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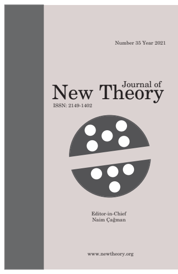

Semi-Invariant Riemannian Submersions with Semi-Symmetric Non-Metric Connection

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Abstract — In this paper, we investigate semi-invariant Riemannian submersion from a Kaehler manifold with semi-symmetric non-metric connection to a Riemannian manifold. We study the geometry of foliations with semi-symmetric non-metric connection. Later, we introduce base manifold to be a local product manifold with semi-symmetric non-metric connection.

Keywords — *Riemannian submersions, semi-invariant submersions, semi-symmetric non-metric connection, Kaehler manifold*

Mathematics Subject Classification (2020) — 53C15, 53C40

1. Introduction

A conventional way to compare two manifolds is by defining smooth maps from one manifold to another. One such map is submersion, whose rank equals to the dimension of the target manifold. Riemannian submersion between Riemannian submanifolds were first introduced by O' Neill and Gray [1, 2]. Later many authors studied different geometric properties of the Riemannian submersions [3], semi-slant submersions [4–6], hemi-slant submersions [7–9], semi-invariant submersions [10–12], anti-invariant submersions [13–15].

On the other hand, Friedmann et al. defined the concept of the semi-symmetric non-metric connection in a differential manifold [16]. Hayden studied metric connection with torsion a Riemannian manifold [17]. Later, Yano investigated a Riemannian manifold with new connection, which is called a semi-symmetric metric connection [18]. Afterwards, Agashe et al. studied semi-symmetric non-metric connection (SSNMC) on a Riemannian manifold [19]. Many author have studied semi-symmetric connection [20–26].

Let M be differentiable manifold with linear connection ∇. Therefore, for all $K, L \in \Gamma(TN)$, we get

$$T(K,L) = \nabla_K L - \nabla_L K - [K,L],$$

where T is torsion tensor of ∇. If the torsion tensor $T = 0$, then the connection ∇ is said to be symmetric, otherwise it is called non-symmetric. Moreover, for all $K, L \in \Gamma(TN)$, the connection ∇ is said to be semi-symmetric if

$$T(K,L) = \eta(L)K - \eta(K)L$$

where η is a 1-form on N. However, ∇ is called metric connection if $\nabla g = 0$ with Riemannian metric g, otherwise it is said to be non-metric.

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In [27], Akyol and Beyendi studied the idea of Riemannian submersion with SSNMC. They investigated O'Neill's tensor fields, obtain derivatives of those tensor fields and compare curvatures of the total manifold, the base manifold and the fibres by computing curvatures.

The main purpose of this paper is to investigate geometry of semi-invariant Riemannian submersion from a Kaehler manifold with SSNMC to a Riemannian manifold.

2. Preliminaries

Definition 2.1. Let $F : (N^n, g_N) \to (B^b, g_B)$ be a submersion between two Riemannian manifolds. Then, F said to be Riemannian submersion if

i. F has maximal rank.

ii. The differential F_* preserves the lenghts of horizontal vectors.

On the other hand, $F^{-1}(k)$ is an $(n-b)$ dimensional submanifold of N, for each $k \in N$ The submanifolds $F^{-1}(k)$ are called fibers. Moreover, vector fields tangent to fibers are called vertical and vector fields orthogonal to fibers are horizantal. A vector field X on N is called basic if X is horizontal and $F_* X_q = X_{\pi_*(q)}$ for all $q \in N$. We determine that $\mathcal{V}$ and $\mathcal{H}$ define projections $kerF_*$ and $(kerF_*)^\perp$, respectively.

