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Research Article

Reversibility of skew Hurwitz series rings

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Abstract

We study the reversibility of skew Hurwitz series at zero as a generalization of an α-rigid ring, introducing the concept of skew Hurwitz reversibility. A ring R is called skew Hurwitz reversible (SH-reversible, for short), if the skew Hurwitz series ring (HR, α) is reversible i.e. whenever skew Hurwitz series $f, g \in (HR, \alpha)$ satisfy $fg = 0$, then $gf = 0$. We examine some characterizations and extensions of SH-reversible rings in relation with several ring theoretic properties which have roles in ring theory.

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Keywords. skew Hurwitz series ring, reversible ring, α-rigid ring, matrix ring

1. Introduction

Throughout this paper, R denotes an associative and commutative ring with identity and α denotes a nonzero and non-identity endomorphism, unless otherwise stated.

Rings of formal power series have been of interest and have had important applications in many areas, one of which has been differential algebra. In a series of papers ([13–15]) Keigher demonstrated that the ring HR of Hurwitz series over a commutative ring R with identity has many interesting applications in differential algebra.

The concept of Hurwitz series was extended by Hassanein in [7] to the ring of skew Hurwitz series. The ring (HR, α) of skew Hurwitz series over a ring R is defined as follows: the elements of (HR, α) are functions $f : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of all natural numbers. The operation of addition in (HR, α) is componentwise and the operation of multiplication is defined, for every $f, g \in (HR, \alpha)$, by

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \quad \text{for all } n \in \mathbb{N}$$

where $\binom{n}{k}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}$ with $n \geq k$ by $\frac{n!}{k!(n-k)!}$.

If one identifies a skew formal power series $\sum_{i=0}^{\infty} a_i x^i \in R[[x; \alpha]]$ with the function f such that $f(n) = a_n$, then multiplication in (HR, α) is similar to the usual product of skew formal power series, except that binomial coefficients appear in each term in the product introduced above. In [21, Proposition 2.1], it has been shown that for any ring R that containing the field of rational numbers $\mathbb{Q}$ and α is an $\mathbb{Q}$-algebra homomorphism of

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R, then the rings (HR, α) and $R[[x;\alpha]]$ are isomorphic. To avoid repetitions some of the results known for skew power series ring $R[[x;\alpha]]$, we assume that R is a ring which does not contain the field of rational numbers throughout this paper.

To each $r \in R$ and $n \in \mathbb{N}$, we associate elements $h_r, h'_n \in (HR, \alpha)$ defined by

$$h_r(x) = \begin{cases} r, & x = 0; \\ 0, & x \neq 0 \end{cases}, \qquad h'_n(x) = \begin{cases} 1, & x = n; \\ 0, & x \neq n \end{cases}$$

where for all $x \in \mathbb{N}$. It is clear that $r \mapsto h_r$ is a ring embedding of R into (HR, α) and also (HR, α) is a ring with identity h_1. Let $\mathrm{supp}(f)$ denote the support of $f \in T$, i.e. $\mathrm{supp}(f) = \{i \in \mathbb{N} \mid 0 \neq f(i) \in R\}$ and $\pi(f)$ denote the minimal element in $\mathrm{supp}(f)$ and $\triangle(f)$ denote the maximal element in $\mathrm{supp}(f)$. The ring (hR, α) of skew Hurwitz polynomials over a ring R is a subring of (HR, α) that consist elements of the form $f \in (HR, \alpha)$ such that $\triangle(f) < \infty$.

Recall that a ring R is called *reduced* if it has no nonzero nilpotent elements. In [5], Cohn introduced the notion of a reversible ring as a generalization of commutativity. A ring R is called *reversible*, if whenever $a, b \in R$ satisfy $ab = 0$, then $ba = 0$. Anderson and Camillo [2] used the notation ZC_2 for reversible rings, while Krempa and Niewieczerzal [18] used the term C_0 for it. Cohn showed that the Köthe Conjecture is true for the class of reversible rings.

