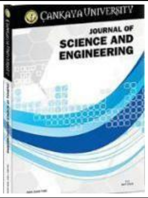
e-ISSN: 2564-7954 CUJSE 17(2): 80-95 (2020) Research Article



Çankaya University

Journal of Science and Engineering

https://dergipark.org.tr/cankujse

CUJSE

Optimized Three-step Hybrid Block Method for Stiff Problems in Ordinary Differential Equations

Muideen O. Ogunniran[1*], Yahaya Haruna[2], Raphael B. Adeniyi[3], Morufu O. Olayiwola[1]

[1] Department of Mathematical Sciences, Osun State University, Nigeria
[2] Department of Mathematics, Saadatu Rimi College of Education Kumbotso, Kano State, Nigeria,
[3] Department of Mathematics, University of Ilorin, Kwara State, Nigeria

Keywords

Intra-step Points
Optimized Hybrid Methods
Singularity
Stiff Problems

Abstract

This paper presents the construction and implementation of a three-step optimized hybrid method for solving stiff system of first order initial value problems of ordinary differential equations. The method contains six implicit formulas which were obtained from a continuous approximation, using shifted chebyshev polynomial as the basis function, via evaluations at six different points on the selected three-step including three optimized intra-step points. The method is consistent, zero-stable and convergent. Numerical experiments are included to show the competitive and superior strength of the proposed method for solving these kinds of problems over similar properties of methods in literature.

1. Introduction

Many applied sciences problems can be formulated into ordinary differential equations. The first order ordinary differential equation of the form appears frequently in applied sciences.

$$y'(t) = f(t, y); y(t_0) = y_0 \tag{1}$$

Solving problems of equation (1) can be done seamlessly by the conventional methods of Euler, its various modifications, Ruge-kutta methods, multi-step methods and recently developed block methods. However, the accuracy of these methods and their rate of convergence recently become points of concerns for researchers especially problems with special properties such as discontinuity and stiffness. Where the conditions for existence and uniqueness of equation (1) are assumed satisfied, our aim is to solve the initial problem of the form (1) and its stiff related problems on a given interval $[t_0, t_3]$ using an Optimized Hybrid Block Method (OHBM). Milne [1] was the researcher to first introduce block methods. Block methods were developed to eradicate prediction, a major drawback, of starting values of predictor-corrector approach. They have been proven to be more efficient in terms of cost implementation, computation time, convergence rate and accuracy [2]. Akinfenwa, Jator and Yao [3] developed a continuous clock Backward Differentiation Formula (BEF) that was effectively used for solving stiff ordinary differential equations. Many more block numerical methods were found in the work of Musa, Suleiman, & Senu [4]: Fully Implicit 3-point Block Extended Backward Differentiation Formula for Stiff Initial Value Problems; Musa, Suleiman, Ismail, Senu & Ibrahim [5]: An Improved 2-point Block Backward

* Corresponding Author: muideen.ogunniran@uniosun.edu.ng
Received: May 7, 2020, Accepted: September 20, 2020

Differentiation Formula for Solving Stiff Initial Value Problems. Sunday, Odekunle, James, & Adesanya [6]: Numerical Solution of Stiff and Oscillatory Differential Equations Using a Block Integrator; Ramos, Kalogiratou, Monovasilis & Simos [7]: A Trigonometrically Fitted Optimized Two-step Hybrid Block Method for Solving Initial Value Problems of the form $y'' = f(x, y, y')$ with Oscillatory Solutions; Ramos [2]: An Optimized Two-step Hybrid Block Method for Solving First-order Initial-value Problems in Ordinary Differential Equations; Ramos & Popescu [8]: How Many k-step Linear Block Methods Exist and Which of them is the Most Efficient One? and Singh, Garg, Kanwar & Ramos [9]: An Efficient Optimized Adaptive Step-size Hybrid Block Method for Integrating Differential System. The stability and possibility of some Runge-kutta methods were shown in solving the stiff system of equations arising from the decomposition of singular Lane-Emden equations as found in Ogunniran, Tayo, Haruna and Adebisi [10].

2. Methods

In this session, we describe the development of a continuous implicit three-step hybrid block method for the solution of Initial Value Problem (IVP) as defined in equation (1) assuming the existence of stiffness in the physical and behaviourial state of the problem. The method is basically on the approximation of the exact solution of (1) at points on the following divisions of intervals a to b of the solutions with fixed step length by a linear combination of the basis functions $\{T_0^*(t), T_1^*(t), T_2^*(t), T_3^*(t), T_4^*(t), T_5^*(t), T_6^*(t), T_7^*(t)\}$ where $T_m^*(t)$ is a family of shifted Chebyshev polynomial of order m.

The shifted chebyshev polynomials of the first kind are orthogonal on the support interval $[0,1]$ with weight function:

$$w(t) = \frac{1}{\sqrt{t - t^2}}$$

and normalized by the requirement that $T_m^*(1) = 1$.

$T_m^*(1) = 1$ satisfies the three-term recurrence relation:

$$T_{m+1}^*(t) = 2(2t - 1)T_m^*(t) - T_{m-1}^*(t), \text{for} m \geq 1.$$

with starting values

$$T_0^*(t) = 1, T_1^*(t) = 2t - 1.$$

2.1. Formulation of the Method

We consider the approximation, $y(t)$ of (1) by a polynomial $u(t)$. The polynomial represents an approach to obtaining the iterative method which is given by an implicit set of equations. We therefore consider the points $t_i, t_{i+1}, t_{i+2}, t_{i+3}$ with the step length $h = t_i - t_{i-1}$ and three intra-step points $t_{r_1} = t_i + r_1 h$, $t_{r_2} = t_i + r_2 h$ and $t_{r_3} = t_i + r_3 h$ with $0 < r_1 < 1$, $1 < r_2 < 2$ and $2 < r_3 < 3$.

For solution of (1), we assume the solution $y(t)$ is approximated by $u(t)$ in the form:

$$y(t) \approx u(t) = \sum_{m=0}^{7} a_m T_m^*(t) \tag{2}$$

where $a_m, m = 1(1)7$ are real unknown parameter to be determined.

