &= (usut^{n'}st-susuts^{n'})t^{m-1}st-usut^{n'}(st^mst-tst^2s^{m-1})\\
&= usut^{n'}stt^{m-1}st-susuts^{n'}t^{m-1}st-usut^{n'}st^mst+usut^{n'}tst^2s^{m-1}\\
&= usut^{n'}tstts^{m-1}-susuts^{n'}t^{m-1}st \quad \text{(by (8))}\\
&\equiv susuts^{n'+1}ts^{m-1}-susuts^{n'}t^{m-1}st\\
&\equiv susuts^{n'}stss^{m-2}-susuts^{n'}t^{m-1}st \quad \text{(by (1))}\\
&\equiv susuts^{n'}tstss^{m-3}-susuts^{n'}t^{m-1}st \quad \text{(by (1))}\\
&\equiv susuts^{n'}ttstss^{m-4}-susuts^{n'}t^{m-1}st \quad \text{(by (1))}\\
&\equiv \cdots\\
&\equiv susuts^{n'}t^{m-1}sts^{m-m}-susuts^{n'}t^{m-1}st\equiv 0.
\end{aligned}$$

Let us consider the intersection compositions of (6) with $(6)-(19)$. We have the following ambiguities.

$(6)\wedge(6): w=ut^nutut^mutui \quad (6)\wedge(7): w=ut^nutus^musu,$
$(6)\wedge(8): w=ut^nutusut^{m'}st, \quad (6)\wedge(9): w=ut^nutusutst^{m'}utu,$
$(6)\wedge(10): w=ut^nutus^mutsut^kst^p, \quad (6)\wedge(18): w=ut^nutuu^{-1},$
$(6)\wedge(11-a): w=ut^nutut^kut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dusu,$
$(6)\wedge(11-b): w=ut^nutut^kuts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^dutu,$
$(6)\wedge(11-c): w=ut^nutut^kut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dutsut^hst^p,$
$(6)\wedge(11-c)^*: w=ut^nutut^kut(A_1B_1A_2B_2\cdots A_N)^\lambda s^dutsut^hst^p,$
$(6)\wedge(12-a): w=ut^nutusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)s^dusu,$
$(6)\wedge(12-b): w=ut^nutusuts(t^{m'}ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^dutu,$
$(6)\wedge(12-c): w=ut^nutusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dutsut^hst^p,$
$(6)\wedge(12-c)^*: w=ut^nutusutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^dutsut^hst^p,$
$(6)\wedge(13-a): w=ut^nutus^kust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dusu,$
$(6)\wedge(13-b): w=ut^nutus^kus(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^dutu,$
$(6)\wedge(13-c): w=ut^nutus^kust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dutsut^hst^p,$
$(6)\wedge(13-c)^*: w=ut^nutus^kust^{m'}ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^dutsut^hst^p,$
$(7)\wedge(6): w=us^nusut^mutu, \quad (9)\wedge(6): w=usutst^{n'}utut^mutu,$
$(11-a)\wedge(6): w=ut^kut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dusut^nutu,$
$(11-b)\wedge(6): w=ut^kuts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^dutut^nutu,$
$(12-a)\wedge(6): w=usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dusut^nutu,$
$(12-b)\wedge(6): w=usuts(t^{n'}ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^dutut^nutu,$
$(13-a)\wedge(6): w=us^kust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dusut^nutu,$
$(13-b)\wedge(6): w=us^kus(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^dutut^nutu,$
$(19)\wedge(6): w=u^{-1}ut^nutu.$

It is seen that these compositions are trivial. Our next composition will be (7) with $(7)-(19)$. The ambiguties of these intersection compositions are the following:

$(7)\wedge(7): w=us^nusus^musu, \quad (7)\wedge(8): w=us^nusut^{m'}st,$
$(7)\wedge(8): w=us^nususut^{m'}st, \quad (7)\wedge(9): w=us^nusutst^{m'}utu,$
$(7)\wedge(9): w=us^nususutst^{m'}utu, \quad (7)\wedge(10): w=us^nusus^kutsut^mst^p,$
$(7)\wedge(18): w=us^nusuu^{-1},$
$(7)\wedge(11-a): w=us^nusut^kut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dusu,$
$(7)\wedge(11-b): w=us^nusut^kuts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^dutu,$
$(7)\wedge(11-c): w=us^nusut^kut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^dutsut^hst^p,$