For a ring R equipped with an endomorphism $\alpha : R \to R$, a *skew polynomial ring* $R[x;\alpha]$ over the coefficient ring R (also called an *Ore extension of endomorphism type*) is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. For any skew polynomial ring $R[x;\alpha]$ of R, we have $\alpha(1) = 1$ since $1.x = x.1 = \alpha(1)x$.

According to Krempa [17], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called *α-rigid* if there exists a rigid endomorphism α of R . Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced by [9, Propositon 5]. Rege and Chhawchharia [22] introduced the notion of an Armendariz ring which is a generalization of a reduced ring. A ring R is called *Armendariz* if whenever any polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m, g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i and j. Armendariz property of a ring is extended to skew polynomial rings by considering the polynomials in $R[x;\sigma]$ instead of $R[x]$ (see [11] and [10] for more details). For an endomorphism σ of a ring R, R is called *α-Armendariz* (resp., *α-skew Armendariz*) if for $p(x) = \sum_{i=0}^{m} a_ix^i$ and $q(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x;\alpha]$, $p(x)q(x) = 0$ implies $a_ib_j = 0$ (resp., $a_i\alpha^i(b_j) = 0$) for all $0 \leq i \leq m$ and $0 \leq j \leq n$.

Kim and Lee showed in [16, Example 2.1] that polynomial rings over reversible rings need not be reversible. By [16, Proposition 2.4], if R is an Armendariz ring, then R is reversible if and only if $R[x]$ is reversible. Based on this result Yang and Liu [24] considered reversible rings over which polynomial rings are reversible and called them *strongly reversible*. By [12, Example 2.1], if R is reversible, then $R[x;\alpha]$ is not reversible. Therefore Jin et. al. [12] called a ring R *strongly α-skew reversible* if the skew polynomial ring $R[x;\alpha]$ is reversible. Another generalization of reversible rings are α-reversible rings introduced by Başer et. al. in [3]. An endomorphism α of a ring R is called right (resp., left) reversible if whenever $ab = 0$ for $a, b \in R$, then $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). A ring R is called right (resp., left) α-reversible if there exists a right (resp., left) reversible endomorphism α of R. R is α-reversible if it is both right and left α-reversible.

Motivated by the above, in this paper, we introduce the notion of a skew Hurwitz reversible ring (SH-reversible, shortly) (see Definition 2.2), which is a generalization of α-rigid rings for an endomorphism α of a ring R and an extension of reversible rings, and study SH-reversible rings and their related properties. We examine the relationships between several classes of rings and SH-reversible rings and investigate some extensions of SH-reversible rings.

2. SH-reversible rings

In this section we introduce a class of rings, called SH-reversible rings. We give relations between SH-reversible rings and some related rings, such as, α-rigid, reversible, α-reversible, mat-reversible, abelian, SHA-ring, etc. Firstly, we begin with the following example which illustrates the need to introduce the reversibility property of skew Hurwitz series rings.

Example 2.1. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Then we know that R is reversible since R is reduced. Let $\alpha : R \to R$ be an endomorphism of R defined by $\alpha((a,b)) = (b,a)$. For $f = h_{(1,0)}$ and $g = h_{(0,1)} + h_{(0,1)}h'_2$ in (HR, α), $fg = 0$ but $gf = h_{(0,1)}h'_2 \neq 0$. Thus (HR, α) is not reversible (and hence not reduced).

Inspired by this example, we can give the following definition.

Definition 2.2. Let R be a commutative ring and α be an endomorphism of R. Then R is called *SH-reversible* if (HR, α) is reversible.

Any α-rigid ring (i.e. $R[x;\alpha]$ is reduced) is SH-reversible by Theorem 2.6. However, there exists a SH-reversible ring which is not α-rigid (see Example 2.7). It is clear that any domain R with a monomorphism α is SH-reversible since R is α-rigid. Note that every subring S with $\alpha(S) \subseteq S$ of an SH-reversible ring is also SH-reversible. We will freely use this fact without references.