Differentiating (2), we have:

$$y'(t) \approx u'(t) = \sum_{m=0}^{7} a_m T^{*\prime}{}_m(t) \tag{3}$$

The evaluations produce a system of 8 algebraic equations in 8 unknown which is given matrix form as below:

$$\begin{pmatrix} T_0^*(t) & T_1^*(t) & T_2^*(t) & T_3^*(t) & T_4^*(t) & T_5^*(t) & T_6^*(t) & T_7^*(t) \\ T^{*\prime}{}_0(t) & T^{*\prime}{}_1(t) & T^{*\prime}{}_2(t) & T^{*\prime}{}_3(t) & T^{*\prime}{}_4(t) & T^{*\prime}{}_5(t) & T^{*\prime}{}_6(t) & T^{*\prime}{}_7(t) \\ T^{*\prime}{}_0(t) & T^{*\prime}{}_1(t) & T^{*\prime}{}_2(t) & T^{*\prime}{}_3(t) & T^{*\prime}{}_4(t) & T^{*\prime}{}_5(t) & T^{*\prime}{}_6(t) & T^{*\prime}{}_7(t) \\ T^{*\prime}{}_0(t) & T^{*\prime}{}_1(t) & T^{*\prime}{}_2(t) & T^{*\prime}{}_3(t) & T^{*\prime}{}_4(t) & T^{*\prime}{}_5(t) & T^{*\prime}{}_6(t) & T^{*\prime}{}_7(t) \\ T^{*\prime}{}_0(t) & T^{*\prime}{}_1(t) & T^{*\prime}{}_2(t) & T^{*\prime}{}_3(t) & T^{*\prime}{}_4(t) & T^{*\prime}{}_5(t) & T^{*\prime}{}_6(t) & T^{*\prime}{}_7(t) \\ T^{*\prime}{}_0(t) & T^{*\prime}{}_1(t) & T^{*\prime}{}_2(t) & T^{*\prime}{}_3(t) & T^{*\prime}{}_4(t) & T^{*\prime}{}_5(t) & T^{*\prime}{}_6(t) & T^{*\prime}{}_7(t) \\ T^{*\prime}{}_0(t) & T^{*\prime}{}_1(t) & T^{*\prime}{}_2(t) & T^{*\prime}{}_3(t) & T^{*\prime}{}_4(t) & T^{*\prime}{}_5(t) & T^{*\prime}{}_6(t) & T^{*\prime}{}_7(t) \\ T^{*\prime}{}_0(t) & T^{*\prime}{}_1(t) & T^{*\prime}{}_2(t) & T^{*\prime}{}_3(t) & T^{*\prime}{}_4(t) & T^{*\prime}{}_5(t) & T^{*\prime}{}_6(t) & T^{*\prime}{}_7(t) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_i \\ f_i \\ f_{i+r_1} \\ f_{i+1} \\ f_{i+r_2} \\ f_{i+2} \\ f_{i+r_3} \\ f_{i+3} \end{pmatrix} \tag{4}$$

To obtain the real coefficients of (2), we impose the following conditions in the evaluations of (2) and (3),

$$\left.\begin{array}{l} y_i = u(t_i);\, i = 0; \\ f_{i+j} = u'(t_{i+j});\, j = 0, r_1, 1, r_2, 2, r_3, 3. \end{array}\right\} \tag{5}$$

which yield the system of 8 equations in 8 unknowns and was solved to have the results of these coefficients of the polynomial $u(x)$ in terms of $y_i, f_i, f_{i+r_1}, f_{i+1}, f_{i+r_2}, f_{i+2}, f_{i+r_3}, f_{i+3}$. After due substitution in (2), the resulting scheme is obtained.

To obtain the appropriate values of r_1, r_2 and r_3, we optimized the local truncation errors of the formulae y_{i+1}, y_{i+2}, and y_{i+3}. In what follows, we have:

$$\left.\begin{array}{l} \mathcal{L}(y(t_{i+1}); h) = \Big(\frac{19\, r_1{}^2 r_2 r_3}{1209600} + \frac{19\, r_2{}^2 r_3 r_1 r}{1209600} + \frac{19\, r_2 r_3{}^2 r_1}{1209600} - \frac{17\, r_1{}^2 r_2}{2419200} \\ - \frac{17\, r_1{}^2 r_3}{2419200} - \frac{17\, r_1 r_3{}^2}{2419200} + \frac{97\, r_1 r_2 r_3}{1209600} - \frac{17\, r_1 r_3{}^2}{2419200} - \frac{17\, r_2{}^2 r_3}{2419200} \\ - \frac{17\, r_2 r_3{}^2}{2419200} + \frac{11\, r_1{}^2}{2822400} - \frac{3\, r_1 r_2}{78400} - \frac{3\, r_1 r_3}{78400} + \frac{11\, r_2{}^2}{2822400} - \frac{3\, r_2 r_3}{78400} + \frac{11\, r_3{}^2}{2822400} + \frac{11\, r_1}{470400} + \frac{11\, r_2}{470400} \\ + \frac{11\, r_3}{470400} - \frac{13}{793800}\Big) y^{(9)} h^9 + \Big(\frac{19\, r_1 r_2 r_3}{151200} - \frac{17\, r_1 r_2}{302400} - \frac{17\, r_1 r_3}{302400} - \frac{17\, r_2 r_3}{302400} + \frac{11\, r_1}{352800} + \frac{11\, r_2}{352800} + \frac{11\, r_3}{352800} \\ - \frac{83}{4233600}\Big) y^{(8)} h^8 \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{l} \mathcal{L}(y(t_{i+2}); h) = \Big(\frac{r_1{}^2 r_2 r_3}{151200} + \frac{r_2{}^2 r_3 r_1}{151200} + \frac{r_2 r_3{}^2 r_1}{151200} + \frac{r_1{}^2 r_3}{151200} + \frac{r_1{}^2 r_3}{151200} + \frac{r_1 r_2{}^2}{151200} + \frac{r_1 r_2 r_3}{18900} + \frac{r_1 r_3{}^2}{151200} + \frac{r_2{}^2 r_3}{151200} + \\ \frac{r_2 r_3{}^2}{151200} - \frac{r_1{}^2}{58800} + \frac{r_1 r_2}{44100} + \frac{r_1 r_3}{44100} - \frac{r_2{}^2}{58800} + \frac{r_2 r_3}{44100} - \frac{r_3{}^2}{58800} - \frac{r_1}{9800} - \frac{r_2}{9800} - \\ \frac{r_3}{9800} + \frac{23}{99225}\Big) y^{(9)} h^9 + \Big(\frac{r_1 r_2 r_3}{18900} + \frac{r_1 r_2}{18900} + \frac{r_1 r_3}{18900} + \frac{r_2 r_3}{18900} - \frac{r_1}{7350} - \frac{r_2}{7350} - \frac{r_3}{7350} + \frac{8}{33075}\Big) y^{(8)} h^8 \end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l} \mathcal{L}(y(t_{i+3}); h) = \Big(\frac{r_1{}^2 r_2 r_3}{44800} + \frac{r_2{}^2 r_3 r_1}{44800} + \frac{r_2 r_3{}^2 r_1}{44800} - \frac{3\, r_1{}^2 r_3}{89600} - \frac{3\, r_1{}^2 r_3}{89600} - \frac{3\, r_1 r_2{}^2}{89600} + \frac{3\, r_1 r_2 r_3}{44800} - \frac{3\, r_1 r_3{}^2}{89600} - \frac{3\, r_2{}^2 r_3}{89600} - \\ \frac{3\, r_2 r_3{}^2}{89600} + \frac{27\, r_1{}^2}{313600} - \frac{9\, r_1 r_2}{78400} - \frac{9\, r_1 r_3}{78400} + \frac{27\, r_2{}^2}{313600} - \frac{9\, r_2 r_3}{78400} + \frac{27\, r_3{}^2}{313600} + \frac{81\, r_1}{156800} + \frac{81\, r_2}{156800} + \\ \frac{81\, r_3}{156800} - \frac{81}{39200}\Big) y^{(9)} h^9 + \Big(\frac{r_1 r_2 r_3}{5600} - \frac{3\, r_1 r_2}{11200} - \frac{3\, r_1 r_3}{11200} - \frac{3\, r_2 r_3}{11200} + \frac{27\, r_1}{39200} + \frac{27\, r_2}{39200} + \frac{27\, r_3}{39200} - \frac{297}{156800}\Big) y^{(8)} h^8 \end{array}\right\} \tag{8}$$

