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n-complete crossed modules and wreath products of groups

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Abstract — In this paper we examine the n-completeness of a crossed module and we show that if $X = (W_1, W_2, \partial)$ is an n-complete crossed module, where $W_i = A_i \, wr B_i$ is the wreath product of groups A_i and B_i, then A_i is at most n-complete, for $i = 1, 2$. Moreover, we show that when $X = (W_1, W_2, \partial)$ is an n-complete crossed module, where A_i is nilpotent and B_i is nilpotent of class n, for $i = 1, 2$, then if A_i is an abelian group, then it is cyclic of order p_i. Also, if $W_i = C_p \, wr C_2$, where p is prime with $p > 3$, $i = 1, 2$, then $X = (W_1, W_2, \partial)$ is not an n-complete crossed module.

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1. Introduction

The notion of crossed module is investigated by Whitehead [1]. After him, many mathematicians applied crossed modules in many directions such as homology and cohomology of groups, algebraic structures, K-theory, and so on. Actor crossed module of algebroid is defined by Alp in [2]. Actions and automorphisms of crossed modules is studied by Norrie [3]. Tensor product modulo n of two crossed modules is introduced by Conduche and Rodriguez-Fernandez [4]. The concepts of q-commutator and q-center of a crossed module (where q is a non-negative integer) is studied by Doncel-Juarez and Crondjean-Valcarcel [5].

Let $X = (T, G, \partial)$ be a crossed module and $X = (T, G, \partial) = \gamma_1(X), \ldots, \gamma_n(X), \ldots$ be the lower central series of $X = (T, G, \partial)$. We define the series $K_1, \ldots, K_n, \ldots$ where K_n consists of the automorphisms of X which induce the identity on the quotient crossed module $\frac{X}{\gamma_{n+1}(X)}$. Now, in this paper, we present the definition of an n-complete crossed module which is an extension of the definition of a semi-complete crossed module.

2. n-commutator crossed submodule

It is well known that an action of the group G on the group T is a homomorphism $G \to Aut(T)$ or, a map $\mu : T \times G \to T$ such that

1. $\mu(t_1 t_2, x) = \mu(t_1, x)\mu(t_2, x)$,

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2. $\mu(t, x_1 x_2) = \mu(\mu(t, x_1), x_2)$,

for all $t_1, t_2 \in T$ and $x, x_1, x_2 \in G$.

As usual, we will consider the notation $\mu(t,x) = {}^x t$ in continue. Indeed, a crossed module [6] is a 4-tuple $X = (T, G, \mu, \partial)$ or 3-tuple (T, G, ∂), where T and G are groups, μ is an action of T on G, and $\partial : G \to T$ is a homomorphism. The map ∂ is called the boundary, and it satisfies the following statements:

1. X Mod 1: $\partial({}^t x) = t^{-1}\partial(x)t$ for all $x \in G$ and $t \in T$.

2. X Mod 2: ${}^{\partial(y)}x = y^{-1}xy$ for all $x, y \in G$.

If T and G are finite groups, then the crossed module is called finite.

Example 2.1. Let G be a group. We denote by RG the crossed module $(G, 1, \mu, \partial)$, where 1 is the trivial subgroup of G, and the action μ and the boundary map ∂ are trivial.

Example 2.2. Let G be a group. We denote by DG the crossed module (G, G, μ, id), where μ is the conjugation action, and $id : x \to x$ is the trivial map.

From the definition, we immediately conclude that $K = \text{Ker}\,\partial$ is a central subgroup of G, $I = \text{im}\,\partial$ is a normal subgroup of T, and obtain the following exact sequence $1 \to K \to G \to T \to C \to 1$, where $C = \frac{T}{I}$ is the cokernel of ∂. Specially, for a finite crossed module we have $|G||C| = |K||T|$ [7]. A morphism $\phi : X \to Y$ between two crossed modules $X = (T_1, G_1, \mu_X, \partial_X)$ and $y = (T_2, G_2, \mu_Y, \partial_Y)$ is a pair (ϕ_1, ϕ_2), where $\phi_1 : T_1 \to T_2$, $\phi_2 : G_1 \to G_2$ are group homomorphisms, and the following relations hold:

$$\partial_Y \circ \phi_2 = \phi_1 \circ \partial_X, \quad \mu_Y \circ (\phi_2 \times \phi_1) = \phi_2 \circ \mu_X.$$