Lemma 2.3. *Let R be an SH-reversible ring which is torsion free as a $\mathbb{Z}$-module. Then we have the following results.*

1. *R is reversible and α-reversible.*
2. *α is a monomorphism of R.*
3. *For any $a, b \in R$ and nonnegative integer m and n, $a\alpha^m(b) = 0 \Leftrightarrow ab = 0 \Leftrightarrow ba = 0 \Leftrightarrow b\alpha^n(a) = 0$*
4. *$\alpha(e) = e$ for any $e^2 = e \in R$.*

Proof. (1) This is clear by the definition of an SH-reversible ring.
(2) Assume that $\alpha(a) = 0$. Then $h'_2 h_a = 0$ in (HR, α). Since R is SH-reversible, we have $h_a h'_2 = 0$ and so $a = 0$.
(3) Let $a\alpha^m(b) = 0$. We have $fg = 0$ for skew Hurwitz series f and g in (HR, α) such that f's mth component is a and g's first component is b. Since R is SH-reversible, then $gf = ba = 0$ and hence $ab = 0$. We get $fg = 0$ for skew Hurwitz series f and g in (HR, α) such that f's first component is a and g's nth component is b, so $b\alpha^n(a) = 0$ by assumption.
(4) Suppose $f, g \in (HR, \alpha)$ are defined as $f = h_{1-e} + h_{\alpha(e-1)}h'_2$ and $g = h_e + h_e h'_2$. Then $fg = 0$ and so $gf = 0$ since R is SH-reversible, hence we have $e\alpha(e) = e$ since R is torsion free as a $\mathbb{Z}$-module. Now suppose $f', g' \in (HR, \alpha)$ are defined as $f' = h_e + h_{\alpha(e)}h'_2$ and $g' = h_{e-1} + h_{e-1}h'_2$. Then $f'g' = 0$ and since R is SH-reversible, we have $g'f' = 0$, and R is torsion free as a $\mathbb{Z}$-module imply that $e\alpha(e) = \alpha(e)$. Therefore $\alpha(e) = e$. □

Skew Hurwitz series rings over reversible rings need not be reversible by Example 2.1. However this property of such rings with the Armendariz condition of skew Hurwitz series ring can be extended to their skew Hurwitz series rings. In [1] Ahmadi et. al., commutative ring R is called *skew Hurwitz serieswise Armendariz* (or an *SHA-ring*), if for every skew Hurwitz series $f = (a_i), g = (b_j) \in (HR, \alpha)$ for all $i \in \mathbb{N}$, $fg = 0$ if and only if $a_i b_j = 0$ for all i, j. The converse of Lemma 2.3(1) is shown in the following proposition.

Proposition 2.4. *Let R be an SHA-ring. Then R is reversible and α-reversible if and only if R is SH-reversible.*

Proof. It is enough to show the neccessity by Lemma 2.3(1). Let $fg = 0$ for $f = (a_i), g = (b_j) \in (HR, \alpha)$. Since R is an SHA-ring, then $a_i b_j = 0$ for each i and j. We obtain that $b_j a_i = 0$ and $b_j \alpha(a_i) = 0$ by the assumption. Therefore $gf = 0$ and so R is SH-reversible. □

We recall the following properties of α-rigid rings.

Lemma 2.5. [6, Lemma 3.2] *Let R be an α-rigid ring and $a, b \in R$, then*

(1) *If $a\alpha^n(a) = 0$ then $a = 0$, for each $n \in \mathbb{N}$.*
(2) *If $ab = 0$ then $a\alpha(b) = 0$.*

Theorem 2.6. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module and α be an endomorphism of R. Then R is α-rigid if and only if R is SH-reversible and R is reduced.*

Proof. Suppose that R is α-rigid. It is clear that R is reduced. Let $fg = 0$ for $f = (a_i), g = (b_j) \in (HR, \alpha)$ for all $i \in \mathbb{N}$. Then we have the following equivalences

$$a_0 b_0 = 0 \tag{2.1}$$
$$a_0 b_1 + a_1 \alpha(b_0) = 0 \tag{2.2}$$
$$a_0 b_2 + 2a_1 \alpha(b_1) + a_2 \alpha^2(b_0) = 0 \tag{2.3}$$
$$a_0 b_3 + 3a_1 \alpha(b_2) + 3a_2 \alpha^2(b_1) + a_3 \alpha^3(b_0) = 0 \tag{2.4}$$
$$\vdots = \vdots$$
$$a_0 b_n + \binom{n}{1} a_1 \alpha(b_{n-1}) + \cdots + \binom{n}{n} a_n \alpha^n(b_0) = 0 \tag{n}$$