Equations (6) - (8) are forced to be of order 8, as such we equate the coefficients of $y^{(8)} h^8$ in equations to zero thus producing a system of 3 algebraic equations as follows:

$$\frac{19\, r_1 r_2 r_3}{151200} - \frac{17\, r_1 r_2}{302400} - \frac{17\, r_1 r_3}{302400} - \frac{17\, r_2 r_3}{302400} + \frac{11\, r_1}{352800} + \frac{11\, r_2}{352800} + \frac{11\, r_3}{352800} - \frac{83}{4233600} = 0 \tag{9}$$

$$\frac{r_1 r_2 r_3}{18900} + \frac{r_1 r_2}{18900} + \frac{r_1 r_3}{18900} + \frac{r_2 r_3}{18900} - \frac{r_1}{7350} - \frac{r_2}{7350} - \frac{r_3}{7350} + \frac{8}{33075} = 0 \tag{10}$$

$$\frac{r_1 r_2 r_3}{5600} - \frac{3\, r_1 r_2}{11200} - \frac{3\, r_1 r_3}{11200} - \frac{3\, r_2 r_3}{11200} + \frac{27\, r_1}{39200} + \frac{27\, r_2}{39200} + \frac{27\, r_3}{39200} - \frac{297}{156800} = 0 \tag{11}$$

Solving these equations results into a symmetric plane curve with respect to the diagonal $r_1 = r_2 = r_3$ which are then solved to obtain a unique solution with the constraints $r_1 = r_2 = r_3$ and producing a unique solution with constraints $0 < r_1 < 1, 1 < r_2 < 2, 2 < r_3 < 3$, thereby producing the optimized intra-step points as follows:

$$\left.\begin{aligned} r_1 &= \frac{3}{2} - \frac{1}{2}\sqrt{5} \approx 0.381966012 \\ r_2 &= \frac{3}{2} \approx 1.5 \\ r_3 &= \frac{3}{2} + \frac{1}{2}\sqrt{5} \approx 2.618033988 \end{aligned}\right\} \tag{12}$$

After substituting the values obtained for r_1, r_2 and r_3, we have the intended scheme as below:

$$N = A + hBf \tag{13}$$

where;

$$N = \begin{pmatrix} 3780(3+\sqrt{5})y_{i+\frac{3}{2}-\frac{1}{2}\sqrt{5}} \\ y_{i+1} \\ y_{i+\frac{3}{2}} \\ y_{i+2} \\ 3780(\sqrt{5}-3)y_{i+\frac{3}{2}+\frac{1}{2}\sqrt{5}} \\ y_{i+3} \end{pmatrix}$$

$$A = \begin{pmatrix} (11340 + 3780\sqrt{5})y_i \\ y_i \\ y_i \\ y_i \\ (3780\sqrt{5} - 11340)y_i \\ y_i \end{pmatrix}$$

$$B = \begin{pmatrix} (516\sqrt{5}+1498) & 3528 + 1242\sqrt{5} & -567 - 1089\sqrt{5} & 1792 + 384\sqrt{5} & 378 - 774\sqrt{5} & 1008 - 270\sqrt{5} & -77 - 9\sqrt{5} \\ \frac{106}{945} & \frac{41}{140} + \frac{2}{15}\sqrt{5} & \frac{151}{420} & -\frac{16}{189} & \frac{11}{420} & \frac{41}{140} - \frac{2}{15}\sqrt{5} & \frac{1}{945} \\ \frac{1037}{8960} & \frac{81}{280} - \frac{81}{640}\sqrt{5} & \frac{5427}{8960} & \frac{8}{35} & -\frac{243}{8960} & \frac{81}{280} - \frac{81}{640}\sqrt{5} & -\frac{13}{8960} \\ \frac{107}{945} & \frac{2}{7} - \frac{2}{15}\sqrt{5} & \frac{58}{105} & \frac{512}{945} & \frac{23}{105} & \frac{2}{7} - \frac{2}{15}\sqrt{5} & \frac{2}{945} \\ 516\sqrt{5} - 1498 & -1008 - 275\sqrt{5} & 567 - 1089\sqrt{5} & -1792 + 384\sqrt{5} & -378 - 774\sqrt{5} & -3528 + 1242\sqrt{5} & 77 - 9\sqrt{5} \\ \frac{4}{35} & \frac{81}{140} & \frac{81}{140} & \frac{16}{35} & \frac{81}{140} & \frac{81}{140} & \frac{4}{35} \end{pmatrix}$$

$$f = \begin{pmatrix} f_i \\ f_{i+\frac{3}{2}-\frac{1}{2}\sqrt{5}} \\ f_{i+1} \\ f_{i+\frac{3}{2}} \\ f_{i+2} \\ f_{i+\frac{3}{2}+\frac{1}{2}\sqrt{5}} \\ f_{i+3} \end{pmatrix}$$

3. Specification of the Method

This section contains the discussion of some main characteristics of the proposed three-step optimized hybrid block method. Rigorous analysis was carried out to obtain the order, consistency, zero-stability, convergence and linear stability of the method.

3.1. Order and Zero-stability

Order and Consistency of the Method
Due to Lambert [6], a numerical scheme is said to be of order $p = k$ if in the difference equation, $\mathcal{L}\{y(x); h\}$; $c_\circ = c_1 = c_2 = \ldots = c_k = 0$ and $c_{k+1} \neq 0$, and the Error constant = c_{k+1}.

Expanding (13) in Taylor series and collecting related terms, we have corresponding formula and it order as given in the table below:

Table 1: Table of Order for Formulae in Method (13)

S/N	Formula	Order
1	$y_{i+\frac{3}{2}-\frac{1}{2}\sqrt{5}}$	8
2	y_{i+1}	8
3	$y_{i+\frac{3}{2}}$	8
4	y_{i+2}	8
5	$y_{i+\frac{3}{2}+\frac{1}{2}\sqrt{5}}$	8
6	y_{i+3}	8

Table 1 above shows the error of the block method is 8 which implies the proposed method is consistent because the necessary and sufficient condition for consistence of a method is that it is of at least order 1.

