http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 3, Pages 666–681 (2022)
DOI:10.31801/cfsuasmas.989344
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: September 1, 2021; Accepted: February 4, 2022

ON THE MAXIMUM MODULUS OF A COMPLEX POLYNOMIAL

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Abstract. In this paper we impose distinct restrictions on the moduli of the zeros of $p(z) = \sum_{v=0}^{n} a_v z^v$ and investigate the dependence of $\|p(Rz) - p(\sigma z)\|$, $R > \sigma \geq 1$ on M_α and $M_{\alpha+\pi}$, where $M_\alpha = \max_{1\leq k\leq n} |p(e^{i(\alpha+2k\pi)/n})|$ and on certain coefficients of $p(z)$. This paper comprises several results, which in particular yields some classical polynomial inequalities as special cases. Moreover, the problem of estimating $p\left(1 - \frac{w}{n}\right)$, $0 < w \leq n$ given $p(1) = 0$ is considered.

1. Introduction

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n over $\mathbb{C}$. Then it is well known that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

The result in (1) is sharp and equality holds when $p(z) = \lambda z^n$, where $\lambda \in \mathbb{C}$.

The inequality (1), known as Bernstein's inequality, was proved by Bernstein [4] in 1926, however it was also proved earlier by Riesz [14]. By the maximum modulus principle, $\max_{|z|\leq 1} |p(z)| = \max_{|z|=1} |p(z)|$ and so if we consider $\|p\| = \max_{|z|=1} |p(z)|$, then inequality (1) can be written as

$$\|p'\| \leq n \|p\|. \tag{2}$$

For $R \geq 1$, the inequality pertaining to the estimate of $\|p\|$ on a large circle

2020 *Mathematics Subject Classification.* 26D10, 30C15, 41A17.
Keywords. Polynomials, definite integral, zeros, inequalities.
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$|z| = R$ given below is well known [11, Problem 269] or [15].

$$\max_{|z|=R} |p(z)| \leq R^n \|p\|, \tag{3}$$

equality holds in (3) when $p(z) = \lambda z^n$, $\lambda \in \mathbb{C}$.

Marden [9], Milovanović et al. [10] and Rahman and Schmeisser [12] have presented an exceptional introduction to this topic. Frappier, Rahman and Ruscheweyh [6] were able to refine (1) under the same hypothesis, by replacing the estimate of the maximum modulus of $|p(z)|$ on a unit circle $|z| = 1$ with the estimate of the maximum modulus of $|p(z)|$ taken over $(2n)^{th}$ roots of unity. The maximum modulus of $|p(z)|$ taken over $(2n)^{th}$ roots of unity may be less than the maximum modulus of $|p(z)|$ on unit circle $|z| = 1$ which is shown by a simple example $p(z) = z^n + ia, a > 0$. In fact they proved that

$$\|p'\| \leq n \max_{1 \leq k \leq 2n} |p(e^{ik\pi/n})|. \tag{4}$$

As an improvement of (4) A.Aziz [2] showed that the maximum modulus of $|p(z)|$ taken over $(2n)^{th}$ roots of unity in (4) can be replaced by maximum modulus of $|p(z)|$ taken over n^{th} roots of the equation $w^n = e^{i\alpha}$. In fact he proved that, for a polynomial $p(z)$ of degree n and for every $\alpha \in \mathbb{R}$,

$$\|p'\| \leq \frac{n}{2}(M_\alpha + M_{\alpha+\pi}), \tag{5}$$

where

$$M_\alpha = \max_{1 \leq k \leq n} |p(e^{i(\alpha+2k\pi)/n})| \tag{6}$$

and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha + \pi$. The result is sharp and equality in (5) holds for the polynomial $p(z) = z^n + re^{i\alpha}, -1 \leq r \leq 1$.

As an application of inequality (5) A.Aziz [2] was able to establish the following refinement of (3).