Use the condition that R is α-rigid and the fact that α-rigid rings are reduced and so reversible, we obtain that $b_0 a_0 = 0$ by Eq. (2.1). Multiply Eq. (2.2) on the left hand side by b_0 and on the right hand side by $\alpha(a_1)$, then $b_0 a_1 \alpha(b_0 a_1) = 0$ and so $b_0 a_1 = 0$; hence $a_1 b_0 = 0$ implies $a_1 \alpha(b_0) = 0$ by Lemma 2.5. From Eq. (2.2) we obtain $a_0 b_1 = 0$. Next multiply Eq. (2.3) on the left hand side by b_0 and on the right hand side by $\alpha^2(a_2)$, then $b_0 a_2 \alpha^2(b_0 a_2) = 0$ and so $b_0 a_2 = 0$; hence $a_2 \alpha^2(b_0) = 0$ by Lemma 2.5. We have an equation

$$a_0 b_2 + 2a_1 \alpha(b_1) = 0. \tag{2.5}$$

Multiply Eq. (2.5) on the left side by b_1 and on the right side $\alpha(a_1)$, then $2b_1 a_1 \alpha(b_1 a_1) = 0$. Since R is torsion free as a $\mathbb{Z}$-module and R is α-rigid, we have $b_1 a_1 = 0$ and so $a_1 \alpha(b_1) = 0$ by Lemma 2.5; hence $a_0 b_2 = 0$. Continuing in this way, we get $a_i b_j = 0$ for each i, j. By assumption and using the Lemma 2.5, we obtain that $gf = 0$, as reguired.
Conversely, assume that $a\alpha(a) = 0$ for $a \in R$. Then we have $fg = 0$ for the skew Hurwitz series $f = h_a h'_2$ and $g = h_a$ in (HR, α). By the assumption we obtain that $gf = h_{a^2} = 0$ and so $a^2 = 0$; hence $a = 0$ since R is reduced. □

By Theorem 2.6, every α-rigid ring is SH-reversible. The following is an example of a non α-rigid ring which is SH-reversible.

Example 2.7. Let $\mathbb{Z}$ be the ring of integers. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$$

Let $\alpha : R \to R$ be an automorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$$

Suppose that $fg = 0$ for $f = \left(\begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix}\right)$ and $g = \left(\begin{pmatrix} c_i & d_i \\ 0 & c_i \end{pmatrix}\right)$ in (HR, α), where $\begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix}, \begin{pmatrix} c_i & d_i \\ 0 & c_i \end{pmatrix} \in R$ for all i, j. From $fg = 0$, we have the following systems of equations:

$$\begin{aligned} a_0c_0 =& 0...(*) \\ a_0d_0 + b_0c_0 =& 0...(**) \end{aligned} \tag{2.6}$$

$$\begin{aligned} a_0c_1 + a_1c_0 =& 0...(*) \\ a_0d_1 + b_0c_1 + a_1d_0 + b_1c_0 =& 0...(**) \end{aligned} \tag{2.7}$$

$$\begin{aligned} a_0c_2 + 2a_1c_1 + a_2c_0 =& 0...(*) \\ a_0d_2 + b_0c_2 - 2a_1d_1 + 2b_1c_1 + a_2d_0 + b_2c_0 =& 0...(**) \\ &\vdots \end{aligned} \tag{2.8}$$

Suppose that $a_0 \neq 0$. From Eq. 2.6(∗), we have $c_0 = 0$ since $\mathbb{Z}$ is a integral domain. Then we obtain that $d_0 = 0$ from Eq. 2.6(∗∗). In Eq. 2.7(∗), we have $a_0c_1 = 0$ and so $c_1 = 0$ since $a_0 \neq 0$. We obtain that $d_1 = 0$ by using these facts from Eq. 2.7(∗∗). From Eq. 2.8(∗), we get $c_2 = 0$ and so $d_2 = 0$ from Eq. 2.8(∗∗). Continuing this process, we obtain $c_i = d_i = 0$ for all i. This yields $gf = 0$. Therefore R is SH-reversible. However, for $0 \neq A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $A\alpha(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 0$ and thus R is not α-rigid.

