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A CLASSIFICATION OF RING ELEMENTS IN SKEW PBW EXTENSIONS OVER COMPATIBLE RINGS

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Abstract. For a skew PBW extension over a right duo compatible ring, we characterize several kinds of their elements such as units, idempotent, von Neumann regular, π-regular and the clean elements. As a consequence of our treatment, we extend several results in the literature for Ore extensions and commutative rings.

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1. Introduction

Following Lam [36], for a ring B, an element $a \in B$ is called *von Neumann regular*, if there exists an element $r \in B$ with $a = ara$. The element $a \in B$ is said to be a π-*regular element* of B, if $a^m r a^m = a^m$, for some $r \in B$ and $m \geq 1$. We define the sets $\mathrm{Idem}(B) = \{a \in B \mid a^2 = a\}$, $\mathrm{vnr}(B) = \{a \in B \mid a$ is von Neumann regular$\}$ and $\pi - r(B) = \{a \in B \mid a$ is $\pi-$regular$\}$. It is clear that $\mathrm{Idem}(B) \subseteq \mathrm{vnr}(B) \subseteq \pi - r(B)$. Now, a ring B is called *von Neumann regular*, if the equality $\mathrm{vnr}(B) = B$ holds. B is said to be π-*regular*, if $\pi - r(B) = B$. B is called *Boolean*, whenever $\mathrm{Idem}(B) = B$. Of course, the implications Boolean $\Rightarrow$ von Neumann regular $\Rightarrow$ π-regular hold.

Now, by Contessa [17], an element $a \in B$ is said to be a *von Neumann local element*, if either $a \in \mathrm{vnr}(B)$ or $1 - a \in \mathrm{vnr}(B)$. Following [44], an element $a \in B$ is a *clean element*, if a is the sum of a unit and an idempotent of B. Let $\mathrm{vnl}(B) = \{a \in B \mid a$ is von Neumann local$\}$ and $\mathrm{cln}(B) = \{a \in B \mid a$ is clean$\}$. If $\mathrm{cln}(B) = B$, then B is said to be a *clean ring* [44]. Examples of clean rings are the exchange rings and semiperfect rings. Several characterizations of clean

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elements in polynomial rings have been established in [33] and [45]. In the case that $\mathrm{vnl}(B) = B$, B is called a *von Neumann local* ring [21] (c.f. [2]).

Now, thinking about the Ore extensions $B[x;\sigma,\delta]$ introduced by Ore [48], where σ is an endomorphism of B and δ is a σ-derivation of B which is an additive map satisfying the equality $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for every elements $a, b \in B$, where the skew-multiplication is given by $xr = \sigma(r)x + \delta(r)$, for all $r \in B$, Krempa [35] called an endomorphism σ of a ring B a *rigid endomorphism*, if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in B$. B is called a *σ-rigid ring*, if there exists a rigid endomorphism σ of B [28]. One can see that any rigid endomorphism of a ring is a monomorphism and also that σ-rigid rings are reduced [28] (recall that a ring B is *reduced*, if it has no nonzero nilpotent elements). Several properties of these rings have been established in the literature (e.g., [27], [28], [29], and [35]). With respect to ideals, according to Hong et al. [29], for an endomorphism σ of a ring B, a σ-ideal I is said to be a *σ-rigid ideal*, if $a\sigma(a) \in I \Rightarrow a \in I$, for $a \in B$. In that paper, the authors investigated relations between the σ-rigid ideals of B and the related ideals of some ring extensions.

As a generalization of σ-rigid rings, in [25] the second author considered compatible rings in the following way (see Annin [3] for more details): a ring B with an endomorphism σ and a σ-derivation δ is called *σ-compatible*, if for each $a, b \in B$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$. B is said to be *δ-compatible*, if for every $a, b \in B$, $ab = 0 \Rightarrow a\delta(b) = 0$. If B is both σ-compatible and δ-compatible, B is said to be *(σ,δ)-compatible*. In [25], Lemma 2.2, it was shown that B is σ-rigid if and only if B is σ-compatible and reduced, which means that the σ-compatible rings are a generalization of σ-rigid ring to the more general case where B is not assumed to be reduced. About ideals, in [20] the second author defined σ-compatible ideals, which are a generalization of σ-rigid ideals, in the following way: an ideal I is called a *σ-compatible ideal*, if for each $a, b \in B$, $ab \in I \Leftrightarrow a\sigma(b) \in I$. Moreover, I is said to be a *δ-compatible ideal*, if for each $a, b \in B$, $ab \in I \Rightarrow a\delta(b) \in I$. If I is both σ-compatible and δ-compatible, I is a *(σ,δ)-compatible ideal*.

Considering Ore extensions, recently, in [21] the first two authors characterized the unit elements, the idempotent elements, the von Neumann regular elements, the π-regular elements and also the von Neumann local elements of an Ore extension $B[x;\sigma,\delta]$ when the base ring B is a right duo ring which is (σ,δ)-compatible. As a matter of fact, they completely characterized the clean elements of the Ore extension ring $B[x;\sigma,\delta]$ when the base ring B is a right duo ring which is (σ,δ)-compatible.

With all above results in mind, our purpose in this paper is to establish analogue characterizations to the established in [21] for Ore extensions but now for a more general kind of noncommutative rings. We are taking about the skew PBW extensions, which are noncommutative rings of polynomial type more general than Ore extensions of injective type (i.e., when σ is injective). These extensions were introduced by Gallego and Lezama [18] with the aim of extending the PBW extensions introduced by Bell and Goodearl [11]. During the last years, several authors have been studying ring and module theoretical properties of these objects (e.g., [5], [6], [22], [23], [24], [38], [41], [47], [51], [56], [60] and [64]). In Section 2 we will say some words about the relations between skew PBW extensions and other families of noncommutative rings of polynomial type considered in the literature.