On the other hand, a Riemannian submersion $F : N \to B$ determines tensor fields T and A on N such that,

$$T(E,F) = T_E F = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{H}F, \tag{1}$$

$$A(E,F) = A_E F = \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{V}F \tag{2}$$

for any $E, F \in \Gamma(TM)$ (see [1]). By virtue of (1) and (2), one can obtain

$$\overset{M}{\nabla}_V W = T_V W + \hat{\nabla}_V W \tag{3}$$

$$\overset{M}{\nabla}_V X = T_V X + \mathcal{H}(\overset{M}{\nabla}_V X) \tag{4}$$

$$\overset{M}{\nabla}_X V = \mathcal{V}(\overset{M}{\nabla}_X V) + A_X V \tag{5}$$

$$\overset{M}{\nabla}_X Y = A_X Y + \mathcal{H}(\overset{M}{\nabla}_X Y) \tag{6}$$

for all $V, W \in \Gamma(kerF_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$. Further, if X is basic, then

$$\mathcal{H}(\overset{M}{\nabla}_V X) = A_X V \tag{7}$$

On the other hand, let N, B be two Rieamannian manifold and $F : N \to B$ is a smooth map. Therefore, the second fundamental form of F is expressed by

$$(\nabla F_*)(K,L) = \overset{B}{\nabla}_K F_* L - F_*(\overset{N}{\nabla}_K L) \tag{8}$$

for $K, L \in \Gamma(TN)$. Moreover, π is said to be a *totally geodesic* map if $(\nabla F_*)(K,L) = 0$ for $K, L \in \Gamma(TN)$ [28].

Now, we recall the definition of Kaehler manifold. Let N be a Hermitian manifold with respect Hermitian structure (J, g) such that

$$J^2 = -I \tag{9}$$

and

$$g(E,F) = g(JE, JF) \tag{10}$$

for all $E, F \in \Gamma(TN)$,where $g(JE,F) = -g(E,JF)$.

A Hermitian manifold is called Kaehler manifold if

$$\nabla J = 0 \tag{11}$$

On the other hand, we define a linear connection $\widetilde{\nabla}$ on Kaehler manifold N such that

$$\widetilde{\nabla}_E F = \nabla_E F + \eta(F)E \tag{12}$$

where $E, F \in \Gamma(TN)$, ∇ is a Levi-Civita connection on N and η is a 1-form with the vector field P on N by

$$\eta(E) = g(E, P)$$

By virute of (12), we arrive that

$$\widetilde{T}(E, F) = \eta(F)E - \eta(E)F$$

and

$$(\widetilde{\nabla}_E g)(F, K) = -\eta(F)g(E, K) - \eta(K)g(E, F)$$

where $\widetilde{T}$ is torsion tensor of $\widetilde{\nabla}$. Then, $\widetilde{\nabla}$ defined a semi-symmetric non metric conection with (12).

Let N be a Kaehler manifold. We using (12), for all $K, L \in \Gamma(TN)$, we get,

$$\begin{aligned}(\nabla_K J)L &= \nabla_K JL - J\nabla_K L \\ &= \widetilde{\nabla}_K JL - \eta(L)K - J\widetilde{\nabla}_K L + \eta(L)JK\end{aligned}$$

Then, using (11) we obtain,

$$(\widetilde{\nabla}_K J)L = \eta(L)JK - \eta(L)K \tag{13}$$

Now, we call O'Neill's tensor fields for SSNMC [27]. For all $K, L \in \Gamma(TN)$, we have,

$$\widetilde{T}_K L = T_K L + \eta(hL)vK$$

and

$$\widetilde{A}_K L = A_K L + \eta(vL)hK$$

Then, using last two equations, we obtain

$$\widetilde{\nabla}_K L = T_K L + v\widetilde{\nabla}_K L \tag{14}$$

$$\widetilde{\nabla}_K X = T_K X + h\widetilde{\nabla}_K X + \eta(X)K \tag{15}$$

$$\widetilde{\nabla}_X K = A_X K + v\widetilde{\nabla}_X K + \eta(K)X \tag{16}$$

$$\widetilde{\nabla}_X Y = A_X Y + h\widetilde{\nabla}_X Y \tag{17}$$

where for all $K, L \in \Gamma(kerF_*), X, Y \in \Gamma((\ker F_*)^\perp)$.

3. Semi-Invariant Riemannian Submersion

Definition 3.1. Let N and B be a Kaehler manifold and Riemannian manifold, respectively. Let us assume that $F : N \to B$ be a Riemannian submersion. Therefore, F is called semi-invariant Riemannian submersion if there is a distribution $D_1 \subseteq kerF_*$ such that

$$kerF_* = D_1 \oplus D_2$$

and

$$JD_1 = D_1, \quad JD_2 \subseteq (\ker F_*)^\perp$$

where D_2 is orthogonal complementary to D_1 in $kerF_*$ ([12]).