$(7)\wedge(11-c)^{*}: w=us^{n}usut^{k}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}st^{p}$,
$(7)\wedge(12-a): w=us^{n}usutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usu$,
$(7)\wedge(12-a): w=us^{n}ususutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usu$,
$(7)\wedge(12-b): w=us^{n}usuts(t^{m'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utu$,
$(7)\wedge(12-b): w=us^{n}ususuts(t^{m'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utu$,
$(7)\wedge(12-c): w=us^{n}usutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}st^{p}$,
$(7)\wedge(12-c)^{*}: w=us^{n}usutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}st^{p}$,
$(7)\wedge(12-c): w=us^{n}ususutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}st^{p}$,
$(7)\wedge(12-c)^{*}: w=us^{n}ususutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}st^{p}$,
$(7)\wedge(13-a): w=us^{n}usus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usu$,
$(7)\wedge(13-b): w=us^{n}usus^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utu$,
$(7)\wedge(13-c): w=us^{n}usus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}st^{p}$,
$(7)\wedge(13-c)^{*}: w=us^{n}usus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}st^{p}$,
$(9)\wedge(7): w=usutst^{n'}utus^{m}usu$,
$(11-a)\wedge(7): w=ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{n}usu$,
$(11-b)\wedge(7): w=ut^{k}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utus^{n}usu$,
$(12-a)\wedge(7): w=usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}usu$,
$(12-b)\wedge(7): w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utus^{m}usu$,
$(13-a)\wedge(7): w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}usu$,
$(13-b)\wedge(7): w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utus^{m}usu$,
$(19)\wedge(7): w=u^{-1}us^{n}usu$.

These composition are trivial. We proceed with intersection compositions of (8) with $(8)-(19)$. The ambiguities are the following:

$(9)\wedge(8): w=usutst^{n'}utusut^{m'}st$, $\quad (8)\wedge(16): w=usut^{n'}stt^{-1}$,
$(11-a)\wedge(8): w=ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{n'}st$,
$(11-a)\wedge(8): w=ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususut^{n'}st$,
$(11-b)\wedge(8): w=ut^{k}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusut^{n'}st$,
$(12-a)\wedge(8): w=usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m'}st$,
$(12-a)\wedge(8): w=usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususut^{m'}st$,
$(12-b)\wedge(8): w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utusut^{m'}st$,
$(13-a)\wedge(8): w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m'}st$,
$(13-a)\wedge(8): w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususut^{m'}st$,
$(13-b)\wedge(8): w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusut^{m'}st$,
$(19)\wedge(8): w=u^{-1}usut^{n'}st$.

These compositions are trivial. Let us check two of them.

$$
\begin{aligned}
(19)\wedge(8) \quad &: \quad w=u^{-1}usut^{n'}st,\\
(f,g)_w &= (u^{-1}u-1)sut^{n'}st-u^{-1}(usut^{n'}st-susuts^{n'})\\
&= u^{-1}usut^{n'}st-sut^{n'}st-u^{-1}usut^{n'}st+u^{-1}susuts^{n'}\\
&= u^{-1}susuts^{n'}-sut^{n'}st\\
&\equiv uu^{-1}susuts^{n'}-usut^{n'}st \quad (\text{by } (8))\\
&\equiv 0.
\end{aligned}
$$

$$
\begin{aligned}
(9)\wedge(8) \quad &: \quad w=usutst^{n'}utusut^{m'}st,\\
(f,g)_w &= (usutst^{n'}utu-sutsutsut^{n'})sut^{m'}st-usutst^{n'}ut(usut^{m'}st-susuts^{m'})\\
&= usutst^{n'}utusut^{m'}st-sutsutsut^{n'}sut^{m'}st-usutst^{n'}utusut^{m'}st+usutst^{n'}utsusuts^{m'}\\
&= usutst^{n'}utsusuts^{m'}-sutsutsut^{n'}sut^{m'}st \quad (\text{by } (12-a))\\
&\equiv sutsutsut^{n'}sustss^{m'-1}-sutsutsut^{n'}sut^{m'}st \quad (\text{by } (1))\\
&\equiv sutsutsut^{n'}sutstss^{m'-2}-sutsutsut^{n'}sut^{m'}st \quad (\text{by } (1))\\
&\equiv \cdots\\
&\equiv sutsutsut^{n'}sut^{m'}sts^{m'-m'}-sutsutsut^{n'}sut^{m'}st\equiv 0.
\end{aligned}
$$