3.2. Zero-stability

This is a property concerning the method when limiting h to zero. Thus as h tends to zero in (13), we have the following system of equations:

$$\left.\begin{array}{l} y_{i+\frac{3}{2}-\frac{1}{2}\sqrt{5}} = y_i \\ y_{i+1} = y_i \\ y_{i+\frac{3}{2}} = y_i \\ y_{i+2} = y_i \\ y_{i+\frac{3}{2}+\frac{1}{2}\sqrt{5}} = y_i \\ y_{i+3} = y_i \end{array}\right\} \quad (14)$$

which can be written in matrix form

$$IY_i - B_0 Y_{i-1} = 0 \quad (15)$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, Y_i = \begin{pmatrix} y_{i+\frac{3}{2}-\frac{1}{2}\sqrt{5}} = y_i \\ y_{i+1} = y_i \\ y_{i+\frac{3}{2}} = y_i \\ y_{i+2} = y_i \\ y_{i+\frac{3}{2}+\frac{1}{2}\sqrt{5}} = y_i \\ y_{i+3} = y_i \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, Y_{i-1} = \begin{pmatrix} y_i \\ y_i \\ y_i \\ y_i \\ y_i \\ y_i \end{pmatrix}$$

According to Lambert [11], a block method is zero-stable if the roots r_k of the first characteristic polynomial $\xi(r) = det|Ir - B_0|$ does not exceed 1 i.e. $|r_k| \leq 1$. The first characteristic polynomial of method (13) is given by

$$r^5(r-1) = 0 \tag{16}$$

The roots of (16) are $r = 0,0,0,0,0,1$ which none of it does not exceed 1, thus Method (13) is zero-stable.

3.3. Convergence

According to Henrici [12], we establish the convergence of the three-step optimized hybrid block method since consistence and zero-stability are necessary and sufficient features for convergence.

3.4. Linear Stability

This is a behaviourial property related to $h > 0$. As in most literature, the linear stability will be analyzed using the Dalquist's test

$$y'(t) = \gamma y(t), \Re(\gamma) < 0 \tag{17}$$

Applying (13) on (17), we have obtained a recurrence equation

$$Y_i = M(z)Y_{i-1} \tag{18}$$

where $M(z)$ is the stability matrix given by

$$M(z) = \frac{B_0 + zC_0}{B_1 - zC_1} \tag{19}$$

where

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

$$C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & (516\sqrt{5}+1498) \\ 0 & 0 & 0 & 0 & 0 & \frac{106}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{1037}{8960} \\ 0 & 0 & 0 & 0 & 0 & \frac{107}{945} \\ 0 & 0 & 0 & 0 & 0 & 516\sqrt{5}-1498 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{35} \end{pmatrix} \tag{22}$$

$$C_1 = \begin{pmatrix} 3528+1242\sqrt{5} & -567-1089\sqrt{5} & 1792+384\sqrt{5} & 378-774\sqrt{5} & 1008-270\sqrt{5} & -77-9\sqrt{5} \\ \frac{41}{140}+\frac{2}{15}\sqrt{5} & \frac{151}{420} & -\frac{16}{189} & \frac{11}{420} & \frac{41}{140}-\frac{2}{15}\sqrt{5} & \frac{1}{945} \\ \frac{81}{280}-\frac{81}{640\sqrt{5}} & \frac{5427}{8960} & \frac{8}{35} & -\frac{243}{8960} & \frac{81}{280}-\frac{81}{640}\sqrt{5} & -\frac{13}{8960} \\ \frac{2}{7}-\frac{2}{15}\sqrt{5} & \frac{58}{105} & \frac{512}{945} & \frac{23}{105} & \frac{2}{7}-\frac{2}{15}\sqrt{5} & \frac{2}{945} \\ -1008-275\sqrt{5} & 567-1089\sqrt{5} & -1792+384\sqrt{5} & -378-774\sqrt{5} & -3528+1242\sqrt{5} & 77-9\sqrt{5} \\ \frac{81}{140} & \frac{81}{140} & \frac{16}{35} & \frac{81}{140} & \frac{81}{140} & \frac{4}{35} \end{pmatrix} \tag{23}$$

and $z = h\gamma$. The eigenvalues of $M(z)$ was obtained as $\xi_0, \xi_0, \xi_0, \xi_0, \xi_0, \xi_k$. The only leading eigenvalue

$$\xi_k = -1/2\,\frac{\begin{array}{l} 383095755\,\sqrt{5}z^6 + 3683945187\,\sqrt{5}z^5 + 2037188145\,z^6 \\ +10983913668\,\sqrt{5}z^4 + 25951773525\,z^5 - 20484255800\,\sqrt{5}z^3 \\ +147227776480\,z^4 - 182103644640\,\sqrt{5}z^2 + 453569705100\,z^3 \\ -96494328000\,z\sqrt{5} + 849828086600\,z^2 - 64329552000\,\sqrt{5} + 1630877976000\,z + 1087251984000 \end{array}}{\begin{array}{l} 1317413079\,\sqrt{5}z^5 - 1079991735\,z^6 - 16482485901\,\sqrt{5}z^4 \\ +8698126980\,z^5 + 49001940865\,\sqrt{5}z^3 - 27029731235\,z^4 \\ -55445935170\,\sqrt{5}z^2 + 112158342075\,z^3 - 48247164000\,z\sqrt{5} \\ -381063854950\,z^2 + 32164776000\,\sqrt{5} + 815438988000\,z - 543625992000 \end{array}} \tag{24}$$

and can further be simplified as

$$\xi_k = -1/2\,\frac{\begin{array}{l} 2893816295.0\,z^6 + 34189325380.0\,z^5 + 171788554100.0\,z^4 + 407765516700.0\,z^3 \\ +442631958400.0\,z^2 + 1415110099000.0\,z + 943406732800.0 \end{array}}{\begin{array}{l} -1079991735\,z^6 + 11643952180.0\,z^5 - 63885690140.0\,z^4 + 221730012900.0\,z^3 \\ -505044735100.0\,z^2 + 707555049600.0\,z - 471703366400.0 \end{array}} \tag{25}$$

is a rational function which was plotted on a contour to obtain the region of absolute stability of the method as displayed below.