This yields the commutativity of the following diagrams:

$$\begin{tikzcd} G_1 \arrow[r, "\partial_X"] \arrow[d, "\phi_2"'] & T_1 \arrow[d, "\phi_1"] \\ G_2 \arrow[r, "\partial_Y"'] & T_2 \end{tikzcd} \qquad \begin{tikzcd} G_1 \times T_1 \arrow[r, "\mu_X"] \arrow[d, "\phi_2 \times \phi_1"'] & G_1 \arrow[d, "\phi_2"] \\ G_2 \times T_2 \arrow[r, "\mu_Y"'] & G_2 \end{tikzcd}$$

Definition 2.3. Suppose that (T, G, ∂) is a crossed module and n is a non-negative integer. We define the notion of n-commutator crossed submodule of (T, G, ∂) as $\partial : D^n_G(T) \to G \neq_n G$, where $D^n_G(T)$ is the subgroup of T generated by the set

$$\left\{ {}^x a a^{-1} b^n \mid x \in G,\ a, b \in T \right\},$$

and in a general case, if N is a normal subgroup of G, then $G \neq_n G$ is the n-commutator subgroup of G and N, i.e., the subgroup generated by the

$$\left\{ [x, a] a'^n \mid x \in G, \quad a, a' \in N \right\}.$$

The n-commutator crossed submodule of (T, G, ∂) is a normal crossed submodule.

Example 2.4. The group G acts on N by conjugation if N is a normal subgroup. The triple (N, G, i) is a crossed module, where i is the inclusions. The n-commutator crossed submodule of (N, G) equals $(G \neq_n$

$N,\ G \neq_n G,\ i)$. This implies that for any group G, the triple (G, G, id) is a crossed module and $(G \neq_n G,\ G \neq_n G,\ id)$ is its n-commutator.

Let (T, G, ∂) be a crossed module with trivial center. According to [3], we can obtain a sequence of crossed modules as follows:

$$(T, G, \partial),\ \mathscr{A}(T, G, \partial), \mathscr{A}(\mathscr{A}(T, G, \partial)), \ldots$$

in which each term embeds in its successor. This sequence is called the actor tower of (T, G, ∂).

We say the crossed module (T, G, ∂) is complete if $Z(T, G, \partial) = 1$ and the canonical morphism $< \eta, \gamma >$: $(T, G, \partial) \to \mathscr{A}(T, G, \partial)$ is an isomorphism. Notice that the crossed module (T, G, ∂) is semi complete if $< \eta, \gamma >$ is an epimorphism. Consequently, a semi complete crossed module with trivial center is complete.

3. n-complete crossed modules

A crossed module (T, G, ∂) is said to be n-complete if n is the smallest positive integer such that K_n is sub-crossed module $I_{nn}(T, G, \partial)$, where $I_{nn}(T, G, \partial)$ is the crossed module of the inner automorphisms of (T, G, ∂).

Proposition 3.1. Let (T, G, ∂) is an n-complete crossed module. Then, T and G are at most n-complete and nilpotent of class at most n.

Example 3.2. If (G, G, i) is an n-complete crossed module, then G is n-complete and nilpotent of class n.

In Proposition 3.3 we give a relation between nilpotent groups and n-complete crossed modules.

Proposition 3.3. If (T, G, ∂) is a crossed module and groups T, G are nilpotent of class at most n, then (T, G, ∂) is an n-complete crossed module for some m with $m \leq n$.

Suppose that (R, K, ∂) is a normal crossed submodule of (T, G, ∂) and (S, H, ∂') is a crossed module such that $(T/R, G/K) \cong (S, H)$, then we call (T, G) an extension of (R, K) by (S, H). If there exists a surjective morphism $\psi = (\psi_1, \psi_2) :\ (X_1, X_2) \to (T, G)$, the trivially (X_1, X_2) is an extension of the crossed module $\ker\phi$ by (T, G). An extension $((X_1, X_2), \psi)$ by (T, G) is n-central extension if $\ker\psi = (\ker\psi_1, \ker\psi_2)$ is contained in $Z^n(X_1, X_2)$.