Let R be a ring and let ${}_RV_R$ be an R-R-bimodule which is an arbitrary ring in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ holds for all $v, w \in V$ and $r \in R$. Then the *ideal extension* $I(R; V)$ of R by V is defined to be the additive abelian group $I(R; V) = R \oplus V$ with multiplication $(r, v)(s, w) = (rs, rw + vs + vw)$. Note that $(HR, \alpha) \cong I(R; A)$ where $A = \{f \in (HR, \alpha) \mid f(0) = 0\}$ by [8, Proposition 2.1]. We can give the following corollary as a result of Theorem 2.6.

Corollary 2.8. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module and α be an endomorphism of R. If R is α-rigid, then the ideal extension $I(R, A)$ of R is reversible, where $A = \{f \in (HR, \alpha) \mid f(0) = 0\}$.*

If we take $\alpha = id_R$, we can give the following corollary, by using the fact that reduced rings are reversible, as a consequence of Theorem 2.6.

Corollary 2.9. [4, Corollary 2.7] *The following assertions are equivalent:*

(1) *The ring R is reduced and is torsion free as a $\mathbb{Z}$-module.*
(2) *The Hurwitz series ring HR is reduced.*
(3) *The Hurwitz polynomial ring hR is reduced.*

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $e^2 = e \in R$ and $a \in R$.

Proposition 2.10. *Let R be an SH-reversible ring and α is an endomorphism of R. Then R is abelian and if $f = (a_i) \in (HR, \alpha)$ is an idempotent for all $i \in \mathbb{N}$, then $a_0 \in R$ is an idempotent and $f = h_{a_0}$.*

Proof. Let $e^2 = e \in R$. Then $h_e h_{1-e} = 0$ and $h_{1-e} h_e = 0$ in (HR, α) and so $h_e h_{1-e} h_r = 0$ and $h_r h_{1-e} h_e = 0$ in (HR, α) for any $r \in R$. Since R is SH-reversible, we have $h_{1-e} h_r h_e =$

0 and $h_e h_r h_{1-e} = 0$ hence $re = ere$ and $er = ere$. These imply that R is abelian. Now let $f^2 = f$, where $f = (a_i) \in (HR, \alpha)$ and $a_i \in R$ for all i. Then we have the following equations:

$$a_0^2 = a_0 \tag{0}$$

$$a_0 a_1 + a_1 \alpha(a_0) = a_1 \tag{1}$$

$$a_0 a_2 + 2a_1 \alpha(a_1) + a_2 \alpha^2(a_0) = a_2 \tag{2}$$

$$\vdots$$

$$\binom{n}{0} a_0 a_n + \binom{n}{1} a_1 \alpha(a_{n-1}) + \cdots + \binom{n}{n} a_n \alpha^n(a_0) = a_n \tag{n}$$

$$\vdots$$

Note that from Eq.(0), a_0 is an idempotent of R and so it is central and $\alpha(a_0) = a_0$ from Lemma 2.3(iv). Then we get the following:

$$2a_1 a_0 = a_1 \tag{1'}$$

$$a_2 a_0 + 2a_1 \alpha(a_1) + a_2 a_0 = a_2 \tag{2'}$$

$$\vdots$$

$$\binom{n}{0} a_n a_0 + \binom{n}{1} a_1 \alpha(a_{n-1}) + \cdots + \binom{n}{n} a_n a_0 = a_n \tag{n'}$$

$$\vdots$$

Multiplying Eq.(1') on the right hand side by $1-a_0$, we obtain $a_1(1-a_0) = 0$, so $a_1 a_0 = a_1$ and hence $a_1 = 0$. Thus, Eq.(2') becomes $2a_2 a_0 = a_2$. Similarly, $2a_2 a_0(1-a_0) = a_2(1-a_0)$ implies $a_2 = 0$. Continuing this procedure yields that $a_i = 0$ for $i \geq 1$. Furthermore (HR, α) is abelian since R is abelian. □