Now we consider composition of intersection of (9) with $(9)-(19)$. We have the ambiguities as follows.

$(9)\wedge(9): w = usutst^{n'}utusutst^{m'}utu, \qquad (9)\wedge(10): w = usutst^{n'}utus^{k}utsut^{m}st^{p},$
$(9)\wedge(18): w = usutst^{n'}utuu^{-1},$
$(9)\wedge(11-a): w = usutst^{n'}utut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usu,$
$(9)\wedge(11-b): w = usutst^{n'}utut^{k}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utu,$
$(9)\wedge(11-c): w = usutst^{n'}utut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}st^{p},$
$(9)\wedge(11-c)^{*}: w = usutst^{n'}utut^{k}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}st^{p},$
$(9)\wedge(12-a): w = usutst^{n'}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usu,$
$(9)\wedge(12-b): w = usutst^{n'}utusuts(t^{m'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utu,$
$(9)\wedge(12-c): w = usutst^{n'}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}st^{p},$
$(9)\wedge(12-c)^{*}: w = usutst^{n'}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}st^{p},$
$(9)\wedge(13-a): w = usutst^{n'}utus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usu,$
$(9)\wedge(13-b): w = usutst^{n'}utus^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utu,$
$(9)\wedge(13-c): w = usutst^{n'}utus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}st^{p},$
$(9)\wedge(13-c)^{*}: w = usutst^{n'}utus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}st^{p},$
$(11-a)\wedge(9): w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{n'}utu,$
$(11-a)\wedge(9): w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{n'}utu,$
$(11-b)\wedge(9): w = ut^{k}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusutst^{n'}utu,$
$(12-a)\wedge(9): w = usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{m'}utu,$
$(12-a)\wedge(9): w = usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{m'}utu,$
$(12-b)\wedge(9): w = usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utusutst^{m'}utu,$
$(13-a)\wedge(9): w = us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{m'}utu,$
$(13-a)\wedge(9): w = us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{m'}utu,$
$(13-b)\wedge(9): w = us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusutst^{m'}utu,$
$(19)\wedge(9): w = u^{-1}usutst^{n'}utu.$

It is seen that these composition are trivial. Our next compositions will be (10) with (10) – (19). The ambiguties of these intersection composition are the following:

$(10)\wedge(14): w = us^{n}utsut^{m}ss^{-1}, \qquad (10)\wedge(16): w = us^{n}utsut^{m}stt^{-1},$
$(19)\wedge(10): w = u^{-1}us^{n}utsut^{m}st^{k'},$
$(11-a)\wedge(10): w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}utsut^{n}st^{p},$
$(11-b)\wedge(10): w = ut^{k}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utus^{n}utsut^{m}st^{p},$
$(12-a)\wedge(10): w = usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}utsut^{n}st^{p},$
$(12-b)\wedge(10): w = usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utus^{m}utsut^{n}st^{p},$
$(13-a)\wedge(10): w = us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}utsut^{n}st^{p},$
$(13-b)\wedge(10): us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utus^{m}utsut^{n}st^{p}.$