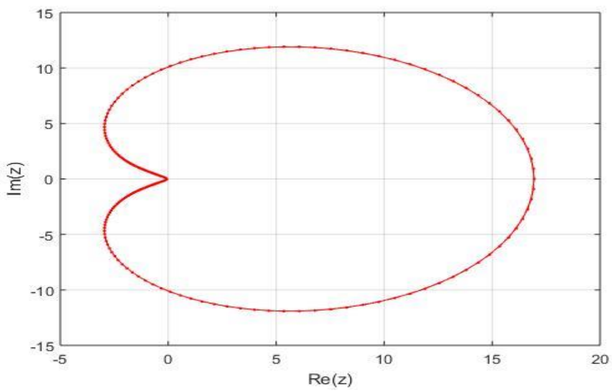


Figure 1. Region of Absolute Stability of the Method

4. Numerical Experiment

Test 5.1:
The Hamiltonian problem in time dependent optimal problem is given as:

$$\left.\begin{array}{l} \min\limits_{u} \int_0^1 u^2(t)dt \\ \text{suchthat} \\ y' = y + u, y(0) = 0, y'(0) = 1. \end{array}\right\} \tag{26}$$

As a constraint that leads to the following initial value problem using Hamiltonian conditions in the conventional way where:

$$\left.\begin{array}{l} H = f + \lambda g \\ f = u^2, g = y + u \\ H_u = 0, H_y = -\lambda' \text{and} H_\lambda = y' \end{array}\right\} \tag{27}$$

and the Hamiltonian conditions lead to

$$y''(t) - y(t) = 0y(0) = 0, y'(0) = 1.$$

The last equation is then reduced to:

$$\left.\begin{array}{l} y'_1 = y_2 \\ y'_2 = -y_1 \\ y_1(0) = 0, y_2(0) = 1, x \in [0,10]. \\ y_1(x) = sinx, y_2(x) = cosx \end{array}\right\} \tag{28}$$

See Table 2 for numerical results.

Table 2: Absolute Errors for Test Problem 5.1 Using Method (13), $\boldsymbol{h = 0.01}$

$\boldsymbol{t}$	$\boldsymbol{y_1(t)}$ **Component**	$\boldsymbol{y_2(t)}$ **Component**
0.0	0.0000	0.0000
0.1	1.0000×10^{-14}	1.0000×10^{-14}
0.2	1.0000×10^{-14}	1.0000×10^{-14}
0.3	1.0000×10^{-14}	1.0000×10^{-14}
0.4	1.0000×10^{-14}	1.0000×10^{-14}
0.5	1.0000×10^{-14}	1.0000×10^{-14}
0.6	1.0000×10^{-14}	1.0000×10^{-14}
0.7	1.0000×10^{-14}	1.0000×10^{-14}
0.8	1.0000×10^{-14}	1.0000×10^{-14}
0.9	1.0000×10^{-14}	1.0000×10^{-14}
1.0	1.0000×10^{-14}	1.0000×10^{-14}

Exact Error=$|y(t) - y(t_i)|$, where $y(t)$ is the exact solution and $y(t_i)$ is results from numerical methods. Computation time = $0.125s$ and Tolerance for convergence = 10^{-14}.

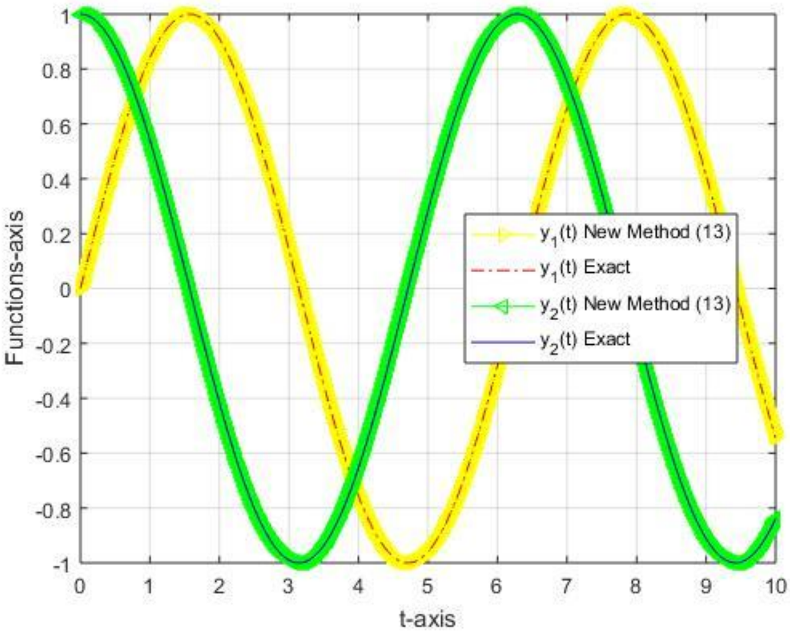


Figure 2. Solution Graph in Comparison with the Exact Solution for Test Problem 5.1

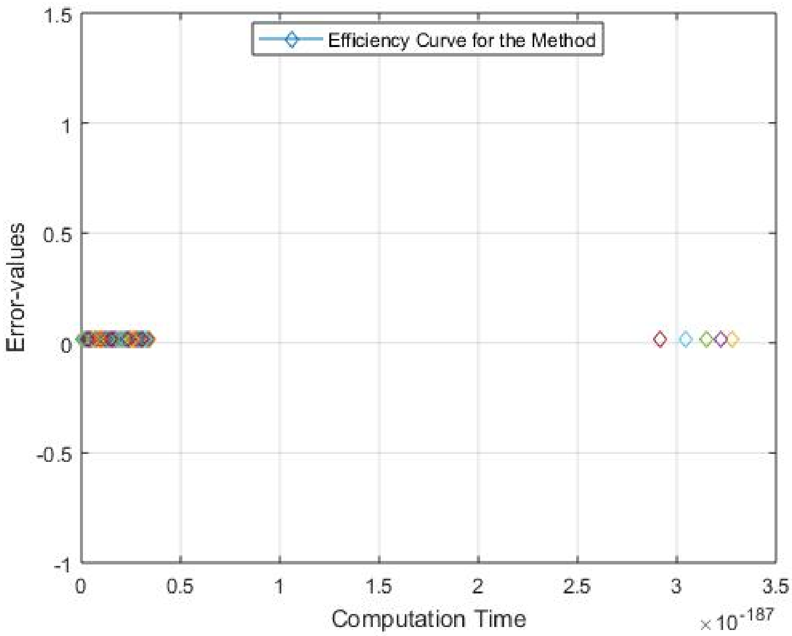


Figure 3. Efficiency Curve of the Method for Test Problem 5.1

Test 5.2:

Ramos [8]: Consider the following linear system given by:

$$\left.\begin{aligned} y'_1(t) &= -21y_1 + 19y_2 - 20y_3, y_1(0) = 1, \\ y'_2(t) &= 19y_1 - 21y_2 + 20y_3, y_2(0) = 0, \\ y'_3(t) &= 40y_1 - 40y_2 - 40y_3, y_3(0) = -1. \end{aligned}\right\} \quad (29)$$

whose exact solution is:

$$\left.\begin{aligned} y_1(t) &= \frac{1}{2}\left(\exp^{-2t} + \exp^{-40t}(cos(40t) + sin(40t))\right), \\ y_2(t) &= \frac{1}{2}\left(\exp^{-2t} - \exp^{-40t}(cos(40t) + sin(40t))\right), \\ y_3(t) &= \frac{1}{2}\exp^{-40t}(cos(40t) + sin(40t)). \end{aligned}\right\} \quad (30)$$

Discussion:

As appear in Ramos [2] among others, we compared the results of our method (OHBM) along side BDF $_6$, a classical BDF method of order six, a continuous six-step BDF method (BHM) both the work of Akinfenwa *et. al.* [3] and Ramos [2]. Different step sizes were considered and the maximum relative errors were compared over the three components $y_1(t), y_2(t)$ and $y_3(t)$. The optimized three-step method presented in this paper has proven to be superior in terms of accuracy and a negligible computation time was used for its execution.