For a polynomial $p(z)$ of degree n, and for every α and $R > 1$

$$\|p(Rz) - p(z)\| \leq \frac{R^n - 1}{2}[M_\alpha + M_{\alpha+\pi}], \tag{7}$$

where M_α is defined by (6) and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha + \pi$. The result is the best possible and equality in (7) holds for $p(z) = z^n + re^{i\alpha}, -1 \leq r \leq 1$.

In the same paper A.Aziz [2] also proved that if $p(z)$ is a polynomial of degree n such that $p(1) = 0$, then for $0 < w \leq n$

$$\left|p\left(1 - \frac{w}{n}\right)\right| \leq \frac{1}{2}\left[1 - \left(1 - \frac{w}{n}\right)^n\right]\{M_0 + M_\pi\}, \tag{8}$$

where M_α is defined by (6). The result is the best possible and equality in (8) holds for $p(z) = z^n - 1$.

The study of mathematical objects associated with Bernstein type inequalities has been very active over the years, many papers are published each year in a variety of journals and different approaches are being employed for different purposes. In the present article we have come up with the similar type of inequalities, their refined and improved forms. If we restrict ourselves to the class of polynomials having no zero in $|z|<1$, then one would expect, the further developments of the upper bound estimate in (1). In fact, P. Erdös conjectured and later P.D. Lax [8] proved that if $p(z)\neq 0$ in $|z|<1$, then

$$\|p'\| \leq \frac{n}{2}\|p\|. \tag{9}$$

The result is best possible and equality holds for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$. In this connection A. Aziz [2], improved the inequality (5) by showing that if $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for every given real α

$$\|p'\| \leq \frac{n}{2}(M_\alpha^2+M_{\alpha+\pi}^2)^{1/2}, \tag{10}$$

where M_α is defined by (6) for all real α. The result is the best possible and equality in (10) holds for $p(z)=z^n+e^{i\alpha}$. Furthermore, A. Aziz [2] also established that if $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for every given real α and $R>1$

$$\|p(Rz)-p(z)\| \leq \frac{R^n-1}{2}[M_\alpha^2+M_{\alpha+\pi}^2]^{1/2}, \tag{11}$$

where M_α is defined by (6). The result is the best possible and equality in (11) holds for $p(z)=z^n+e^{i\alpha}$. By estimating the minimum modulus of $|p(z)|$ on the unit circle inequality (11) was refined and generalized by Ahmad [1]. In fact proved the following result.
If $p(z)$ is a polynomial of degree n having all its zero in $|z|\geq 1$ and $m=\min_{|z|=1}|p(z)|$, then for all real λ and $R>r\geq 1$

$$\|p(Rz)-p(rz)\| \leq \frac{R^n-r^n}{2}[M_\lambda^2+M_{\lambda+\pi}^2-2m^2]^{1/2}, \tag{12}$$

where M_λ is defined by (6). Just replace argument α of z simply by λ, unless otherwise stated. In the same paper Ahmad [1] also proved that if $p(z)$ is a polynomial of degree n having all its zero in $|z|\geq k\geq 1$ and $m=\min_{|z|=1}|p(z)|$, then for all real λ and $R>r\geq 1$

$$\|p(Rz)-p(rz)\| \leq \frac{R^n-r^n}{\sqrt{2(1+k^2)}}[M_\lambda^2+M_{\lambda+\pi}^2-2m^2]^{\frac{1}{2}}, \tag{13}$$

where M_λ is defined by (6).