3. Extensions of SH-reversible rings

Recall that for a ring R and an endomorphism σ of R, an ideal I of R is called a *σ-ideal* if $\sigma(I) \subseteq I$. If I is a σ-ideal of R, then $\bar{\sigma} : R/I \to R/I$ defined by $\bar{\sigma}(a + I) = \sigma(a) + I$ for $a \in R$ is an endomorphism of R/I. Following [7, Remark 3.1], every right (resp. left) ideal I of R corresponds a right (resp. left) ideal (HI, α) in (HR, α) where $(HI, \alpha) = \{f \in (HR, \alpha) \mid a_n \in I \text{ for all } n \in \mathbb{N}\}$.

Proposition 3.1. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module, α be an automorphism of R and I be an α-ideal of R. If R/I is an SH-reversible ring and I is an α-rigid ring without identity, then R is SH-reversible.*

Proof. Let $fg = 0$ for $f = (a_i), g = (b_j) \in (HR, \alpha)$. Then we have $\bar{f}\bar{g} = \bar{0}$ in $(H(R/I), \bar{\alpha})$ where $\bar{f} = (a_i + I), \bar{g} = (b_j + I)$. By assumption since R/I is SH-reversible, we obtain that $\bar{g}\bar{f} = \bar{0}$, i.e., $gf \in I$. Since I is α-rigid, then (HI, α) is reduced by [7, Proposition 2.11]. Hence $(gf)^2 = 0$ in (HI, α) and so $gf = 0$. □

Let α_i be an endomorphism of a ring R_i for each $i \in I$. Then the map $\alpha : \prod_{i \in I} R_i \to \prod_{i \in I} R_i$ defined by $\alpha((a_i)) = (\alpha_i(a_i))$ for $(a_i) \in \prod_{i \in I} R_i$ is endomorphism of $\prod_{i \in I} R_i$. The proof of the following lemma is obtained by routine computations.

Lemma 3.2. *Let R_i be a ring with an endomorphism α_i for each $i \in I$. Then the following statements are equivalent:*

(1) *R_i is SH-reversible for each $i \in I$.*

(2) *The direct product* $\prod_{i\in I} R_i$ *is* SH*-reversible.*
(3) *The direct sum* $\bigoplus_{i\in I} R_i$ *is* SH*-reversible.*

Proof. It is enough to show that (1) $\Rightarrow$ (2). Suppose that R_i is SH-reversible for each $i \in \Gamma$. Let $fg = 0$ for $f = (f_n), g = (g_m)$ where $f_n = (a_i^{(n)})$ and $g_m = (b_i^{(m)})$ for all n, m and for each $i \in I$. Since $H(\prod_{i\in I} R_i) \cong \prod_{i\in I} H(R_i)$, we have $fg = 0$ for $f = ((a_n^{(i)}))$ and $g = ((b_m^{(i)}))$ in $\prod_{i\in I} H(R_i)$. Then $(a_n^{(i)})(b_m^{(i)}) = 0$ in (HR_i, α_i). Since R_i is SH-reversible for each $i \in I$, we have $(b_m^{(i)})(a_n^{(i)}) = 0$ and so $gf = 0$. Therefore the direct product of R_i is SH-reversible. □

A ring R is called *local* if $R/J(R)$ is a division ring, where $J(R)$ denotes the Jacobson radical of R. R is called *semilocal* if $R/J(R)$ is semisimple Artinian and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Note that local rings are abelian and semilocal (see [19] for details). In [20], Paykan showed that R is a local ring iff (HR, α) is local, and R is semiperfect iff (HR, α) is semiperfect. We can give the following proposition.