See Table 3 for Numerical Results and Comparison.

Table 3: Numerical Comparison of Maximum Relative Error for Test Problem 5.2 Using Different Methods

Steps	BDF $_6$	Akinfenwa et. al. [3]	Ramos [2]	OHBM
20	2.000×10^{-1}	4.700×10^{-2}	8.360×10^{-3}	2.943×10^{-5}
40	2.600×10^{-1}	2.100×10^{-3}	4.009×10^{-4}	1.250×10^{-6}
80	2.600×10^{-3}	1.400×10^{-4}	6.785×10^{-6}	2.721×10^{-13}
160	9.100×10^{-5}	7.500×10^{-6}	1.156×10^{-7}	4.262×10^{-15}
320	1.800×10^{-6}	1.700×10^{-7}	1.853×10^{-9}	7.707×10^{-18}
640	3.300×10^{-8}	3.000×10^{-9}	2.901×10^{-11}	3.000×10^{-21}
Computation Time	NA	NA	NA	$0.03125s$

Relative Error, MaxRerr=$\max_i \frac{|y(t)-y(t_i)|}{y(t)}$, where $y(t)$ is the exact solution and $y(t_i)$ is results from numerical methods. NA mean NOT APPLICABLE as the author did not consider the execution times.

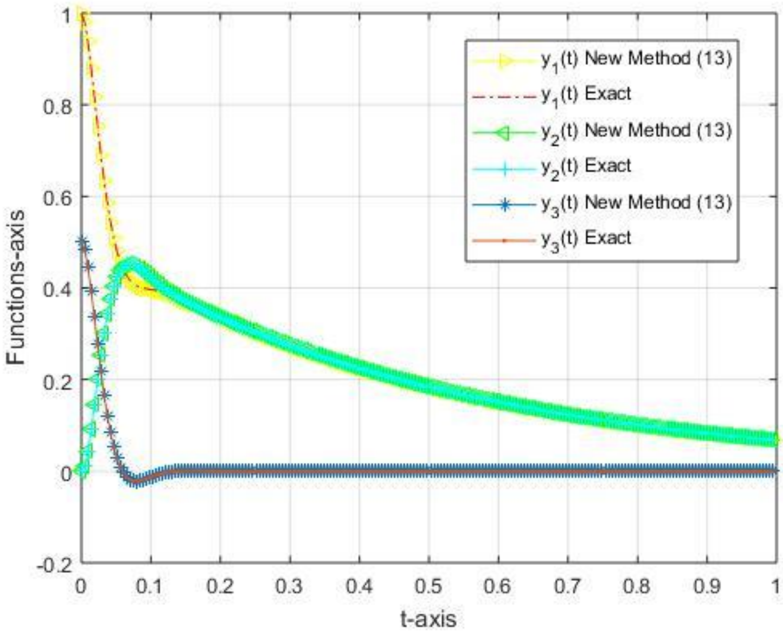


Figure 4. Solution Graph in Comparison with the Exact Solution for Test Problem 5.2

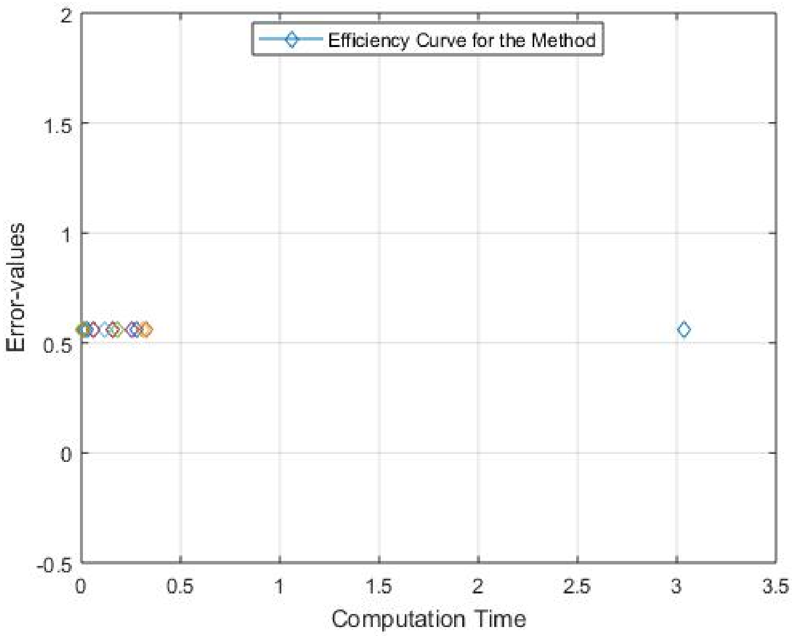


Figure 5. Efficiency Curve of the Method for Test Problem 5.2

Test 5.3:
Singh *et. al.* [9]; Van der Pol System:

$$y'_1 = y_2, y'_2 = \frac{y_2(1-{y'_1}^2)-y_1}{\mu};$$
$$y_1(0) = 2, y_2(0) = -\frac{2}{3} + \frac{10}{81}\mu - \frac{292}{2187}\mu^2 - \frac{1814}{19683}\mu^3; \mu = 10^{-1}. \tag{31}$$

Discussion:
This problem is approximated over the interval $[0,0.55139]$ for $\mathrm{h} = 10^{-3}$ and 10^{-4} and comparison was done with RADAU: This code is based on implicit Runge–Kutta methods (Radau-IIa) with variable order (1, 5, 9, 13) and step size control. This code is specifically designed for solving stiff systems, ode15s: A variable-step, variable-order (VSVO) IVP solver. This code is based on the numerical differentiation formulas (NDFs) of orders 1 to 5. This code is a built-in ODE solver in MATLAB which is also specifically designed for solving stiff systems, Singh et. al. [12]: The efficient optimized adaptive step-size hybrid block method and the proposed method (OHBM). The values $\mathrm{y}_1 = 1.5633739442300918$ and $\mathrm{y}_2 = -1.0000208318542727$ as obtained from Singh et. al. [9] was used as a reference values for numerical experiment. The great performance and superior strength claims of the proposed optimized block hybrid method were established and confirmed as in the numerical results presented in Table 4. See Table 4 for numerical results.