Throughout the paper, the *zero-divisors* of a ring B, denoted by $Z(B)$, are the elements $a \in B$ such that there exists a nonzero element $b \in B$ with $ab = 0$ or $ba = 0$. The set of all units, the prime radical, the upper nil radical, the Levitzki radical, the set of all nilpotent elements and the Jacobson radical of B are denoted by $U(B)$, $\mathrm{Nil}_*(B)$, $\mathrm{Nil}^*(B)$, $\mathrm{L}-\mathrm{rad}(B)$ and $J(B)$, respectively. Let us recall that a ring B is *reversible*, if $ab = 0$ implies $ba = 0$, for $a, b \in B$. B is *semicommutative*, if $ab = 0$ implies $aBb = 0$, for $a, b \in B$. B is called *2-primal*, if $\mathrm{Nil}_*(B) = \mathrm{Nil}(B)$ (this notion was introduced by Birkenmeier [12]). In [62], Proposition 1.11, Shin proved that a ring B is 2-primal if and only if every minimal prime ideal P of B is *completely prime* (i.e., B/P is a domain). A ring B is *weakly 2-primal*, if $\mathrm{Nil}(B) = L{-}\mathrm{rad}(B)$. A ring B is *NI*, if $\mathrm{Nil}(B) = \mathrm{Nil}^*(B)$. The following relations are well-known: reduced $\Rightarrow$ reversible $\Rightarrow$ semicommutative $\Rightarrow$ 2-primal $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI, but the converses do not hold (see [16] and [34]). A ring B is said to be *right* (respectively, *left*) *duo*, if every right (respectively, left) ideal is an ideal. The importance of the study of all these classes of rings is due to their importance in the Köthe's conjecture (see [10] and [42]).

To finish this introduction, we describe the structure of the article. In Section 2 we recall some useful results about skew PBW extensions for the rest of the paper. In Section 3 we establish key facts about (Σ, Δ)-compatible rings which are important in the proofs of the results obtained in the following sections. Precisely, in Section 4 we characterize the units of a skew PBW extension over a right duo (Σ, Δ)-compatible ring, while in Section 5 we establish relations between the idempotent, von Neumann regular and local, and clean elements of a right duo (Σ, Δ)-compatible ring R and those elements corresponding of a skew PBW extension A over R. The results obtained in Sections 4 and 5 generalize corresponding results presented by the first two authors in [21] for Ore extensions of injective type. We have to say that the techniques used here are fairly standard and follow the same path as other text on the subject, and hence the results presented here are new for skew PBW extensions and all they are similar to others existing in the literature. Our paper can be considered as a modest contribution to the study of ring elements of noncommutative rings of polynomial type which can not be expressed as Ore extensions but as skew PBW extensions. Finally, in the future work, we consider a possible topic of research concerning modules over skew PBW extensions.

2. Skew PBW extensions

Skew PBW extensions are a direct generalization of PBW extensions introduced by Bell and Goodearl [11]. They also are strictly more general than Ore extensions of injective type (see [58], Example 1, for a list of noncommutative rings which are skew PBW extensions but not Ore extensions). Nevertheless, as time went by, we and others realized that these extensions also generalize several families of noncommutative rings appearing in representation theory, Hopf algebras, quantum groups, noncommutative algebraic geometry and other algebras of interest in the context of theoretical physics (e.g., [39] and [54] for more details). Next, we mention briefly some of them: (1) Universal enveloping algebras of finite dimensional Lie algebras. (2) Almost normalizing extensions defined by McConnell and Robson [43]. (3) Solvable polynomial rings introduced by Kandri-Rody and Weispfenning [31]. (4) Diffusion algebras studied by Isaev, Pyatov, and Rittenberg [30]. (5) 3-dimensional skew polynomial algebras studied by Rosenberg [61] (see also [55]). The advantage of skew PBW extensions is that they do not require the coefficients to commute with the variables and, moreover, the coefficients need not come from a field (see Definition 2.1). In fact, the skew PBW extensions share examples of algebras with generalized Weyl algebras defined by Bavula [8] (also known as hyperbolic algebras defined by Rosenberg [61]), with G-algebras introduced by Apel [4] and some PBW algebras defined by Bueso et al., [15], (both G-algebras and PBW algebras take coefficients in fields and assume that coefficients commute with variables), Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul and augmented Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, some graded skew Clifford algebras and others (e.g., [13], [39], [63] and [64]). As we can see, skew PBW extensions include a considerable number of noncommutative rings of polynomial type, so a classification of ring elements of these extensions will establish results for algebras not considered before and, of course, it will cover also several treatments in the literature.

Definition 2.1 ([18], Definition 1). Let R and A be rings. We say that A *is a skew PBW extension of* R (also called a σ-PBW extension of R), which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same multiplicative identity element.

(ii) There exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis given by $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, and $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \le i \le n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements x_i, x_j, there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Proposition 2.2 ([18], Proposition 3). *Let A be a skew PBW extension of R. For each $1 \le i \le n$, there exist an injective endomorphism $\sigma_i : R \to R$ and an σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$. We denote $\Sigma := \{\sigma_1, \dots, \sigma_n\}$, and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Definition 2.3 ([18], Definition 4). Let A be a skew PBW extension of R.

(a) A is called *quasi-commutative*, if the conditions (iii) and (iv) in Definition 2.1 are replaced by (iii'): for each $1 \le i \le n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; (iv'): for any $1 \le i, j \le n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(b) A is called *bijective*, if σ_i is bijective for each $1 \le i \le n$, and $c_{i,j}$ is invertible, for any $1 \le i < j \le n$.

Remark 2.4 ([18], Section 3). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) Consider the families Σ and Δ in Proposition 2.2. Throughout the paper, for any element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will write $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^\alpha = \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes composition, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) Given the importance of monomial orders in the proofs of the results presented in Section 4, next we recall some key facts about these for skew PBW extensions.

Let $\succeq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we will write $x^\alpha \succ x^\beta$. If f is a nonzero element of A, then f can be expressed uniquely as $f = a_0 + a_1X_1 + \cdots + a_mX_m$, where every X_i is a monomial with $a_i \in R$, and $X_m \succ \cdots \succ X_1$ (eventually, we will use expressions as $f = a_0 + a_1Y_1 + \cdots + a_mY_m$, with $a_i \in R$, and $Y_m \succ \cdots \succ Y_1$). With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_mX_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^m$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. Thus, we extend $\succeq$ to $\mathrm{Mon}(A) \cup \{0\}$.