Example 3.2. Let F be a submersion. We denote that

$$F: \begin{array}{ccc} \mathbb{R}^6 & \longrightarrow & \mathbb{R}^3 \\ (x_1, x_2, x_3, x_4, x_5, x_6) & & (\frac{x_1+x_2}{\sqrt{2}}, \frac{x_3+x_6}{\sqrt{2}}, \frac{x_4+x_5}{\sqrt{2}}) \end{array}$$

Then, it follows that

$$kerF_* = span\{V_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2},\ V_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_6}, V_3 = -\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}\}$$

and

$$(kerF_*)^{\perp} = span\{H_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},\ H_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6},\ H_3 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$$

Hence we have $JV_1 = -V_1$, $JV_2 = H_3$ and $JV_3 = -H_2$. Thus it follows that $D_1 = span\{V_1\}$ and $D_2 = span\{H_2, H_3\}$. On the other hand, we arrive that,

$$g_{\mathbb{R}^3}(F_*H_2, F_*H_2) = g_{\mathbb{R}^6}(H_2, H_2), \quad g_{\mathbb{R}^3}(F_*H_3, F_*H_3) = g_{\mathbb{R}^6}(H_3, H_3)$$

where $g_{\mathbb{R}^3}$ and $g_{\mathbb{R}^6}$ determine metrics of $\mathbb{R}^3$ and $\mathbb{R}^6$, respectively. Then, F is semi-invariant Riemannian submersion.

Let $F : (N, J, g) \to (B, g)$ be a semi-invariant Riemannian submerion such that N and B are Kaehler manifold and Riemannian manifold respectively. For all $K \in \Gamma(TN)$, we write

$$E = \mathcal{V}E + \mathcal{H}E$$

where $\mathcal{V}E \in \Gamma(kerF_*)$ and $\mathcal{H}E \in \Gamma((\ker F_*)^{\perp})$. Then, for all $K \in \Gamma(kerF_*)$, we write

$$JK = \phi K + \omega K \tag{18}$$

where $\phi K \in \Gamma(D_1)$ and $\omega K \in \Gamma(JD_2)$.

Since F is a semi-invariant Riemannian submersion, we can determine

$$(\ker F_*)^{\perp} = JD_2 \oplus \mu$$

where JD_2 and μ are complementary to each other. Similarly, $x \in \Gamma((\ker F_*)^{\perp})$, we get

$$JX = BX + CX \tag{19}$$

where $BX \in \Gamma(D_2)$ and $CX \in \Gamma(\mu)$.

4. Geometry of Distributions

We note that, for brevity we use a abbreviation "F is a semi-invariant Riemannian submersion with SSNMC" for $F : (N, J, g) \to (B, g)$ be a semi-invariant Riemannian submersion from Kaehler manifold with SSNMC M and Riemannian manifold N.

Theorem 4.1. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution D_1 is integrable if and only if we have

$$\begin{aligned} g_B(F_*(T_V BZ + h\widetilde{\nabla}_V CZ), F_*(\omega U)) - g_B(F_*(T_U BZ + h\widetilde{\nabla}_U CZ), F_*(\omega V)) &= g_N(v\widetilde{\nabla}_U BZ + T_U CZ, \phi V) \\ &\quad - g_N(v\widetilde{\nabla}_V BZ + T_V CZ, \phi U) \\ &\quad + 2g_N(\phi U, V)\eta(Z) \end{aligned}$$

for all $U, V \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

PROOF. Firstly, we using (10) and (13). For all $U, V \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we arrive that

$$g_N(\widetilde{\nabla}_U V, Z) = g_N(\widetilde{\nabla}_U JV, JZ) \tag{20}$$

By virtue of (13) and (20), we get,

$$g_N([U,V], Z) = g_N(\widetilde{\nabla}_U JV, JZ) - g_N(\widetilde{\nabla}_V JU, JZ)$$

After some calculations, we conclude,

$$g_N([U,V], Z) = -g_N(JV, \widetilde{\nabla}_U JZ) + g_N(JU, \widetilde{\nabla}_V JZ)$$