Proposition 3.3. *Let* R *be a ring and* α *be an endomorphism of* R*. Then we have the following.*

(i) R *is* SH*-reversible and semiperfect if and only if* $R = \bigoplus_{i=1}^{n} R_i$ *such that each* R_i *is local and an* SH*-reversible ring, where* α_i *is an endomorphism of* R_i *for all* $i = 0, 1, \dots n$*.*
(ii) *Let* e *be a central idempotent of* R*. Then* eR *and* $(1-e)R$ *are* SH*-reversible if and only if* R *is* SH*-reversible.*

Proof. (i) Suppose that R is SH-reversible and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \dots, e_n\}$ of local idempotents whose sum is 1 by [**?**, Corollary 3.7.2]. Then $R = \sum_{i=1}^{n} e_i R$ such that e_iRe_i is a local ring for all $i = 1, \dots, n$. Since R is SH-reversible, then R is abelian from Proposition 2.10 and $e_iRe_i = e_iR$. Also, by Lemma 2.3(iv), $\alpha(e_iR) \subseteq e_iR$ for all $i = 1, \dots, n$. Then e_iR is SH-reversible and local subring of R, where α_i is an endomorphism of e_iR induced by α. Conversely, let R be a finite direct sum of SH-reversible local rings R_i for all $i = 0, 1, \dots n$. Then R is semiperfect since local rings are semiperfect and R is SH-reversible by Lemma 3.2.
(ii) The proof is clear by Lemma 3.2 since $R \cong eR \oplus (1-e)R$. □

Let R be a ring. Define $V_n = \sum_{i=1}^{n-1} E_{i,j+1}$, for $n \geq 2$, where $E_{i,j}$ is the matrix units for all i, j. Consider the ring

$$T(R,n) = RI_n + RV_n + RV_n^2 + \cdots + RV_n^{n-1};$$

$$T(R,n) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & a_0 \end{pmatrix} \mid a_i \in R \right\}$$

in [1]. If R is a commutative ring, then $T(R,n)$ is also a commutative ring. Let α be an endomorphism of R, then for each n, $\bar{\alpha} : T(R,n) \to T(R,n)$, given by $\bar{\alpha}([a_{ij}]) = [\alpha(a_{ij})]$ is an endomorphism. On the other hand, Veldsman introduced *mat*-reversible rings in [23] as follows: let R be an identity ring and $\mathbb{M}_k(R, x^k)$ be the Barnett matrix ring over R determinated by the polynomial $h(x) = x^k \in R[x]$, $k \geq 1$. This means $\mathbb{M}_k(R, x^k) \cong \frac{R[x]}{<x^k>}$ is just the regular representation of the ring $\frac{R[x]}{<x^k>}$. In particular, $\mathbb{M}_k(R, x^k)$ is the ring of all $k \times k$ matrices of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ 0 & a_0 & a_1 & \cdots & a_{k-3} & a_{k-2} \\ 0 & 0 & a_0 & \cdots & a_{k-4} & a_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & a_0 \end{bmatrix}$$

with entries a_i in R. The ring $\mathbb{M}_k(R, x^k)$ of $k \times k$ matrices over R can be defined without requiring that R has an identity which we will henceforth do. For $k \geq 1$, we then say a ring R is *mat-k-reversible* provided the ring $\mathbb{M}_k(R, x^k)$ is reversible, i.e.,

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} \\ 0 & a_0 & a_1 & \cdots & a_{k-2} \\ 0 & 0 & a_0 & \cdots & a_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix} \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{k-1} \\ 0 & b_0 & b_1 & \cdots & b_{k-2} \\ 0 & 0 & a_0 & \cdots & b_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 \end{bmatrix} = 0$$

implies

$$\begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{k-1} \\ 0 & b_0 & b_1 & \cdots & b_{k-2} \\ 0 & 0 & a_0 & \cdots & b_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} \\ 0 & a_0 & a_1 & \cdots & a_{k-2} \\ 0 & 0 & a_0 & \cdots & a_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix} = 0$$

for any two matrices from $\mathbb{M}_k(R, x^k)$. A ring R is called *mat-reversible* if it is *mat-k*-reversible for all $k \geq 1$. In the other words; a ring R is called *mat-reversible* if $T(R, n)$ is reversible for all n. Now we can give the following proposition.