Following [18], Definition 11, if $\succeq$ is a total order on $\mathrm{Mon}(A)$, we say that $\succeq$ is a *monomial order* on $\mathrm{Mon}(A)$, if the following conditions hold:

- For every $x^\beta, x^\alpha, x^\gamma, x^\lambda \in \mathrm{Mon}(A)$, $x^\beta \succeq x^\alpha$ implies $\mathrm{lm}(x^\gamma x^\beta x^\lambda) \succeq \mathrm{lm}(x^\gamma x^\alpha x^\lambda)$ (the total order is compatible with multiplication).
- $x^\alpha \succeq 1$, for every $x^\alpha \in \mathrm{Mon}(A)$.
- $\succeq$ is degree compatible, i.e., $|\beta| \succeq |\alpha| \Rightarrow x^\beta \succeq x^\alpha$.

In [18], monomial orders are also called *admissible orders*. The condition (iii) of the previous definition is needed in the proof of the fact that every monomial order on $\mathrm{Mon}(A)$ is a well order, that is, there are not infinite decreasing chains in $\mathrm{Mon}(A)$ (see [18], Proposition 12). Nevertheless, this hypothesis is not really needed to get a well ordering if a more elaborated argument, based upon Dickson's Lemma, is developed (see [9], Theorem 4.62 or [14], Propositions 1.2 and 1.20).

The importance of considering monomial orders on $\mathrm{Mon}(A)$ can be appreciated in [18] where the Gröbner theory for left ideals of skew PBW extensions was studied.

The result established in [15], Chapter 2 or [14], Theorem 1.2, for PBW rings in the sense of [15] motivated the following result for skew PBW extensions. Connections with filtered rings and their corresponding graded rings can be found in [39].

Proposition 2.5 ([18], Theorem 7)**.** *If A is a polynomial ring with coefficients in R and the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(i) *For each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(ii) *For each* $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, *there exist unique elements* $c_{\alpha,\beta} \in R$ *and* $p_{\alpha,\beta} \in A$ *such that* $x^\alpha x^\beta = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}$, *where* $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$, *or* $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ *if* $p_{\alpha,\beta} \neq 0$.

Remark 2.6. With respect to the Proposition 2.5, we have two observations:

(i) ([49], Proposition 2.9) If $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $r \in R$, then

$$
\begin{aligned}
x^\alpha r &= x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r = x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}\left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n-j}\delta_n(\sigma_n^{j-1}(r))x_n^{j-1}\right)\\
&+ x_1^{\alpha_1}\cdots x_{n-2}^{\alpha_{n-2}}\left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j}\delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r)))x_{n-1}^{j-1}\right)x_n^{\alpha_n}\\
&+ x_1^{\alpha_1}\cdots x_{n-3}^{\alpha_{n-3}}\left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j}\delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r))))x_{n-2}^{j-1}\right)x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}\\
&+ \cdots + x_1^{\alpha_1}\left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2-j}\delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))))x_2^{j-1}\right)x_3^{\alpha_3}x_4^{\alpha_4}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}\\
&+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r))))x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad \sigma_j^0 := \mathrm{id}_R \quad \text{for} \quad 1 \leq j \leq n.
\end{aligned}
$$

(ii) ([49], Remark 2.10) Using (i), it follows that for the product $a_iX_ib_jY_j$, if $X_i := x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$ and $Y_j := x_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}}$, then

$$
\begin{aligned}
a_iX_ib_jY_j &= a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ \cdots + a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b_j)}x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b_j}x^{\beta_j}.
\end{aligned}
$$

In this way, when we compute every summand of $a_iX_ib_jY_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i.

Several examples of skew PBW extensions can be found in [39] and [59].

To finish this section, we include one more definition and a result about quotient rings of skew PBW extensions.

Definition 2.7 ([37], Definition 2.1)**.** Let R be a ring, $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ a finite set of endomorphisms of R and $\Delta := \{\delta_1, \dots, \delta_n\}$ a finite set of Σ-derivations. If I is an ideal of R, I is called Σ-invariant, if $\sigma_i(I) \subseteq I$, for every $1 \leq i \leq n$. Δ-invariant ideals are defined similarly. If I is both Σ and Δ-invariant, we say that I is (Σ, Δ)-invariant.

Proposition 2.8 ([37], Proposition 2.6)**.** *Let A be a skew PBW extension of a ring R and I a (Σ, Δ)-invariant ideal of R. Then:*

(1) *IA is an ideal of A and $IA \cap R = I$. IA is proper if and only if I is proper. Moreover, if for every $1 \leq i \leq n$, σ_i is bijective and $\sigma_i(I) = I$, then $IA = AI$.*

(2) *If I is proper and $\sigma_i(I) = I$, for every $1 \leq i \leq n$, then A/IA is a skew PBW extension of R/I. Moreover, if A is bijective, then A/IA is bijective.*

(3) *Let R be left (right) Noetherian and σ_i bijective, for every $1 \leq i \leq n$. Then $\sigma_i(I) = I$, for every i and $IA = AI$. If I is proper and A is bijective, then A/IA is a bijective skew PBW extension of R/I.*

3. (Σ, Δ)-compatible rings

Following Krempa [35], an endomorphism σ of a ring B is said to be *rigid*, if $a\sigma(a) = 0$ implies $a = 0$, for $a \in B$. A ring B is said to be σ-rigid, if there exists a rigid endomorphism σ of B. It is clear that any rigid endomorphism of a ring is a monomorphism, and σ-rigid rings are reduced ([28], p. 218). Properties of σ-rigid rings have been studied by several authors (c.f. [35] and [28]). With this in mind, in [3], it is said that B is σ-*compatible*, if for every $a, b \in B$, we have $ab = 0$ if and only if $a\sigma(b) = 0$; B is said to be δ-*compatible*, if for each $a, b \in B$, $ab = 0 \Rightarrow a\delta(b) = 0$. If B is both σ-compatible and δ-compatible, B is called (σ, δ)-*compatible*. In this case, the endomorphism σ is injective. Since one can appreciate the relation between these notions and σ-rigid rings, in [25], Lemma 2.2, it was shown that a ring B is (σ, δ)-compatible and reduced if and only if B is σ-rigid. Hence σ-compatible rings generalize σ-rigid rings for the case B is not assumed to be reduced. The natural task for us is to extend this notion of compatibility to a more general context of Ore extensions of injective type, that is, the family of skew PBW extensions; this is precisely the content of Definition 3.2. Before, we recall the notion of Σ-rigid ring introduced by the third author.