We know that F is a semi-invariant Riemannian submersion, by virtue of (19), (14), (15) and (18), we conclude that,

$$\begin{aligned} g_N([U,V], Z) \;=\; & -g_N(\phi V, v\widetilde{\nabla}_U BZ + T_U CZ) - g_N(wV, T_U BZ + h\widetilde{\nabla}_U CZ) - \eta(CZ) g_N(JV, U) \\ & + g_N(\phi U, v\widetilde{\nabla}_V BZ + T_V CZ) + g_N(wU, T_V BZ + h\widetilde{\nabla}_V CZ) + \eta(CZ) g_N(JU, V) \end{aligned}$$

which gives proof. □

Theorem 4.2. Let F be a semi-invariant Riemannian submersion with SSNMC . Therefore, the distribution D_2 is integrable if and only if we have

$$g_B(F_*(T_Z \phi U + h\widetilde{\nabla}_Z wU), F_*(JW)) = g_B(F_*(T_W \phi U + h\widetilde{\nabla}_W wU), F_*(JZ))$$

for all $U \in \Gamma(D_1)$ and $Z, W \in \Gamma(D_2)$.

PROOF. By virtue of (12), (10) and (13), we get

$$g_N([Z,W], U) = g_N(\widetilde{\nabla}_Z JW, JU) - g_N(\widetilde{\nabla}_W JZ, JU)$$

for all $U \in \Gamma(D_1)$ and $Z, W \in \Gamma(D_2)$. Therefore, we conclude

$$g_N([Z,W], U) = -g_N(JW, \widetilde{\nabla}_Z JU) + g_N(JZ, \widetilde{\nabla}_W JU)$$

Then, using (18), (14) and (15), we arrive,

$$g_N([Z,W], U) = -g_N(JW, T_Z \phi U + h\widetilde{\nabla}_Z wU) + g_N(JZ, T_W \phi U + h\widetilde{\nabla}_W wU)$$

which proves assertion. □

Theorem 4.3. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution D_1 defines a totally geodesic foliation on N if and only if we have

$$F_*(\widetilde{\nabla}_U JV) \in \Gamma(\mu)$$

and

$$\begin{aligned} g_B(F_*(T_U \phi V), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_U wV), F_*(CX)) \;=\; & g_N(v\widetilde{\nabla}_U \phi V + T_U wV, BX) \\ & + g_N(U, BX)\eta(wV) \end{aligned}$$

for all $U, V \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $X \in \Gamma(\ker F_*^{\perp})$.

PROOF. We know that, D_1 defines a totally geodesic foliation on M if and only if $g_N(\widetilde{\nabla}_U V, Z) = 0$ and $g_N(\widetilde{\nabla}_U V, X) = 0$, for all $U, V \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $X \in \Gamma(\ker F_*^{\perp})$.

Then, using (13) and (10), we get,

$$g_N(\widetilde{\nabla}_U V, Z) = g_N(\widetilde{\nabla}_U JV, JZ)$$

Since $E = \mathcal{V}E + \mathcal{H}E$, for all $E \in \Gamma(TM)$, we have

$$g_N(\widetilde{\nabla}_U V, Z) = g_N(\mathcal{H}\widetilde{\nabla}_U JV, JZ)$$

Therefore, F is a semi-invariant Riemannian submersion and character of F, we arrive that,

$$g_N(\widetilde{\nabla}_U V, Z) = g_B(F_*(\mathcal{H}\widetilde{\nabla}_U JV), F_*(JZ))$$

Moreover, using (13) and (10), we get

$$g_N(\widetilde{\nabla}_U V, X) = g_N(\widetilde{\nabla}_U JV, JX)$$

By virtue of (14) and (15), we get

$$g_N(\widetilde{\nabla}_U V, X) = g_N(T_U\phi V + h\widetilde{\nabla}_U wV, CX) + g_N(v\widetilde{\nabla}_U\phi V, BX) + \eta(wV)g_N(U, BX)$$

or

$$g_N(\widetilde{\nabla}_U V, X) = g_B(F_*(T_U\phi V + h\widetilde{\nabla}_U wV), F_*(CX)) + g_M(v\widetilde{\nabla}_U\phi V, BX) + \eta(wV)g_N(U, BX)$$