It is seen that these compositions are trivial. To finish with the case of compositions of intersection, we need to consider (11) – (19) with (11) – (19). The ambiguities are as follows:

$$(11-a)\wedge(11-a) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{n'}usu,$$
$$(11-a)\wedge(11-b) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{n'}utu,$$
$$(11-a)\wedge(11-c) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}utsut^{h}st^{p},$$
$$(11-a)\wedge(11-c)^{*} \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{m'}utsut^{h}st^{p},$$
$$(11-a)\wedge(12-a) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}usu,$$
$$(11-a)\wedge(12-a) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}usu,$$
$$(11-a)\wedge(12-b) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{m'}utu,$$
$$(11-a)\wedge(12-b) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{m'}utu,$$
$$(11-a)\wedge(12-c) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}utsut^{h}st^{p},$$
$$(11-a)\wedge(12-c)^{*} \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{n'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{m'}utsut^{h}st^{p},$$
$$(11-a)\wedge(12-c) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}utsut^{h}st^{p},$$
$$(11-a)\wedge(12-c)^{*} \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{n'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{m'}utsut^{h}st^{p},$$
$$(11-a)\wedge(13-a) \;:\; w = ut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}usu,$$

$$(11-a)\wedge(13-b) \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus^m us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^{n'} utu,$$
$$(11-a)\wedge(13-c) \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus^m ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^{m'} utsut^h st^p,$$
$$(11-a)\wedge(13-c)^* \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus^m ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^{m'} utsut^h st^p,$$
$$(11-a)\wedge(18) \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usuu^{-1},$$
$$(11-b)\wedge(11-a) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utut^m ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^{n'} usu,$$
$$(11-b)\wedge(11-b) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utut^m uts^{n'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^h utu,$$
$$(11-b)\wedge(11-c) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utut^m ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^{n'} utsut^h st^p,$$
$$(11-b)\wedge(11-c)^* \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utut^m ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^{n'} utsut^h st^p,$$
$$(11-b)\wedge(12-a) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^h usu,$$
$$(11-b)\wedge(12-b) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusuts(t^{n'} ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^h utu,$$
$$(11-b)\wedge(12-c) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^q utsut^h st^p,$$
$$(11-b)\wedge(12-c)^* \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utusutst^{n'} ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^q utsut^h st^p,$$
$$(11-b)\wedge(13-a) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utus^m ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^h usu,$$
$$(11-b)\wedge(13-b) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utus^m us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^{n'} utu,$$
$$(11-b)\wedge(13-c) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utus^m ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^q utsut^h st^p,$$
$$(11-b)\wedge(13-c)^* \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utus^m ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^q utsut^h st^p,$$
$$(11-b)\wedge(18) \quad : \quad w = ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^d utuu^{-1},$$
$$(11-c)\wedge(14) \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d utsut^h ss^{-1},$$
$$(11-c)^*\wedge(14) \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^d utsut^h ss^{-1},$$
$$(11-c)\wedge(16) \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d utsut^h stt^{-1},$$
$$(11-c)^*\wedge(16) \quad : \quad w = ut^k ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^d utsut^h stt^{-1},$$
$$(12-a)\wedge(11-a) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^{m'} usu,$$
$$(12-a)\wedge(11-b) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^h utu,$$
$$(12-a)\wedge(11-c) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^{m'} utsut^h st^p,$$
$$(12-a)\wedge(11-c)^* \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usut^k ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^{m'} utsut^h st^p,$$
$$(12-a)\wedge(12-a) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usutst^{m'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^h usu,$$
$$(12-a)\wedge(12-a) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususutst^{m'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^h usu,$$
$$(12-a)\wedge(12-b) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usuts(t^{m'} ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^h utu,$$
$$(12-a)\wedge(12-b) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususuts(t^{m'} ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^h utu,$$
$$(12-a)\wedge(12-c) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usutst^{m'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^q utsut^h st^p,$$
$$(12-a)\wedge(12-c)^* \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usutst^{m'} ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^q utsut^h st^p,$$
$$(12-a)\wedge(12-c) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususutst^{m'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^q utsut^h st^p,$$
$$(12-a)\wedge(12-c)^* \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d ususutst^{m'} ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^q utsut^h st^p,$$
$$(12-a)\wedge(13-a) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus^k ust^{m'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^h usu,$$
$$(12-a)\wedge(13-b) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus^k us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^{m'} utu,$$
$$(12-a)\wedge(13-c) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus^k ust^{m'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^q utsut^h st^p,$$
$$(12-a)\wedge(13-c)^* \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usus^k ust^{m'} ut(A_1B_1A_2B_2\cdots A_N)^\lambda s^q utsut^h st^p,$$
$$(12-a)\wedge(18) \quad : \quad w = usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^d usuu^{-1},$$
$$(12-b)\wedge(11-a) \quad : \quad w = usuts(t^{n'} ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^d utut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^\lambda s^h usu,$$
$$(12-b)\wedge(11-b) \quad : \quad w = usuts(t^{n'} ut)^c(A_1B_1A_2B_2\cdots A_N)^\lambda t^d utut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^\lambda t^h utu,$$