For Definitions 3.1 and 3.2, consider the notation presented in Remark 2.4 (i) about compositions of endomorphisms and compositions of derivations.

Definition 3.1 ([49], Definition 3.2)**.** Let B be a ring and Σ a family of endomorphisms of B. Σ is called a *rigid endomorphisms family*, if $r\sigma^{\alpha}(r) = 0$ implies $r = 0$, for every $r \in B$ and $\alpha \in \mathbb{N}^n$. A ring B is said to be Σ-*rigid*, if there exists a rigid endomorphisms family Σ of B.

Note that if Σ is a rigid endomorphisms family, then every element $\sigma_i \in \Sigma$ is a monomorphism. In fact, Σ-rigid rings are reduced rings: if B is a Σ-rigid ring and $r^2 = 0$ for $r \in B$, then we obtain the equalities $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$, i.e., $r\sigma^{\alpha}(r) = 0$ and so $r = 0$, that is,

B is reduced (note that there exists an endomorphism of a reduced ring which is not a rigid endomorphism, see [28], Example 9). Σ-rigid rings have been investigated in several papers (e.g., [47], [50], [58] and [59]).

The second and the third author introduced independently the notion of compatibility for skew PBW extensions as the following definition shows. Consider the family of injective endomorphisms Σ and the family Δ of Σ-derivations in a skew PBW extension A of a ring R (see Proposition 2.2).

Definition 3.2 ([22], Definition 3.1; [57], Definition 3.2)**.** Consider a ring R with a finite family of endomorphisms Σ and a finite family of Σ-derivations Δ. Following the notation established in Remark 2.4 (i), we have the following: R is said to be Σ-*compatible*, if for each $a, b \in R$, $a\sigma^{\alpha}(b) = 0$ if and only if $ab = 0$, for every $\alpha \in \mathbb{N}^n$; R is said to be Δ-*compatible*, if for each $a, b \in R$, $ab = 0$ implies $a\delta^{\beta}(b) = 0$, for every $\beta \in \mathbb{N}^n$. If R is both Σ-compatible and Δ-compatible, R is called or (Σ, Δ)-*compatible*.

Example 3.3. Next, we present remarkable examples of σ-PBW extensions over (Σ, Δ)-compatible rings (see [22] or [39] for a detailed definition and reference of every example).

(a) If A is a skew PBW extension of a reduced ring R where the coefficients commute with the variables, that is, $x_i r = r x_i$, for every $r \in R$ and each $i = 1, \dots, n$, or equivalently, $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$, for every i, then it is clear that R is (Σ, Δ)-compatible. Some examples of constant σ-PBW extensions are the following: PBW extensions defined by Bell and Goodearl (which include the classical commutative polynomial rings, universal enveloping algebra of a Lie algebra, and others); some operator algebras (for example, the algebra of linear partial differential operators, the algebra of linear partial shift operators, the algebra of linear partial difference operators, the algebra of linear partial q-dilation operators, and the algebra of linear partial q-differential operators); the class of diffusion algebras; Weyl algebras; additive analogue of the Weyl algebra; multiplicative analogue of the Weyl algebra; some quantum Weyl algebras as $A_2(J_{a,b})$; the quantum algebra $U'(\mathfrak{so}(3, \Bbbk))$; the family of 3-dimensional skew polynomial algebras (there are exactly fifteen of these algebras, see [55]); Dispin algebra $U(osp(1, 2))$; Woronowicz algebra $W_v(\mathfrak{sl}(2, \Bbbk))$; the complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$; q-Heisenberg algebra $\mathbf{H}_n(q)$; the Hayashi algebra $W_q(J)$, and more.

(b) We also encounter examples of σ-PBW extensions (which are not constant) over (Σ, Δ)-compatible rings. Let us see: (i) the quantum plane $O_q(\Bbbk^2)$; the algebra of q-differential operators $D_{q,h}[x, y]$; the mixed algebra D_h; the operator differential rings; the algebra of differential operators $D_{\mathbf{q}}(S_{\mathbf{q}})$ on a quantum space $S_{\mathbf{q}}$, and more.

(c) Several algebras of quantum physics can be expressed as skew PBW extensions: Weyl algebras, additive and multiplicative analogue of the Weyl algebra, quantum Weyl algebras, q-Heisenberg algebra, and others. See [54] or [59] for a detailed list of examples.

Proposition 3.4 shows that (Σ, Δ)-compatible rings are a generalization of Σ-rigid rings introduced in [49], Definition 3.2.

Proposition 3.4 ([22], Lemma 3.5; [57], Proposition 3.4)**.** *Let Σ be a family of endomorphisms of a ring R, and let Δ be a family of Σ-derivations of R. If R is Σ-rigid, then R is (Σ, Δ)-compatible.*

The following example illustrates that the converse of Proposition 3.4 is false.

Example 3.5 ([26], Example 2.2)**.** Let δ be a σ-derivation of B, where B is a σ-rigid ring. Consider

$$B_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in B \right\},$$

the subring of the upper triangular matrix $T_3(B)$. The endomorphism σ of B is extended to the endomorphism $\overline{\sigma} : B_3 \to B_3$ defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of B is also extended to $\overline{\delta} : B_3 \to B_3$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. Then $\overline{\delta}$ is a $\overline{\sigma}$-derivation of B_3, and we have the following facts: B_3 is a $(\overline{\sigma}, \overline{\delta})$-compatible ring, (ii) B_3 is not $\overline{\sigma}$-rigid.

Next, we investigate some key properties of (Σ, Δ)-compatible rings.

Proposition 3.6 ([22], Lemma 3.3; [57], Proposition 3.8)**.** *Let R be a (Σ, Δ)-compatible ring. For every $a, b \in R$, we have:*

(i) *If $ab = 0$, then $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$, for each $\theta \in \mathbb{N}^n$.*
(ii) *If $\sigma^{\beta}(a)b = 0$ for some $\beta \in \mathbb{N}^n$, then $ab = 0$.*
(iii) *If $ab = 0$, then $\sigma^{\theta}(a)\delta^{\beta}(b) = \delta^{\beta}(a)\sigma^{\theta}(b) = 0$, for every $\theta, \beta \in \mathbb{N}^n$.*

As we saw before, Σ-rigid rings are contained strictly in (Σ, Δ)-compatible rings. Nevertheless, Proposition 3.7 shows the importance of reduced rings in the equivalence of both families of rings.