which gives our assertion. □

Theorem 4.4. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution D_2 defines a totally geodesic foliation on N if and only if we have

$$g_B(F_*(T_Z BX), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_Z CX), F_*(CX)) = -g_N(v\widetilde{\nabla}_Z BX + T_Z CX, BX) - \eta(CX)g_N(Z, BX)$$

and

$$\begin{aligned} g_B(F_*(T_Z\phi U), F_*(CW)) + g_B(F_*(h\widetilde{\nabla}_Z wU), F_*(CW)) \quad &= \quad g_N(v\widetilde{\nabla}_Z\phi U + T_Z wU, BW) \\ &+ g_N(Z, BW)\eta(wU) - g_N(Z, \phi U)\eta(wW) \end{aligned}$$

for all $Z, W \in \Gamma(D_2), U \in \Gamma(D_1)$ and $X \in \Gamma(\ker F_*^{\perp})$.

PROOF. For all $Z, W \in \Gamma(D_2), X \in \Gamma(\ker F_*^{\perp})$, using (10), and (13), we conclude,

$$g_N(\widetilde{\nabla}_Z W, X) = g_N(\widetilde{\nabla}_Z JW, JX)$$

Then, from (19), (14) and (15), we have,

$$g_N(\widetilde{\nabla}_Z W, X) = g_N(T_Z BX + v\widetilde{\nabla}_Z BX + T_Z CX + h\widetilde{\nabla}_Z CX + \eta(CX)Z, BX + CX)$$

We know that F is semi-invariant Riemannian submersion, we conclude,

$$g_N(\widetilde{\nabla}_Z W, X) = g_B(F_*(T_Z BX + h\widetilde{\nabla}_Z CX), F_*(CX)) + g_N(v\widetilde{\nabla}_Z BX + T_Z CX, BX) + \eta(CX)g_N(Z, BX)$$

Moreover, for all $Z, W \in \Gamma(D_2), U \in \Gamma(D_1)$, using (10), and (13),

$$g_N(\widetilde{\nabla}_Z W, U) = -g_N(JW, \widetilde{\nabla}_Z JU)$$

By virtue of (19), (14) and (15), imply that

$$g_N(\widetilde{\nabla}_Z W, U) = -g_N(BW + CW, T_Z\phi U + v\widetilde{\nabla}_Z\phi U + T_Z wU + h\widetilde{\nabla}_Z wU + \eta(wU)Z) - \eta(JW)g_N(Z, JU)$$

Since F is semi-invariant Riemannian submersion, we arrive,

$$\begin{aligned} g_N(\widetilde{\nabla}_Z W, U) \quad &= \quad -g_B(F_*(T_Z\phi U + h\widetilde{\nabla}_Z wU), F_*(CW)) - g_N(BW, v\widetilde{\nabla}_Z\phi U + T_Z wU) \\ &- \eta(wU)g_N(BW, Z) - \eta(wW)g_N(Z, \phi U) \end{aligned}$$

which give proof. □

Corollary 4.5. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the fibers of F are the locally product Riemannian manifold of leaves of D_1 and D_2 if and only if

$$F_*(\widetilde{\nabla}_U JV) \in \Gamma(\mu),$$

$$\begin{aligned} g_B(F_*(T_U\phi V), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_U wV), F_*(CX)) &= g_N(v\widetilde{\nabla}_U\phi V + T_U wV, BX) \\ &\quad + g_N(U, BX)\eta(wV) \end{aligned}$$

and

$$g_B(F_*(T_Z BX), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_Z CX), F_*(CX)) = -g_N(v\widetilde{\nabla}_Z BX + T_Z CX, BX) - \eta(CX)g_N(Z, BX)$$

$$\begin{aligned} g_B(F_*(T_Z\phi U), F_*(CW)) + g_B(F_*(h\widetilde{\nabla}_Z wU), F_*(CW)) &= g_N(v\widetilde{\nabla}_Z\phi U + T_Z wU, BW) \\ &\quad + g_N(Z, BW)\eta(wU) - g_M(Z, \phi U)\eta(wW) \end{aligned}$$

for all $U, V \in \Gamma(D_1)$, $Z, W \in \Gamma(D_2)$ and $X \in \Gamma(\ker F_*^{\perp})$.