Let (M, G, μ) and (N, G, ν) be two crossed modules, and consider the pullback

$$\begin{tikzcd} M \times_G N \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & G \end{tikzcd}$$

Then, $M \times_G N = \{(a, b) \mid a \in M,\ b \in N,\ \mu(a) = \nu(b)\}$. If we write $\alpha = \mu\pi_1 = \nu\pi_2$, then for $c \in M \times_G N,\ a \in M,\ b \in N$, we get

$$^{\pi_1(c)}a = {}^{\alpha(c)}a = {}^{\pi_2(k)}a,\ {}^{\pi_1(c)}b = {}^{\alpha(c)}b = {}^{\pi_2(c)}b.$$

The tensor product $M \otimes^q N$ is defined as the group generated by the symbols $a \otimes b$ and $\{c\}$, $a \in M$, $b \in N$, $c \in M \times_G N$, with the following relations:

1. $a \otimes bb' = (a \otimes b)({}^b a \otimes {}^b b')$.
2. $aa' \otimes b = ({}^a a' \otimes {}^a b)(a \otimes b)$.
3. $\{c\}(a \otimes b)\{c\}^{-1} = {}^{\alpha(c)^q} a \otimes {}^{\alpha(c)^q} b$.

4. $[\{c\},\{c'\}] = \pi_1(c)^q \otimes \pi_2(c')^q$.

5. $\{cc'\} = \{c\}\left(\prod_{i=1}^{q-1}(\pi_1(c)^{-1} \otimes ({}^{\alpha(c)^{1-q+i}}\pi_2(c'))^i)\right)\{c'\}$.

6. $\{(a^b a^{-1}, {}^a b b^{-1})\} = (a \otimes b)^q$.

Note that the structure of the tensor product mode q is bifunctorial. Under this conditions there exists an action of G on $M \otimes^q N$ defined as follows:

$${}^x(a \otimes b) = {}^x a \otimes {}^x b,\ {}^x\{c\} = \{{}^x c\}$$

$a \in M$, $b \in N$, $c \in M \times_G N$, $x \in G$. The group M (resp. N) acts on $M \otimes^q N$ through the homomorphism μ (respectively ν) and if $a \in M$, $b \in N$, $c \in M \times_G N$, then

$${}^a\{c\} = (a \otimes \pi_2 c^q)\{c\},\ \ {}^b\{c\} = \{c\}(\pi_1 c^{-q} \otimes b).$$

Now let (T, G, ∂) and (G, G, id) be crossed modules. We can consider the tensor product $T \otimes^q G$, it was first defined by Brown. In this case $T \times_G G \cong T$, $\pi_1 = id_T$, $\pi_2 = \partial$. Similarly, we consider $G \otimes^q G$. Then, we have the following crossed modules:

$$\begin{array}{llll}
(T \otimes^q G, T, \lambda), & \lambda(t \otimes g) = t^g t^{-1}, & \lambda(\{t\}) = t^q, & t \in T,\ g \in G; \\
(T \otimes^q G, T, \lambda'), & \lambda'(t \otimes g) = [\partial(t), g], & \lambda'(\{t\}) = \partial(t)^q, & t \in T,\ g \in G; \\
(G \otimes^q G, G, \xi), & \xi(g \otimes h) = [g, h], & \xi(\{g\}) = g^q, & g, h \in G.
\end{array}$$

Theorem 3.4. If (T, G, ∂) is an n-complete crossed module, then $(T \otimes^n G, G \otimes^n G, (\lambda, \epsilon))$ is an n-complete extension by (T, G, ∂).

The restricted standard wreath product $W = AwrB$ of two groups A and B is the splitting extension of the direct power A^B by the group B, with B acting on A^B according to the rule, if $b \in B$ then $f^b(x) = f(xb^{-1})$ for all $f \in A^B$, $x \in B$. The base group A^B is characteristic in W, in all cases, except when A is of order 2, or is a dihedral group of order $4k+1$ and B is of order 2. In the following it is assumed that A^B is characteristic in W. The next theorem is of great importance for the sequel. But first we need the following results from [8].

Proposition 3.5. [8] If $\alpha \in Aut(A)$, we define $\alpha^* \in Aut(W)$ by $(bf)^{\alpha^*} = bf^{\alpha^*}$ for all $b \in B$, $f \in \mathscr{F}$, where $f^{\alpha^*}(x) = \big(f(x)\big)^{\alpha}$, for all $x \in B$, then the group A^* of all such automorphisms is isomorphic to $Aut(A)$.

Proposition 3.6. [8] If $\beta \in Aut(B)$, we define $\beta^* \in Aut(W)$ by $(bf)^{\beta^*} = b^{\beta} f^{\beta^*}$ for all $b \in B$, $f \in \mathscr{F}$, where $f^{\beta^*}(x) = f(x^{\beta^{-1}})$ for all $x \in B$, then the group B^* of all such automorphisms is isomorphic to $Aut(B)$.