Proposition 3.7 ([22], Lemma 3.5; [57], Theorem 3.9)**.** *If A is a skew PBW extension of a ring R, then the following statements are equivalent: (i) R is reduced and (Σ, Δ)-compatible. (ii) R is Σ-rigid. (iii) A is reduced.*

Lemma 3.8 ([22], Lemma 3.6; [57], Lemma 3.11)**.** *Let A be a skew PBW extension of a (Σ, Δ)-compatible ring R. If $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$, $r \in R$, and $fr = 0$, then $a_ir = 0$, for every i.*

For the next proposition, we assume that the elements $d_{i,j} \in R$ in Definition 2.1 (iv) are central in R.

Proposition 3.9 ([50], Theorem 2.11)**.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of a reversible and (Σ, Δ)-compatible ring R, then for every element $f = \sum_{i=0}^{m} a_iX_i \in A$, $f \in \mathrm{Nil}(A)$ if and only if $a_i \in \mathrm{Nil}(R)$, for each $1 \leq i \leq m$.*

Proposition 3.10. *If R is a reversible ring which is (Σ, Δ)-compatible, and e is an idempotent element of R, then $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for every $i = 1, \dots, n$.*

Remark 3.11. The notion of compatibility has been very useful in the study of different ring theoretical properties of skew PBW extensions, for example see [51], [52], [59] and [60].

4. Units

In this section we characterize the units of a skew PBW extension over a right duo (Σ, Δ)-compatible ring (see [19], [25], [52] and [56] for some classes of rings which satisfy these conditions in the context of Ore extensions and skew PBW extensions, respectively). With this in mind, we establish analogue results to the obtained by the first two authors in [21] for the case of Ore extensions.

Proposition 4.1. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a (Σ, Δ)-compatible ring R. If $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$ and $c, r \in R$, then $fr = c$ if and only if $a_0r = c$ and $a_ir = 0$, for every $1 \leq i \leq n$.*

Proof. The case $m = 0$ is clear. Consider $fr = c$, with $m \geq 1$. For the expression $f = a_0 + a_1X_1 + \cdots + a_mX_m$, the equality $fr = c$ implies that $a_m\sigma^{\alpha_m}(r) = 0$, since $a_mX_mr = a_m[\sigma^{\alpha_m}(r)X_m + p_{\alpha_m,r}]$, and thus $a_mr = 0$, by the Σ-compatibility of R. Now, Proposition 3.6 (3) guarantees that $a_mX_mr = 0$. Induction on m gives us $a_ir = 0$, for $1 \leq i \leq m$ and so $a_iX_ir = 0$, for $i = 0, 1, \dots, m$.

The converse follows from Proposition 3.6. □

Proposition 4.2. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a Σ-rigid ring R. If f, g are nonzero elements of A with $fg = c \in R$, given by $f = \sum_{i=0}^{m} a_iX_i$ and $g = \sum_{j=0}^{t} b_jY_j$, respectively, then $a_0b_0 = c$ and $a_ib_j = 0$, for every i, j with $i + j \geq 1$.*

Proof. We use induction on the sum $m + t$. If $m = 0$ or $t = 0$, then the assertion holds from Proposition 4.1. Let $m, t \geq 1$. Suppose that the result is true for all the smaller values than $m + t$. If we consider the expression for fg, then we can see that $a_m\sigma^{\alpha_m}(b_t) = 0$, and so $a_mb_t = 0 = b_ta_m$, since R is Σ-compatible and reduced. Then, $b_tc = b_tfg = (b_ta_0 + b_ta_1X_1 + \cdots + b_ta_{m-1}X_{m-1})g = f_1g$. Since $m - 1 + t \leq m + t - 1$, the induction hypothesis implies that $b_ta_0b_t = \cdots = b_ta_{m-1}b_t = 0$ whence $a_0b_t = \cdots = a_{m-1}b_t = 0$ (R is reduced). In this way, $c = fg = f(b_0 + b_1X_1 + \cdots + b_{t-1}X_{t-1}) = fg_1$. Again, the induction hypothesis guarantees that $a_0b_0 = c$ and $a_ib_j = 0$, for $1 \leq i + j \leq m + t$, which concludes the proof. □

Proposition 4.3. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a semicommutative (Σ, Δ)-compatible ring R, and let $f = a_0+a_1X_1+\cdots+a_mX_m$, $g = b_0+b_1Y_1+\cdots+b_tY_t \in A$ be nonzero elements of A with $fg = c \in R$ and $a_m, b_t \neq 0$. If $t \geq 1$, then there exists $s \geq 1$ such that $fb_t^s = 0$.*

Proof. Note that if $m = 0$, then $f = a_0$, and using that $fg = c$, it follows that $a_0b_0 = c$ and $a_0b_j = 0$, for every $j \geq 1$, and so $fb_m = 0$. Consider $m \geq 1$. Again, since $fg = c$, we obtain $a_m\sigma^{\alpha_m}(b_t) = 0$, whence $a_mb_t = 0$ (Proposition 3.6). Hence, $cb_t = fgb_t = (a_0 + a_1X_1 + \cdots + a_{m-1}X_{m-1})gb_t$, by the semicommutativity of R and Remark 2.6 (ii). Consider two cases:

Case 1. If $b_t\sigma^{\alpha_t}(b_t) = 0$, then $b_t^2 = 0$, whence $fb_t^2 = 0$.

Case 2. If $b_t\sigma^{\alpha_t}(b_t) \neq 0$, then $gb_t \neq 0$, and using induction hypothesis there exists $r_1 \geq 1$ with $(a_0 + a_1X_1 + \cdots + a_{m-1}X_{m-1})(b_t\sigma^{\alpha_t}(b_t))^{r_1} = 0$. Therefore, $f(b_t\sigma^{\alpha_t}(b_t))^{r_1} = 0$, whence $fb_t^{2r_1} = 0$ (Lemma 4.1). Taking $s = 2r_1$ is the desired value. □

Proposition 4.4. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a semicommutative (Σ, Δ)-compatible ring R, and let $f = a_0+a_1X_1+\cdots+a_mX_m$, $g = b_0+b_1Y_1+\cdots+b_tY_t \in A$ be nonzero elements of A with $fg = c \in R$ and $a_m, b_t \neq 0$. If $m \geq 1$, then there exists $s \geq 1$ such that $a_m^sg = 0$.*

Proof. The proof uses a similar argument to the established in the proof of Proposition 4.3. □

For the next theorem, the first important result of the paper, let I_g be the right ideal of R generated by the coefficients of nonzero elements of a skew PBW extension A.