Theorem 4.6. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution $\ker F_*^{\perp}$ is integrable if and only if we have

$$A_Y CX - A_X CY + v\widetilde{\nabla}_Y BX - v\widetilde{\nabla}_X BY \notin \Gamma(D_1)$$

and

$$\begin{aligned} g_B(F_*(A_Y BX), F_*(wZ)) + g_B(F_*(h\widetilde{\nabla}_Y CX, F_*(wZ)) &= -g_N(v\widetilde{\nabla}_Y BX + A_Y CX, \phi Z) \\ &\quad + \eta(X)g_N(Y - wY, wZ) - \eta(Y)g_N(X - wX, wZ) \end{aligned}$$

for all $X \in \Gamma(\ker F_*^{\perp}), Z \in \Gamma(D_2)$ and $U \in \Gamma(D_1)$.

PROOF. We using (12), (10) and (13), for all $X, Y \in \Gamma(\ker F_*^{\perp}), U \in \Gamma(D_1)$, we have

$$g_N([X, Y], U) = g_N(\widetilde{\nabla}_X JY, JU) - g_N(\widetilde{\nabla}_Y JX, JU)$$

Then, using (19), (16) and (17), we arrive,

$$g_N([X, Y], U) = -g_N(-v\widetilde{\nabla}_X BY - A_X CY + v\widetilde{\nabla}_Y BX + A_Y CX, U)$$

Moreover, for $Z \in \Gamma(D_2)$, by (12), (10) and (13), we get

$$\begin{aligned} g_N([X, Y], Z) &= g_N(\widetilde{\nabla}_X JY, JZ) - \eta(Y)g_N(X - JX, JZ) \\ &\quad - g_N(\widetilde{\nabla}_Y JX, JZ) + \eta(X)g_N(Y - JY, JZ) \end{aligned}$$

Therefore, by virtue of (18), (15) and (16), we conclude that

$$\begin{aligned} g_N([X, Y], Z) &= -g_N(A_Y BX + h\widetilde{\nabla}_Y CX, wZ) - g_N(v\widetilde{\nabla}_Y BX + A_Y CX, \phi Z) \\ &\quad - \eta(Y)g_N(X - wX, wZ) + \eta(X)g_N(Y - wY, wZ) \end{aligned}$$

On the other hand, F is semi-invariant Riemannian submersion, therefore we get,

$$\begin{aligned} g_N([X, Y], Z) &= -g_B(F_*(A_Y BX + h\widetilde{\nabla}_Y CX), F_*(wZ)) - g_N(v\widetilde{\nabla}_Y BX + A_Y CX, \phi Z) \\ &\quad - \eta(Y)g_N(X - wX, wZ) + \eta(X)g_N(Y - wY, wZ) \end{aligned}$$

which conclude proof. □

Theorem 4.7. Let F be a semi-invariant Riemannian submersion with SSNMC . Therefore, the distribution $\ker F_*^{\perp}$ defines a totally geodesic foliation on N if and only if we have

$$v\widetilde{\nabla}_X BY + A_X CY \in \Gamma(D_2)$$

and

$$g_B(F_*(h\widetilde{\nabla}_X CY), F_*(wZ)) = -g_N(A_X BY, JZ)$$

for all $X, Y \in \Gamma(\ker F_*^{\perp}), U \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

PROOF. We using (10), (13) and (19), for all $X, Y \in \Gamma(\ker F_*^{\perp}), U \in \Gamma(D_1)$, we have

$$g_N(\widetilde{\nabla}_X Y, U) = g_N(\widetilde{\nabla}_X BY, JU) + g_M(\widetilde{\nabla}_X CY, JU)$$

Therefore, using (15), (16) and (10), we arrive

$$g_N(\widetilde{\nabla}_X Y, U) = -g_N(J(v\widetilde{\nabla}_X BY + A_X CY), U)$$