Theorem 3.7. [8]

1. The automorphism group of the wreath product W of two groups A and B can be expressed as a product, $Aut(W) = KI_1B^*$, where

   - K is the subgroup of $Aut(W)$ consisting of those automorphisms which leave B element wise fixed.
   - I_1 is the subgroup of $Aut(W)$ consisting of those inner automorphisms corresponding to transformation by elements of the base group $\mathscr{F}$.

- B^* is defined as in Proposition 3.5.

2. The group K can be written as A^*H, where
   - A^* is defined as in Proposition 3.6.
   - H is the subgroup of $Aut(W)$ consisting of those automorphisms which leave both B and diagonal element wise fixed.

3. The subgroups A^*HI_1, HI_1B^*, HI_1, and I_1 are normal in $Aut(W)$ and $Aut(W)$ is splitting extension of A^*HI_1 by B^*. Furthermore, A^* intersects HB^* trivially.

In the following it is assumed that $W_1 = A_1 wr B_1$ and $W_2 = A_2 wr B_2$ are two standard wreath products of groups.

Theorem 3.8. If $X = (W_1, W_2, \partial)$ is an n-complete crossed module, then A_i is at most n-complete, for $i = 1,2$.

Proof.
If $(\alpha, \beta) \in K_n(X)$, then $\alpha \in K_n(A_1)$ and $f \in A_1^{B_1}$. Hence, $f^{\alpha^*}(x) = \big(f(x)\big)^{\alpha} = f(x)u_x$ for $x \in B_1$ and $u_x \in \gamma_{n+1}(A_1)$. If $g_1 \in A_1^{B_1}$, $g_1(x) = u_x$ for all $x \in B_1$, then $f^{\alpha^*}(x) = \big(fg_1(x)\big)$ for all $x \in B_1$. Therefor, $f^{\alpha^*} = fg_1$, where $g_1 \in \gamma_{n+1}(W_1)$. Since W_1 is n-complete, it follows that $K_n(W_1) \leq I(W_1)$ and so $\alpha^* \in I(W_1)$. But according to [9], $\alpha^* \in I(W_1)$ if and only if $\alpha \in I(A_1)$. Hence, $K_n(A_1) \leq I(A_1)$. The proof for $K_n(A_2) \leq I(A_2)$ is similar.

Theorem 3.9. If $X = (W_1, W_2, \partial)$ is an n-complete crossed module, then B_i is nilpotent of class at most n, for $i = 1,2$.

Proof.
If $L(B_1)$ and $L(B_2)$ are the left regular representation of the groups B_1, B_2 respectively, then for each element $l_b \in L(B_1)$, $b \in B_1$, there corresponds an automorphism l_b^* of W_1 defined by $(cf)^{l_b^*} = cf^{l_b^*}$ for all $c \in B_1$, $f \in {A_1}^{B_1}$, where $f^{l_b^*}(x) = f(bx)$ for all $x \in B_1$.
If $f_1 \in {A_1}^{B_1}$ such that $f_1(1) = a$, $f_1(x) = 1$ for all $x \in B_1$, $x \neq 1$ and $b \in B_1$, $b \neq 1$, then $f_1^{l_b^*}(b^{-1}) = f_1(1) = a$ and $f_1^{l_b^*}(x) = f_1(bx) = 1$ for all $x \neq b^{-1}$.
Moreover, we obtain $f_1^{l_b^*} = f_1 g$, where $g(1) = a^{-1}$, $g(b^{-1}) = a$, $g(x) = 1$ for all $x \in B$, $x \neq 1, b^{-1}$. Also, by [10] for the element $g \in A_1^{B_1}$, $g = [b^{-1}, \varphi]$, where $\varphi \in A_1^{B_1}$ with $\varphi(1) = g(1)$ and $\varphi(x) = 1$ for all $x \neq 1$.
Now, if $X_i \in B_1$, we define the element $f_{x_i} \in A_1^{B_1}$ by $f_{x_i}(x_i) = a$ and $f_{x_i}(d) = 1$ for all $d \in B$, $d \neq x_i$, then $(f_{x_i})^{l_b^*} = f_{x_i} g^{x_i}$. If $b \in \gamma_n(B_1)$, then l_b^* belongs to the group $K_n(W_1) \leq I(W_1)$. But $b \in Z(B_1)$ if and only if $l_b^* \in I(W_1)$. So, the group B_1 is nilpotent of class at most n, and similarly B_2 is nilpotent of class at most n.