Theorem 4.5. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a right duo (Σ, Δ)-compatible ring R. If $f = \sum_{i=0}^{m} a_i X_i$ and $g = \sum_{j=0}^{t} b_j Y_j$ are nonzero elements of A such that $fg = c \in R$, then there exist a nonzero element $r \in I_g$ and an element $a \in R$ with $fr = ca$. In the case that b_0 is a unit in R, then $a_1, a_2, \ldots, a_m$ are nilpotent.*

Proof. Note that if $f = a_0$ then $a_0 g = c$, which implies that $a_0 b_0 = c$ and $a_0 b_j = 0$, for every $j \geq 1$. Hence, $r = b_0$ and $a = 1$. Consider $f = \sum_{i=0}^{m} a_i X_i$ with $i \geq 1$. Let us prove the assertion by using induction on i. If $t = 0$, then $g = b_0 \neq 0$. Since $fg = (a_0 + a_1 X_1 + \cdots + a_m X_m) b_0 = c$, we obtain that $a_0 b_0 = c$ and $a_i b_0 = 0$, for every $i \geq 1$ (Proposition 4.1). If this is the case, $r = b_0$ and $a = 1$ as we wish. Suppose that $t \geq 1$ and that the assertion is true for all polynomials of degree less than t. We consider two cases:

Case 1: If $a_m X_m g = 0$, then $a_m b_j = 0$, for each $0 \leq j \leq m$ by Proposition 4.1, and so $a_m I_g = 0$. This means that $c = fg = (a_0 + a_1 X_1 + \cdots + a_{m-1} X_{m-1}) g$, and so there is $0 \neq r \in I_g$ and $a_1 \in R$ such that $(a_0 + a_1 X_1 + \cdots + a_{m-1} X_{m-1}) r_1 = c a_1$. Since $a_m X_m r_1 = 0$, it follows that $f r_1 = c a_1$. Hence $r = r_1$ and $a_1 = a$, as we wanted.

Case 2: Let $a_m X_m g \neq 0$. Then there is $0 \leq j < m$ such that $a_m b_j \neq 0$ (j is the greatest). By Proposition 4.4, there exist $t \geq 2$ such that $a_m^s b_j = 0$, but $a_m^{s-1} b_j \neq 0$. As R is right duo, there is $b \in R$ such that $a_m^{s-1} b_j = b_j b$. Consider $g_1 = gb$. Then $fg_1 = fgb = cb$ and $\langle 0\rangle \neq I_{g_1} \subseteq I_g$. Now $a_m X_m$ annihilates coefficients of g_1 from j-th to t-th. So, after repeating this process a finite number of times, it is concluded that there are $0 \neq k \in A$ and $r_1 \in R$ such that $\deg(k) < t$, $fh = cr_1$ and $a_m X_m k = 0$. Now, the result follows from the Case 1.

Let b_0 be a unit in R. We will prove that $a_1, a_2, \ldots, a_m$ are all nilpotent. Since R is right duo, R is semicommutative, and so $\mathrm{Nil}(R)$ is an ideal of R, by [40], Lemma 3.1. Thus $\overline{R} = R/\mathrm{Nil}(R)$ is a reduced ring. Since R is (Σ, Δ)-compatible, $\mathrm{Nil}(R)$ is a (Σ, Δ)-compatible ideal of R, by Proposition 3.6 (see [23], Definition 3.1 for the notion of (Σ, Δ)-compatible ideal). Hence $\overline{R}$ is $\overline{\Sigma}$-rigid, as one can check using a similar reasoning to the used in [20], Proposition 2.1 and having in mind Proposition 3.7. In this way, by Proposition 2.8, $\overline{A} := A/IA$ is a skew PBW extension of $R/\mathrm{Nil}(R)$. Since $fg = c \in R$, it follows that $\overline{f}\overline{g} = \overline{c}$ in $\overline{A}$. Thus

$\overline{a_0}\overline{b_0} = \overline{c}$ and $\overline{a_i}\overline{b_j} = 0$, for each $i+j \geq 1$, by Proposition 4.1. Hence $\overline{a_i} = 0$, for each $i \geq 1$, since $\overline{b_0}$ is a unit. Therefore a_i is nilpotent, for each $i \geq 1$ as desired. □

Proposition 4.6. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring R, then we obtain that* $\mathrm{Nil}(R)A = L{-}\mathrm{rad}(A) = \mathrm{Nil}(A)$.

Proof. It is well-known that a right duo ring is 2-primal, so we have that R is 2-primal. In this way, A is a 2-primal ring ([41], Corollary 3.10). Now, since every 2-primal ring is a weakly 2-primal ring, then $\mathrm{Nil}(A) = L{-}\mathrm{rad}(A)$. Again, using [41], Corollary 3.10, we have $\mathrm{Nil}(R)A = \mathrm{Nil}(A)$, since R is a 2-primal (Σ, Δ)-compatible ring. Hence, $\mathrm{Nil}(R)A = L - \mathrm{rad}(A)$. □

The next theorem is the second important result of the paper. As we will see, it establishes a relation between units of a right duo (Σ, Δ)-compatible ring R and a skew PBW extension over R.

Theorem 4.7. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring R, then an element $f = \sum_{i=0}^{m} a_i X_i \in A$ is a unit of A if and only if a_0 is a unit and a_i is nilpotent, for $i = 1, \dots, m$.*

Proof. Consider $\overline{R} = R/\mathrm{Nil}(R)$. Proposition 3.7 guarantees that $\overline{R}$ is $\overline{\Sigma}$-rigid, and it is clear that $\overline{R}$ is right duo. Conversely, let $f = \sum_{i=0}^{m} a_i X_i$ be a unit of A. There exists an element $g \in A$ with $fg = gf = 1$, whence $\overline{f}\overline{g} = 1$. Lemma 4.2 guarantees that $\overline{a_0}$ is a unit element of $\overline{R}$, while Theorem 4.5 establishes that the elements $a_1, \dots, a_m$ are nilpotent. Having in mind that $\mathrm{Nil}(R) \subseteq J(A)$ (Proposition 4.6), it follows that $a_0 \in U(R)$.