Proposition 3.4. *Let R be a commutative ring such that it is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. If R is α-rigid, then (HR, α) is mat-reversible.*

Proof. Suppose that R is α-rigid. Let $AB = 0$ for

$$A = \begin{bmatrix} (a_i^{(0)}) & (a_i^{(1)}) & (a_i^{(2)}) & \cdots & (a_i^{(n-1)}) \\ 0 & (a_i^{(0)}) & (a_i^{(1)}) & \cdots & (a_i^{(n-2)}) \\ 0 & 0 & (a_i^{(0)}) & \cdots & (a_i^{(n-3)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (a_i^{(0)}) \end{bmatrix}, B = \begin{bmatrix} (b_i^{(0)}) & (b_i^{(1)}) & (b_i^{(2)}) & \cdots & (b_i^{(n-1)}) \\ 0 & (b_i^{(0)}) & (b_i^{(1)}) & \cdots & (b_i^{(n-2)}) \\ 0 & 0 & (b_i^{(0)}) & \cdots & (b_i^{(n-3)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (b_i^{(0)}) \end{bmatrix}$$

in $T((HR, \alpha), n)$. Then we have the following equalities:

$$\begin{aligned} (a_i^{(0)})(b_i^{(0)}) &= 0 \\ (a_i^{(0)})(b_i^{(1)}) + (a_i^{(1)})(b_i^{(0)}) &= 0 \\ &\vdots \\ (a_i^{(0)})(b_i^{(n-1)}) + (a_i^{(1)})(b_i^{(n-2)}) + \cdots + (a_i^{(n-1)})(b_i^{(0)}) &= 0 \end{aligned}$$

in (HR, α). Since R is α-rigid, then (HR, α) is reduced by [1, Proposition 4.1], and so it is reversible. By using the technique in Theorem 2.6, we obtain that $BA = 0$. Therefore (HR, α) is *mat*-reversible. □

Theorem 3.5. *Let R be a commutative ring such that it is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. If R is α-rigid, then $T(R, n)$ is SH-reversible for each positive integer n.*

Proof. Suppose that R is α-rigid. We take the map $\Psi : (HT(R,n),\bar{\alpha}) \to T((HR,\alpha),n)$, given by $\Psi(f) = [f_{ij}]$, where $f(m) = A_m \in T(R,n)$ and $f_{ij}(m) = (a_{ij}^{(m)})$ and $a_{ij}^{(m)}$ is the (i,j)th entry of A_m for each m is defined in [1]. It is easy to see that Ψ is an isomorphism. We assume that $fg = 0$ for $f,g \in (HT(R,n),\bar{\alpha})$ where $f(m) = A_m = [a_{ij}^m]$ and $g(m) = B_m = [b_{ij}^m]$ for each m. Hence, by the above isomorphism, we have $[f_{ij}][g_{ij}] = 0$ for some $[f_{ij}],[g_{ij}] \in T((HR,\alpha),n)$. Thus $[g_{ij}][f_{ij}] = 0$ by Proposition 3.4 and so $gf = 0$. Therefore $T(R,n)$ is SH-reversible as required. □

Corollary 3.6. *If R is α-rigid, then the ring $\frac{R[x]}{<x^n>}$ is SH-reversible.*

Proof. Since $T(R,n) \cong \frac{R[x]}{<x^n>}$ for each positive integer n, it is clear by Theorem 3.5. □

Given a ring R and a bimodule ${}_RM_R$, the *trivial extension* of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + r_2m_1).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Let α be an endomorphism of R. We can extend α to an endomorphism $\bar{\alpha} : T(R,R) \to T(R,R)$ defined by $\bar{\alpha}\left(\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \alpha(r) & \alpha(s) \\ 0 & \alpha(r) \end{pmatrix}$. Since $T(R,2) = T(R,R)$ for $n = 2$, we can give the following corollary.

Corollary 3.7. *If R is α-rigid, then the trivial extension $T(R,R)$ of R is SH-reversible.*

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