Theorem 3.10. If $X = (W_1, W_2, \partial)$ is an n-complete crossed module, and B_i is nilpotent of class n, for $i = 1,2$, then A_i is directly indecomposable.

Proof.
Suppose that $A_i = U_i \times V_i$ is a non trivial direct decomposition of A_i for $i = 1,2$. If $f \in A_1^{B_1}$, then $f(x) = u_{1x} v_{1x}$ for all $x \in B_1$, where $u_{1x} \in U_1$ and $v_{1x} \in V_1$. If $g_f \in A_1^{B_1}$, $g_f(x) = u_{1x}$ for all $x \in B_1$ and $x \in \gamma_n(B_1) \leq Z(B_1)$, $z \neq 1$, then $\eta : W_1 \to W_1$ by $(bf)^{\eta} = bf[g_f, z]$ is a map. Since $g_{fh} = g_f g_h$ and $g_f^y = g_{f^g}$ for all $f, h \in A_1^{B_1}$, $y \in B_1$, it follows that η is an outer automorphism of W_1 with $\eta \in K_n(W_1)$ and is a contradiction.

Theorem 3.11. If $X = (W_1, W_2, \partial)$ is an n-complete crossed module, where A_i is finite nilpotent and B_i is nilpotent of class n, then A_i is a p_i-group, (p_i is prime) for $i = 1,2$.

Proof.
By Theorem 3.10, the proof is straightforward.

Theorem 3.12. Let $X = (W_1, W_2, \partial)$ is an n-complete crossed module, where A_i is nilpotent and B_i is nilpotent of class n, for $i = 1,2$. If A_i is abelian group, then it is cyclic of order p_i.

Proof.
By Theorem 3.11, A_i is p_i-group. But A_i is abelian, and so A_i is cyclic of order p^r for some positive integer r. Now, we show that $r = 1$.
If r is not equal to 1, we choose an element $x \in \gamma_n(B_1)$, $x \neq 1$ and we define a mapping $\eta : W_1 \to W_1$ by $(bf)^\eta = bf[f,x]^p$, η is an automorphism of W_1 belonging to the group $K_n(W_1)$ by [9]. Since $r > 1$ and η is an outer automorphism, it follows that W_1 is not n-complete. Hence, $r = 1$, and A_2 is cyclic of order p_i, accordingly.

Corollary 3.13. If $X = (W_1, W_2, \partial)$ is an n-complete crossed module and A_i is finite nilpotent and B_i nilpotent of class n, for $i = 1,2$, then A_i is cyclic of prime order.

Now, we give examples of non n-complete crossed module. Let $W = AwrB$ be the restricted wreath product of A by B. The set $\sigma(f) = \{x \in B | f(x) \neq 1\}$ is the support of $f \in A^B$. Map $\pi : A^B \to \frac{A}{A'}$ given by

$$\pi(f) = \prod_{x \in \sigma(f)} f(x)A'$$

is well defined and obviously a homomorphism satisfying $\pi(f^b) = \pi(f)$ for all $b \in B$.

Proposition 3.14. [10] The derived subgroup W' of W is $W' = B'M$, where $M = Ker\pi$.

Theorem 3.15. If $W_i = C_p wr C_2$, where p is prime with $p > 3$, $i = 1,2$, then $X = (W_1, W_2, \partial)$ is not n-complete crossed module.

Proof.
If $W_1 = A_1 wr B_1$, then $W_1' = B_1' M_1$, where $M_1 = \{f | f \in A_1^{B_1}, \pi(f) \in A_1'\}$. But $B_1 = C_2$, $|M_1||A_1|^{|B_1|} = p^2$ and so $|M_1| = p$. W_1 is not nilpotent and thus $\gamma_n(W_1) = M_1$ for all $n \in Z^+$, $n \geq 2$. If $A_1 = C_p =< a >$, $B_1 = C_2 =< b >$, then $f_1 = (a^{p-1}, a^2)$, $f_2 = (a^2, a^{p-1})$, $g_1 = (a,1)$, $g_2 = (1,a)$, instill the mapping $g_1 \to f_1$, $g_2 \to f_2$ which can be extended to an automorphism γ of $A_1^{B_1}$, which commutes with the automorphism of $A_1^{B_1}$ induced by the element $b \in B_1$, since $A_1^{B_1} =< f_1, f_2 >=< g_1, g_2 >$ and $A_1^{B_1}$ is elementary abelian of rank 2 and $p \neq 3$. Thus, the automorphism γ can be extended to an automorphism of W_1, which fixes B_1 element wise [8]. On the other hand, we have