Conversely, let a_0 be a unit element and $a_1, \dots, a_m$ be nilpotent elements of R. Proposition 4.6 shows that the element $a_1X_1 + \cdots + a_mX_m$ belongs to $\mathrm{Nil}(R)A = L - \mathrm{rad}(A)$, and from [36], Lemma 10.32, we know that $L - \mathrm{rad}(A) \subseteq J(A)$, and so $a_1X_1 + \cdots + a_mX_m \in J(A)$. This shows that the element $f = a_0 + a_1X_1 + \cdots + a_mX_m$ is a unit element of A and hence the proof ends. □

Before we state Corollary 4.8, we consider the following: if $A = \sigma(R)\langle x_1, \dots, x_n\rangle$, and S is a subset of R, then SA will denote the set of elements of A with coefficients in S, that is,

$$SA = \{a_0 + a_1X_1 + \cdots + a_mX_m \in A \mid a_i \in S, \text{ for all } 1 \leq i \leq n\}.$$

Proposition 4.6 and Theorem 4.7 imply the following result.

Corollary 4.8. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring R, then* $U(A) = U(R) + \mathrm{Nil}(A)X = U(R) + (\mathrm{Nil}(R)A)X$.

Our purpose in this section is to determinate several relations between the idempotent, von Neumann regular and local, and clean elements of a right duo (Σ, Δ)-compatible ring R and those elements corresponding of a skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$. As in Section 4, we follow the ideas presented by the first two authors in [21] for the context of Ore extensions (see also [7]).

Before, we recall that if I is a nil ideal in a ring B (i.e., $I \subseteq J(B)$), and $a \in B$ is such that $\overline{a} \in \overline{B} = B/I$ is an idempotent element, then there exists an idempotent element $e \in aB$ with $\overline{e} = \overline{a} \in \overline{B}$, by [36], Theorem 21.28 (e.g., [32]).

Proposition 4.9. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring R, and $f = \sum_{i=0}^{m} a_iX_i$ is an idempotent element of A, then $a_i \in \mathrm{Nil}(R)$, for every i, and there exists an idempotent element $e \in R$ such that $\overline{a_0} = \overline{e}$ in $R/\mathrm{Nil}(R)$.*

Proof. Since R is a right duo and (Σ, Δ)-compatible ring, $\mathrm{Nil}(R)$ is a (Σ, Δ)-compatible ideal of R. Hence, by [22], Lemma 3.5 or [57], Theorem 3.9, $R/\mathrm{Nil}(R)$ is a $\overline{\Sigma}$-rigid ring. Now, by Proposition 2.8, $\overline{A} := A/\mathrm{Nil}(R)A$ is a skew PBW extension of $R/\mathrm{Nil}(R)$. Since $f^2 = f \in A$, it follows that $\overline{f}^2 = \overline{f} \in \overline{A}$. Using that $\overline{R}$ is $\overline{\Sigma}$-rigid, we obtain that R/I is Σ-skew Armendariz, by [53], Proposition 3.4. Now, having in mind that R/I is $(\overline{\Sigma}, \overline{\Delta})$-compatible, we can conclude that $\overline{a_i} = \overline{0}$, for each $1 \leq i \leq n$, and $\overline{a_0^2} = \overline{a_0}$. It means that $\overline{f} = \overline{a_0} \in R/\mathrm{Nil}(R)$. Therefore, [36], Theorem 21.28 implies that $\overline{a_0} = \overline{e} \in R/\mathrm{Nil}(R)$, for some idempotent $e \in B$. $\square$

The next assertion is the third important result of the paper.

Theorem 4.10. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring R, and $f = \sum_{i=0}^{m} a_iX_i$ is an idempotent element of A, then $f = a_0$.*

Proof. By Proposition 4.9, $a_i \in \mathrm{Nil}(R)$, for each $1 \leq i \leq n$, and there is an idempotent element $e \in R$ and a nilpotent element $w \in R$ such that $a_0 = e + w$. Assume that $f = e + f'$, where $f' = w + a_1X_1 + \cdots + a_mX_m$. Thus, $f' \in \mathrm{Nil}(A)$, by Proposition 3.9. Since e is an idempotent element of R, and R is right duo (Σ, Δ)-compatible, it follows that $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for every $i = 1, \dots, n$. Hence, by a similar way as used in the proof of [22], Theorem 3.3, one can prove that $f' = 0$ and so $f = a_0$ is an idempotent element of A. This completes the proof. $\square$

From results above we have immediately the following assertions.

Corollary 4.11. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring R, then* $\mathrm{Idem}(A) = \mathrm{Idem}(R)$.

Corollary 4.12. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring R, then A is an Abelian ring.*

For the next result, Theorem 4.14, which is the fourth important result of the paper, before we need the following facts about Abelian rings.

Proposition 4.13 ([21], Proposition 4.2)**.** *Let B be an Abelian ring and $a \in B$. Then the following statements are equivalent:*

(1) $a \in \mathrm{vnr}(B)$;
(2) $ava = a$, *for some* $v \in U(B)$;
(3) $a = ve$, *for some* $v \in U(B)$ *and* $e \in \mathrm{Idem}(B)$;
(4) $ab = 0$, *for some* $b \in \mathrm{vnr}(B) \setminus \{0\}$, *with* $a + b \in U(B)$;
(5) $ab = 0$, *for some* $b \in B$, *with* $a + b \in U(B)$.

Theorem 4.14. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring, then $\mathrm{vnr}(A)$ consists of the elements of the form $\sum_{i=0}^{m} a_iX_i$ where $a_0 = ue$, $a_i \in e(\mathrm{Nil}(R))$, for every $i \geq 1$, some $u \in U(R)$ and $e \in \mathrm{Idem}(R)$.*

Proof. From Corollary 4.12 we know that A is an Abelian ring, while Proposition 4.13 establishes that $\mathrm{vnr}(A) = \{fe \mid f \in U(A)\}$. Now, using that $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for $i = 1, \dots, n$, where e is an idempotent element of R, Corollaries 4.8 and 4.11 guarantee the result. □

Theorem 4.15 is the fifth important result of the paper.