$$(12-b)\wedge(11-c) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utut^{k}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(12-b)\wedge(11-c)^{*} \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utut^{k}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(12-b)\wedge(12-a) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{h}usu,$$

$$(12-b)\wedge(12-b) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utusuts(t^{m'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{h}utu,$$

$$(12-b)\wedge(12-c) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(12-b)\wedge(12-c)^{*} \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(12-b)\wedge(13-a) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{h}usu,$$

$$(12-b)\wedge(13-b) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utus^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{q}utu,$$

$$(12-b)\wedge(13-c) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(12-b)\wedge(13-c)^{*} \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utus^{k}ust^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(12-b)\wedge(18) \quad : \quad w=usuts(t^{n'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{d}utuu^{-1},$$

$$(12-c)\wedge(14) \quad : \quad w=usutst^{n'}(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}ss^{-1},$$

$$(12-c)^{*}\wedge(14) \quad : \quad w=usutst^{n'}(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}ss^{-1},$$

$$(12-c)\wedge(16) \quad : \quad w=usutst^{n'}(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}utsut^{h}stt^{-1},$$

$$(12-c)^{*}\wedge(16) \quad : \quad w=usutst^{n'}(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{d}utsut^{h}stt^{-1},$$

$$(13-a)\wedge(11-a) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}usu,$$

$$(13-a)\wedge(11-b) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{h}utu,$$

$$(13-a)\wedge(11-c) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(11-c)^{*} \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usut^{m}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(12-a) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{h}usu,$$

$$(13-a)\wedge(12-a) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{h}usu,$$

$$(13-a)\wedge(12-b) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usuts(t^{m'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{h}utu,$$

$$(13-a)\wedge(12-b) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususuts(t^{m'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{h}utu,$$

$$(13-a)\wedge(12-c) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(12-c)^{*} \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(12-c) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(12-c)^{*} \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}ususutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(13-a) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{h}usu,$$

$$(13-a)\wedge(13-b) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{h}utu,$$

$$(13-a)\wedge(13-c) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(13-c)^{*} \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usus^{m}ust^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-a)\wedge(18) \quad : \quad w=us^{k}ust^{n'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{d}usuu^{-1},$$

$$(13-b)\wedge(11-a) \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utut^{m}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{m'}usu,$$

$$(13-b)\wedge(11-b) \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utut^{m}uts^{m'}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{h}utu,$$

$$(13-b)\wedge(11-c) \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utut^{m}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-b)\wedge(11-c)^{*} \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utut^{m}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-b)\wedge(12-a) \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{h}usu,$$

$$(13-b)\wedge(12-b) \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusuts(t^{m'}ut)^{c}(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^{h}utu,$$

$$(13-b)\wedge(12-c) \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-b)\wedge(12-c)^{*} \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utusutst^{m'}ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^{q}utsut^{h}st^{p},$$

$$(13-b)\wedge(13-a) \quad : \quad w=us^{k}us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^{d}utus^{m}ust^{m'}ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^{h}usu,$$