$$g_1^\gamma = (a,1)^\gamma = (a^{p-1}, a^2) = (a,1)(a^{p-2}, a^2),$$
$$g_2^\gamma = (1,a)^\gamma = (a^2, a^{p-1}) = (1,a)(a^2, a^{p-2}),$$

and $(a^{p-2}, a^2), (a^2, a^{p-2}) \in M_1 = \gamma_n(W_1)$, $n \geq 2$, so $\gamma \in K_n(W_1)$, $n \geq 2$ and γ is an outer automorphism. Hence $W_1 = C_p wr C_2$ is not n-complete. Therefore, $X = (W_1, W_2, \partial)$ is not n-complete crossed module.

Theorem 3.16. If $W_i = C_p wr B_i$, where p is prime with $p > 3$, $i = 1,2$, and B_i is nilpotent of class n with $k_i = |B_i| \geq 3$, $i = 1,2$, then $X = (W_1, W_2, \partial)$ is not n-complete crossed module.

Proof.
The group $A_i^{B_i}$ is an elementary abelian p-group, since A_i is $A_i = C_p =< a_i >$. The set $g_{x_i} \in A_i^{B_i}$ for all $x_i \in B_i = \{x_1, ..., x_{k_j}\}$ with $g_{x_i}(x_i) = a_i$, $g_{x_i}(x_j) = 1$, $x_j \neq x_i$ is a basis of $A_i^{B_i}$. Now, if we consider the mapping

$g_{x_i} \to f_{x_i} = g_{x_i}[b_1, g_{x_i}] = g_{x_i}{}^2(g_{x_i}{}^{-1})^{b_1}$ for all $x_i \in B_i$, where $b_1 \in \gamma_n(B_i)$, then this mapping is extended to an automorphism $\overline{\gamma}$ of $A_i^{B_i}$, since the set f_{x_i}, $x_i \in B_i$ is a basis of $A_i^{B_i}$. On the other hand, since $C_p \cong Z_p$ and $p > 3$, it follows that the determinant of matrix

$$\begin{bmatrix} 2 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ & & & & \ddots & & & \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

is not zero in Z_p, where the element 2 is in the main diagonal and in each row and column we have once the element -1. But $\overline{\gamma}$ can be extended to an automorphism γ of the group W_i, which fixes B_i element wise, since the automorphism $\overline{\gamma}$ of $A_i^{B_i}$ commutes with the automorphisms of $A_i^{B_i}$ which are induced by the elements of the group B_i. The automorphism γ is an outer automorphism with $\gamma \in K_n(W_i)$. So, $X = (W_1, W_2, \partial)$ is not n-complete crossed module.

Proposition 3.17. [11] The wreath product $W = C_2 wr B$ is not n-complete, where B is finite abelian with $m = |B| \geq 4$ and m is an odd number.

Theorem 3.18. If $W_i = C_2 wr B_i$, where B_i is finite abelian with $m_i = |B_i| \geq 4$, $i = 1,2$, and m_i is an odd number, then $X = (W_1, W_2, \partial)$ is not n-complete crossed module.

Proof.

By Proposition 3.17, the proof is straightforward.

We have assumed up to this point that subgroup A^B is characteristic in $W = AwrB$. Now, we investigate the case of W in which A is a special dihedral group and B is of order 2. At this case A^B is not characteristic in W. We recall that D_m is $D_m =< a, b \mid a^m = 1, b^2 = 1, (ab)^2 = 1 >$.

Theorem 3.19. [11] The standard wreath product $W = D_n wr C_2$ is semi complete if and only if $n = 3$.

Theorem 3.20. Let $W = D_m wr C_2$, where $m = 2k+1$, $k \in N$, and C_2 is the cyclic group of order 2. Then, the crossed module $X = (W, W, i)$ is n-complete if and only if $m = 3$.

Proof.

In this case, we know that for the lower central series of the group D_m, is $\gamma_{k+1}(D_m) =< a^{2^k} >$, for all $k = 1, 2, \ldots$. Since m is an odd number, it follows that

$$\gamma_2(D_m) = \gamma_3(D_m) = \cdots = \gamma_k(D_m) = \gamma_{k+1}(D_m) = \cdots.$$

If the crossed module (W, W, i) is n-complete, then by Theorem 3.8, the group D_m is at most n-complete. This means that D_m is semi complete [12], and this is true if and only if $m = 3$.

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