Table 4. Numerical Results for Test Problem 5.3 at $\boldsymbol{t = 0.55139}$

$\boldsymbol{h}$	Method	MaxErr	Cmpt. Time (s)
10^{-3}	RADAU	4.6122×10^{-6}	0.014
	ode15s	2.9788×10^{-5}	0.018
	Singh et. al. (2019)	5.0900×10^{-8}	0.011
	OHBM	1.9930×10^{-10}	0.010
10^{-4}	RADAU	6.0016×10^{-7}	0.016
	ode15s	4.8015×10^{-6}	0.021
	Singh et. al. (2019)	2.8070×10^{-9}	0.014
	OHBM	2.0117×10^{-12}	0.014

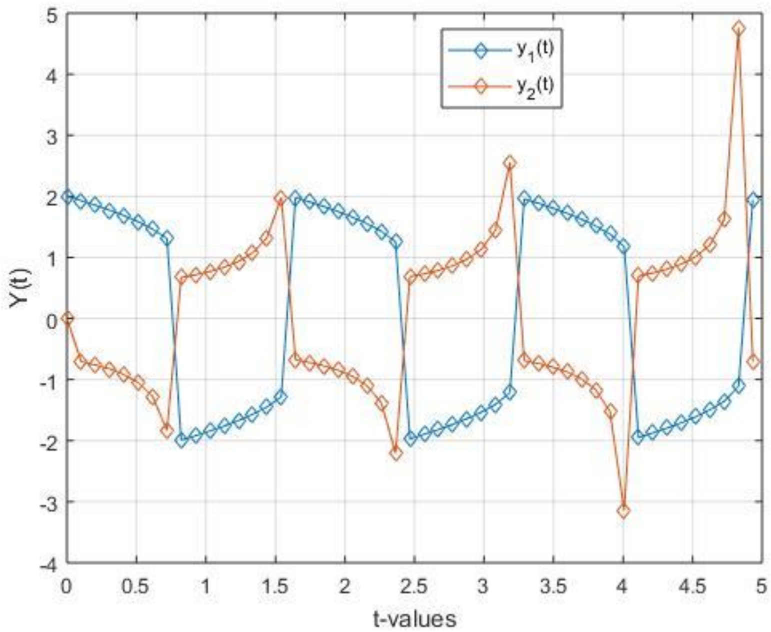


Figure 6. Solution Graph for Test Problem 5.3

Test 5.4:
Ramos [2]: Consider the following problem

$$y'(t) = -10ty; y(0) = 1;$$
$$y(t) = \exp(-5t^2) \quad (32)$$

has appeared in Ibrahim *et. al.* [13], Musa *et. al.* [4], Musa *et. al.* [5] and Ramos [2] on the interval [0,10] using numerous step length. The table of results for this problem shows the maximum absolute errors on the solution interval for different methods as reported by these literature; the 2-point block backward differentiation formula (BBDF) of Ibrahim *et. al.* [13], the 2-point improved block backward differentiation formula (IBBDF) of Musa *et. al.* [5], the 3-point block extended backward differentiation formula (3BEBDF) of Musa *et. al.* [4], the block hybrid method (BHM) of Ramos [2] and the optimized hybrid block method as presented in this paper.

See Table 5 for numerical results.

Table 5. Numerical Comparison of Errors for Test Problem 5.4 at $t = 10$

Step size (h)	**Ibrahim et. al. [13]**	**Musa et. al. [5]**	**Musa et. al. [4]**	**Ramos [2]**	**Exact Error OHBM**
10^{-2}	2.4760×10^{-2}	1.4981×10^{-3}	1.2408×10^{-2}	7.1970×10^{-13}	4.6918×10^{-16}
10^{-3}	2.8661×10^{-2}	1.5115×10^{-5}	7.3642×10^{-4}	7.1986×10^{-19}	9.8193×10^{-21}

The proposed method shows its superiority over existing method of similar properties. Computation time for the proposed method is $0.078s$.

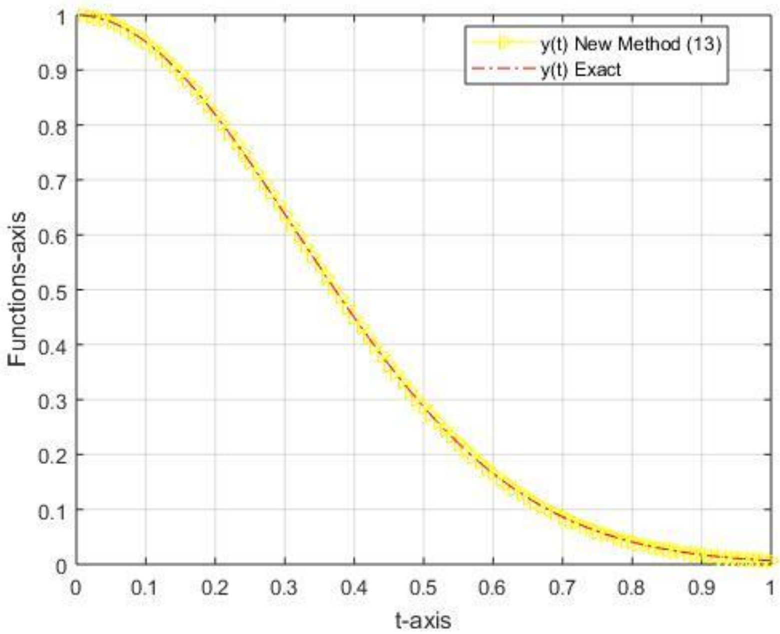


Figure 7. Solution Graph in Comparison with the Exact Solution for Test Problem 5.4

Test 5.5:

Ramos & Popsce [8]: We consider here a stiff parabolic equation with initial and boundary conditions given by:

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}; x \in [0,1], t \in [0,1]$$
$$y(0,t) = y(1,t) = 0; y(x,0) = \sin\pi x + \sin(q\pi x); q > 1 \tag{33}$$

whose exact solution is:

$$yv(x,t) = \exp^{-\pi^2 t}\sin(\pi x) + \exp^{-q^2\pi^2 t}\sin(q\pi x). \tag{34}$$

Taking on the space domain a discrete evenly spaced mesh

$$\gamma: \{x_0 \leq x_1 \cdots \leq x_{i+1} = b\} \tag{35}$$

in such a way the $\forall x_i \in \gamma$,

$$f'(x_i) = \frac{f'(x_{i+1}) - 2f'(x_i) + f'(x_{i-1})}{(\Delta x)^2} \tag{36}$$

setting $y_i(t) = y(x_i, t), fori = 1, \cdots, N$ with $y_0(t) = y(0,t), y_{N+1}(t) = y(1,t) = 0$, the problem may be approximated by the form:

$$\frac{dy}{dt} = By(t) \tag{37}$$

$$y(t_0) = (y_1(t), \cdots, y_n(t))^T \tag{38}$$

and B is a tridiagonal matrix:

$$B = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \tag{39}$$

where $\Delta x = \frac{1}{N+1}$

See Table 6 for Error Comparison of Methods.