Theorem 4.15. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a right duo (Σ, Δ)-compatible ring, then*

$$\pi - r(A) = \left\{\sum a_iX_i \in A \mid a_0 \in \pi - r(R), a_i \in \mathrm{Nil}(R),\ \text{for } i \geq 1\right\}.$$

Proof. By assumption R is a right duo ring, so Corollary 4.12 implies that A is an Abelian ring. Now, Proposition 4.6 establishes that $\mathrm{Nil}(R)A = L{-}\mathrm{rad}(A) = \mathrm{Nil}(A)$, which means that $\mathrm{Nil}(A)$ is a two-sided ideal of A. In this way, Theorem 4.14 and

[21], Corollary 5.6 (1) show that

$$\begin{aligned}\pi-r(A) &= \mathrm{vnr}(A)+\mathrm{Nil}(A)\\ &= \Big\{\sum a_iX_i+\sum b_jY_j \mid a_0=ue, a_i\in e(\mathrm{Nil}(R)),\ \text{for every } i\geq 1,\\ &\quad b_j\in \mathrm{Nil}(R),\ \text{for every } j\geq 0,\ \text{for some } u\in U(R) \text{ and } e\in \mathrm{Idem}(R)\Big\}\\ &= \Big\{\sum a_iX_i\in A \mid a_0=ue+w,\ \text{for some } u\in U(R), e\in \mathrm{Idem}(R),\\ &\quad w\in \mathrm{Nil}(R); a_i\in \mathrm{Nil}(R),\ \text{for every } i\geq 1\Big\}\\ &= \Big\{\sum a_iX_i\in A \mid a_0\in \pi-r(R), a_i\in \mathrm{Nil}(R),\ \text{for every } i\geq 1\Big\}\end{aligned}$$

which concludes the proof. □

Next, we characterize von Neumann local elements of a skew PBW extension over a right duo ring R which is (Σ,Δ)-compatible. Our Theorem 4.16 is the sixth important result of the paper.

Theorem 4.16. *If $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension over a right duo (Σ,Δ)-compatible ring, then $\mathrm{vnl}(A)$ consists of the elements of the form $\sum_{i=0}^m a_iX_i$, where either $a_0=ue$ or $a_0=1-ue$, $a_i\in e(\mathrm{Nil}(R))$, for every $i\geq 1$, some element $u\in U(R)$ and $e\in \mathrm{Idem}(R)$.*

Proof. It follows from Theorem 4.14 and [21], Theorem 6.1 (2). □

The last theorem of the paper characterizes clean elements of skew PBW extensions over right duo (Σ,Δ)-compatible rings.

Theorem 4.17. *If $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension over a right duo (Σ,Δ) compatible ring, then*

$$\mathrm{cln}(A)=\Big\{\sum a_iX_i\in A \mid a_0\in \mathrm{cln}(R) \text{ and } a_i\in \mathrm{Nil}(R), \text{for } i\geq 1\Big\}.$$

Proof. The result follows from Proposition 4.6 and Corollaries 4.8 and 4.11. □

Example 4.18. Remarkable examples of skew PBW extensions over right duo (Σ,Δ)-compatible rings can be found in [53], [54], [56] and [64]. In this way, the results obtained in Sections 4 and 5 can be illustrated with every one of these noncommutative rings. More precisely, if A is a skew PBW extension over a reduced ring R where the coefficients commute with the variables, that is, $x_ir=rx_i$, for every $r\in R$ and each $i=1,\dots,n$, or equivalently, $\sigma_i=\mathrm{id}_R$ and $\delta_i=0$, for every i

(these extensions were called *constant* in [56], Definition 2.6 (a)), then it is clear that R is a Σ-rigid ring. Some examples of these extensions are the following: (i) PBW extensions defined by Bell and Goodearl (which include the classical commutative polynomial rings, universal enveloping algebra of a Lie algebra, and others); some operator algebras (for example, the algebra of linear partial differential operators, the algebra of linear partial shift operators, the algebra of linear partial difference operators, the algebra of linear partial q-dilation operators, and the algebra of linear partial q-differential operators) (ii) Solvable polynomial rings introduced by Kandri-Rody and Weispfenning (iii) 3-dimensional skew polynomial algebras (e.g., [55] and [61]) (iv) Some of the G-algebras introduced by Apel (v) Some PBW algebras defined by Bueso et. al. (vi) Some Calabi-Yau and skew Calabi-Yau algebras (vii) Some Koszul and quadratic algebras. A detailed reference of every one of these algebras can be found in [39], [58] and [63]. Of course, we also encounter examples of skew PBW extensions which are not constant (see [39] for the definition of each one of these algebras): the quantum plane $O_q(\mathbb{k}^2)$; the Jordan plane; the algebra of q-differential operators $D_{q,h}[x,y]$; the mixed algebra D_h; the operator differential rings and the algebra of differential operators $D_{\mathbf{q}}(S_{\mathbf{q}})$ on a quantum space $S_{\mathbf{q}}$. Last, but not least, our results can also be applied to the noncommutative rings considered by Artamonov et al., [6].

5. Future work

The notion of (σ,δ)-compatibility has been considered in the study of modules over Ore extensions (e.g., [1] and [3]). For instance, in [1] the authors introduced the notions of skew-Armendariz modules and skew quasi-Armendariz modules which are generalizations of σ-Armendariz modules and extend the classes of non-reduced skew-Armendariz modules. They obtained different properties of modules over the ring of coefficients and the corresponding Ore extension. Now, recently, the notion of (Σ,Δ)-compatibility in the context of modules over skew PBW extensions has been considered by the third author in [51] with the aim of obtaining similar results to those established in [1] for Ore extensions, and also in [46] with the purpose of characterizing the associated prime ideals over these extensions generalizing the treatment developed in [3] for Ore extensions. Having this in mind and considering the results obtained in this paper, we think as a future work to investigate a classification of several types of elements in modules over these extensions.

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