$$
\begin{aligned}
(13-b)\wedge(13-b) &: \quad w = us^k us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^d utus^m us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^h utu,\\
(13-b)\wedge(13-c) &: \quad w = us^k us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^d utus^m ust^{m'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^q utsut^h st^p,\\
(13-b)\wedge(13-c)^* &: \quad w = us^k us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^d utus^m ust^{m'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^q utsut^h st^p,\\
(13-b)\wedge(18) &: \quad w = us^k us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^d utuu^{-1},\\
(13-c)\wedge(14) &: \quad w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d utsut^h ss^{-1},\\
(13-c)^*\wedge(14) &: \quad w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^d utsut^h ss^{-1},\\
(13-c)\wedge(16) &: \quad w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d utsut^h stt^{-1},\\
(13-c)^*\wedge(16) &: \quad w = us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^d utsut^h stt^{-1},\\
(19)\wedge(11-a) &: \quad w = u^{-1}ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu,\\
(19)\wedge(11-b) &: \quad w = u^{-1}ut^k uts^{m'} us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^d utu,\\
(19)\wedge(11-c) &: \quad w = u^{-1}ut^k ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d utsut^h st^p,\\
(19)\wedge(11-c)^* &: \quad w = u^{-1}ut^k ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^d utsut^h st^p,\\
(19)\wedge(12-a) &: \quad w = u^{-1}usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu,\\
(19)\wedge(12-b) &: \quad w = u^{-1}usuts(t^{n'} ut)^c(A_1B_1A_2B_2\cdots A_N)^{\lambda}t^d utu,\\
(19)\wedge(12-c) &: \quad w = u^{-1}usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d utsut^h st^p,\\
(19)\wedge(12-c)^* &: \quad w = u^{-1}usutst^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^d utsut^h st^p,\\
(19)\wedge(13-a) &: \quad w = u^{-1}us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu,\\
(19)\wedge(13-b) &: \quad w = u^{-1}us^k us(B_1A_1B_2A_2\cdots B_NA_N)^{\lambda}t^d utu,\\
(19)\wedge(13-c) &: \quad w = u^{-1}us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d utsut^h st^p,\\
(19)\wedge(13-c)^* &: \quad w = u^{-1}us^k ust^{n'} ut(A_1B_1A_2B_2\cdots A_N)^{\lambda}s^d utsut^h st^p,\\
(14)\wedge(15) &: \quad w = ss^{-1}s, \qquad (15)\wedge(14): w = s^{-1}ss^{-1},\\
(16)\wedge(17) &: \quad w = tt^{-1}t, \qquad (17)\wedge(16): w = t^{-1}tt^{-1},\\
(18)\wedge(19) &: \quad w = uu^{-1}u, \qquad (19)\wedge(18): w = u^{-1}uu^{-1}.
\end{aligned}
$$

These composition are trivial. Let us check one of them as follows.

$$
\begin{aligned}
(19)\wedge(12-a) \quad &: \quad w = u^{-1}usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu,\\
(f,g)_w \quad &= \quad (u^{-1}u-1)sutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu\\
&- \quad u^{-1}(usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu - sutsutsut^{n'}(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}sus^d)\\
&= \quad u^{-1}usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu - sutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu\\
&- \quad u^{-1}usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu + u^{-1}sutsutsut^{n'}(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}sus^d\\
&= \quad u^{-1}sutsutsut^{n'}(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}sus^d - sutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu\\
&\equiv \quad uu^{-1}sutsutsut^{n'}(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}sus^d - usutst^{n'} ut(A_1B_1A_2B_2\cdots A_NB_N)^{\lambda}s^d usu\\
&\equiv \quad 0.
\end{aligned}
$$

It remains to check including compositions of relations $(1)-(19)$. But it is seen that there are no any compositions of this type. Hence the result follows. □

Now let R be the set of relations $(1)-(19)$ and $C(u)$ be a normal form of a word $u \in G_{24}$. By using the Composition-Diamond Lemma 2.1 and Theorem 3.2 , the normal form for the braid group associated with the complex reflection group G_{24} can be given as follows:

Corollary 3.3. *$C(u)$ has a form*

$$Wt^{\alpha_1}u^{\alpha_2}t^{\alpha_3}W's^{\varepsilon_1}u^{\varepsilon_2}s^{\varepsilon_3}W'',$$

where W, W', W'' are R- irreducible words and $0 \le \alpha_i, \varepsilon_i \le n$, $1 \le i \le 3$.

By considering Corollary 3.3, we have the following other consequence of our main result.

Corollary 3.4. *The word problem for the braid group associated with the complex reflection group G_{24} is solvable.*

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