Table 6. Maximum Absolute Errors at $\boldsymbol{t = 1}$ for Test Problem 5.5

$\boldsymbol{k}$	**Ramos & Popescu [8]**	**OHBM**
2	1.7×10^{-4}	1.2369×10^{-5}
3	1.1×10^{-4}	1.2368×10^{-5}
5	1.0×10^{-4}	4.4149×10^{-5}
10	1.0×10^{-4}	4.4148×10^{-5}
Computation Time	NA	$0.125s$

For $\Delta x = 0.05$, Maximum Absolute Error=$\max_{0\le i\le N}|ExactSolutionatt = 1 - NumericalResultatt = 1andspacex_i|$.

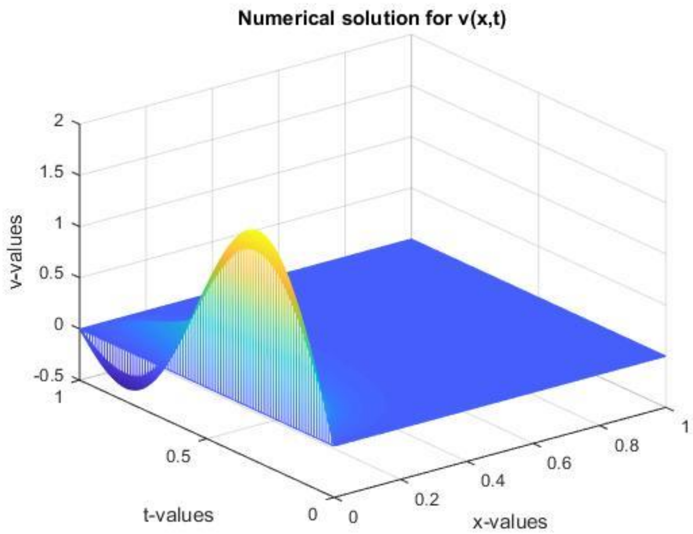


Figure 8. Solution Graph for Test Problem 5.5

5. Discussion of Results and Conclusion

5.1. Computational Details

In the implementation of the derived methods, a system of nonlinear equations must be solved in order to obtain the desired approximation. To solve these nonlinear systems, a Newton-Krylov solver, nsoli.m or a modified Newton solver, nsold.m was used. It is important to point that the numerical methods were programmed via MATLAB 9.2 version on a personal computer with the following specifications:

- System name- HP Pavilion x360 Convertible
- Processor- Intel(R) Core(TM) i3-7100U CPU @ 2.40GHz
- Installed memory (RAM)- 8.00GB
- System Type- 64-bits Operating System, x64-based processor
- Operating system- Windows 10

5.2. Conclusion

An analysis of an optimized three-step hybrid block methods has been extensively carried out. Some numerical experiments have been presented to demonstrate the performance of the method considered. The numerical results of the experiments establish the efficiency of the new method and its superiority over similar characteristics of methods in literature.

Acknowledgements

The authors appreciate the anonymous reviewers for their work and criticism that greatly improve the manuscript.

Declaration of Competing Interest

The authors declare that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Muideen O. Ogunniran: Conceptualization, Methodology, Software, Data curation, Writing-Original draft preparation.
Yahaya Haruna: Visualization, Investigation, Writing-Reviewing and Editing, Software, Validation.
Raphael B. Adeniyi: Investigation and Supervision.
Morufu O. Olayiwola: Investigation and Supervision.

References

[1] W. E. Milne, *Numerical Solution of Differential Equations*, John Wiley and Sons, New York, 1953.
[2] H. Ramos, "An Optimized Two-step Hybrid Block Method for Solving First-order Initial-value Problems in ODEs," *Differential Geometry-Dynamic System*, vol. 19, pp. 107-118, 2017.
[3] O. A. Akinfenwa, S. N. Jator, and N. M. Yao, "Continuous Block Backward Differentiation Formula for Solving Stiff Ordinary Differential Equations," *Computers and Mathematics with Applications* , vol. 65, pp. 996-1005, 2013.
[4] H. Musa, M. B. Suleiman, and N. Senu, Fully "Implicit 3-point Block Extended Backward Differentiation Formula for Stiff Initial Value Problems," *Applied Mathematical Sciences*, vol. 6, pp. 4211-4228, 2012.
[5] H. Musa, M. B. Suleiman, F. Ismail, N. Senu, and Z, B. Ibrahim, "An Improved 2-point Block Backward Differentiation Formula for Solving Stiff Initial Value Problems," *AIP Conference Proceedings*, 2013, no. 1522, 211.
[6] J. Sunday, M. R. Odekunle, A. A. James, and A. O. Adesanya, "Numerical Solution of Stiff and Oscillatory Differential Equations Using a Block Integrator," *British Journal of Mathematics and Computer Science*, vol. 4, no. 17, pp. 2471-2481, 2014.
[7] H. Ramos, Z. Kalogiratou, Th. Monovasilis, and T. E. Simos, "A Trigonometrically Fitted Optimized Two-step Hybrid Block Method for Solving Initial Value Problems of the form $y'' = f(x, y, y')$ with Oscillatory Solutions," *AIP Conference Proceedings*, 2015, no. 1648, 810007.
[8] H. Ramos and P. Popescu, "How Many k-step Linear Block Methods Exist and which of them is Most Efficient and Simplest One?," *Applied Mathematics and Computation*, vol. 36, pp. 296-309, 2018.
[9] G. Singh, A. Garg, V. Kanwar, and H. Ramos, "An Efficient Optimized Adaptive Step-size Hybrid Block Method for Integrating Differential Systems," *Applied Mathematics and Computation*, vol. 362, 124567, 2019.
[10] M. O. Ogunniran, O. A. Tayo, Y. Haruna, and A. F. Adebisi, "Linear Stability Analysis of Runge-Kutta Methods for Singular Lane-Emden Equations," *Journal of Nigerian Society of Physical Sciences*, vol. 2, pp. 134 – 140, 2020.
[11] J. D. Lambert, *Computational Methods in Ordinary Differential Equations*, John Wiley and Sons, New York, 1991.
[12] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley and Sons, New York, 1962.
[13] Z. B. Ibrahim, K. I. Othman, and M. Suleiman, "Implicit r-block Backward Differentiation Formula for Solving First-order Stiff ODEs," *Applied Mathematics Computation*, vol. 186, pp. 558-565, 2007.