Moreover, for $Z \in \Gamma(D_2)$, using (10), (19),(16) and (17), we conclude,

$$g_N(\widetilde{\nabla}_X Y, Z) = g_N(A_X BY + v\widetilde{\nabla}_X BY + \eta(BY)X + A_X CY + h\widetilde{\nabla}_X CY, JZ)$$

Also, character of F, we obtain

$$g_N(\widetilde{\nabla}_X Y, Z) = g_B(F_*(h\widetilde{\nabla}_X CY), F_*(wZ)) + g_N(A_X BY, JZ)$$

which completes proof. □

Theorem 4.8. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the distribution $\ker F_*$ defines a totally geodesic foliation on N if and only if we have

$$v\widetilde{\nabla}_K L + T_K wL + \eta(wL)E \in \Gamma(D_1)$$

and

$$\begin{aligned} g_B(F_*(T_K\phi L), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_K wL), F_*(CX)) &= -g_N(v\widetilde{\nabla}_K \phi L, BX) - g_N(T_K wL, BX) \\ &\quad -\eta(\phi L) g_N(K, BX) \end{aligned}$$

for all $K, L \in \Gamma(\ker F_*)$ and $X \in \Gamma(\mu)$.

PROOF. We know that $\ker F_*$ denote a totally geodesic foliation on N if and only if $g_N(\widetilde{\nabla}_K L, X) = 0$ and $g_N(\widetilde{\nabla}_K L, JZ) = 0$ for all $Z \in \Gamma(D_2)$, $K, L \in \Gamma(\ker F_*)$ and $X \in \Gamma(\mu)$.

Then, by (10), (13) and (18), we get

$$g_N(\widetilde{\nabla}_K L, X) = g_N(\widetilde{\nabla}_K \phi L, JX) + g_N(\widetilde{\nabla}_K wL, JX)$$

Therefore, using (14), (15) and (19), we have

$$g_N(\widetilde{\nabla}_K L, X) = g_N(v\widetilde{\nabla}_K \phi L + T_K wL + \eta(\phi L)K, BX) + g_N(T_K \phi L + h\widetilde{\nabla}_K wL, CX)$$

We know that F is semi-invariant Riemannian submersion, we arrive

$$g_N(\widetilde{\nabla}_K L, X) = g_N(v\widetilde{\nabla}_K \phi L + T_K wL + \eta(\phi L)K, BX) + g_B(F_*(T_K \phi L + h\widetilde{\nabla}_K wL), F_*(CX))$$

Moreover, from (10), (13) ,(18), (14) and (15), we obtain,

$$g_N(\widetilde{\nabla}_K L, JZ) = g_N(v\widetilde{\nabla}_K L + T_K wL + \eta(wL)E, Z)$$

which give proof. □

Corollary 4.9. Let F be a semi-invariant Riemannian submersion with SSNMC. Therefore, the total space M is a locally product manifold of the leaves of $\ker F_*^{\perp}$ and $\ker F_*$ if and only if

$$v\widetilde{\nabla}_X BY + A_X CY \in \Gamma(D_2),$$

$$g_B(F_*(h\widetilde{\nabla}_X CY), F_*(wZ)) = -g_N(A_X BY, JZ)$$

and

$$v\widetilde{\nabla}_K L + T_K wL + \eta(wL)E \in \Gamma(D_1),$$

$$\begin{aligned} g_B(F_*(T_K\phi L), F_*(CX)) + g_B(F_*(h\widetilde{\nabla}_K wL), F_*(CX)) &= -g_N(v\widetilde{\nabla}_K \phi L, BX) - g_N(T_K wL, BX) \\ &\quad -\eta(\phi L) g_N(K, BX) \end{aligned}$$

for all $X, Y \in \Gamma(\ker F_*^{\perp}), K, L \in \Gamma(\ker F_*)$ and $Z \in \Gamma(D_2)$.

5. Conclusion

Riemannian submersions and SSNMC have an important application for many sciences such as physics and mathematical physics. Researchers have increased studies on this field from different areas in recent years. In this paper, the idea of examining Riemann submersion with different connections is emphasized. We defined and studied Riemannian submersions with SSNMC for the first time. We investigated geometry of foliatons with SSNMC. The works on this subject will be useful tools for the applications of Riemannian submersion with different connections.

Conflicts of Interest

The author declares no conflict of interest.

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