While establishing the inequality analogous to (11) for the class of polynomials having all zeros in $|z|\leq k, k\leq 1$, M. H. Gulzar [7] proved that if $p(z)$ is a polynomial of degree n having all its zero in $|z|\leq k\leq 1$, then for all real λ and $R>1$

$$\|p(Rz)-p(z)\| \leq \frac{R^n-1}{\sqrt{2(1+k^{2n})}}[M_\lambda^2+M_{\lambda+\pi}^2]^{\frac{1}{2}}, \tag{14}$$

where M_λ is defined by (6) and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ in M_λ. While seeking the generalization of (14). Formerly, in the same paper Ahmad [1] proved that if $p(z)$ is a polynomial of degree n having all its zero in $|z| \leq k \leq 1$, then for all real λ and $R > r \geq 1$

$$\|p(Rz) - p(rz)\| \leq \frac{R^n - r^n}{\sqrt{2(1+k^{2n})}}[M_\lambda^2 + M_{\lambda+\pi}^2]^{\frac{1}{2}}. \tag{15}$$

We conclude this section by stating the following result for the case when $p(z)$ has no zero in $|z| < k, k \leq 1$.
If $p(z)$ is a polynomial of degree n and $p(z)$ has no zero in $|z| < k, k \leq 1$, then for every real α and $R > 1$

$$\|p(Rz) - p(z)\| \leq \frac{R^n - 1}{\sqrt{2(1+k^{2n})}}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}}, \tag{16}$$

provided $|p'(z)|$ and $|q'(z)|$ attain maximum at the same point on $|z| = 1$, where $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$. The result is best possible and equality in (16) holds for $p(z) = z^n + k^n$. This result is ascribed to Rather and Shah [13].

2. Lemmas

Lemma 1. *If $p(z)$ is a polynomial of degree n having all its zeros $|z| \leq k \leq 1$, then for all real λ*

$$|p'(z)| \leq \frac{n}{2^{\frac{1}{2}}(1+k^{2n})^{\frac{1}{2}}}[M_\lambda^2 + M_{\lambda+\pi}^2]^{\frac{1}{2}}.$$

This lemma is a special case of the result due to M.H.Gulzar [7].

Lemma 2. *If $P(z)$ is a polynomial of degree n, then for $R \geq 1$*

$$\max_{|z|=R}|p(z)| \leq R^n\|p\| - 2\frac{(R^n-1)}{n+2}|a_0| - |a_1|\left[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right], \text{ for } n > 2 \tag{17}$$

and

$$\max_{|z|=R}|p(z)| \leq R^2\|p\| - \frac{(R-1)}{2}[(R+1)|a_0| + (R-1)|a_1|], \text{ for } n = 2. \tag{18}$$

The above lemma is ascribed to Dewan et.al [5].

Lemma 3. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \geq k \geq 1$, then for $|z| = 1$*

$$k|p'(z)| \leq |np(z) - zp'(z)| - nm,$$

where $m = \min_{|z|=k}|p(z)|$.

Lemma 3 is a special case of a result due to A. Aziz and N. A. Rather [3].

Lemma 4. *If $p(z)$ is a polynomial of degree n, then for $|z|=1$ and for every real λ*

$$|p'(z)|^2 + |np(z) - zp'(z)|^2 \leq \frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].$$

The above lemma is due to A.Aziz [2].

Lemma 5. *If $p(z)$ is a polynomial of degree n which has no zeros in $|z|<k$, $k \geq 1$ and $m = \min_{|z|=k}|p(z)|$ then for every real α*

$$\|p'\| \leq \frac{n}{\sqrt{2(1+k^2)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}},$$

where M_α is defined by (6).

Lemma 6. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z|<k$, $k \leq 1$ and $m = \min_{|z|=k}|p(z)|$, then for $|z|=1$*

$$k^n\|p'\| + nm \leq \|q'\|,$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

Lemmas 5 and 6 are due to Rather and Shah [13].

3. Main Results

In this paper we first prove the generalization of inequality (7) which is ascribed to A.Aziz [2]. More precisely we prove the following result.

Theorem 1. *If $p(z)$ is a polynomial of degree n, then for every real α and $R > \sigma \geq 1$*

$$\begin{aligned}\|p(Rz) - p(\sigma z)\| \leq{}& \frac{R^n - \sigma^n}{2}[M_\alpha + M_{\alpha+\pi}] - \frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n} - (R-\sigma)\right)\\ &- 2|a_2|\left[\frac{(R^n-\sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2}-\sigma^{n-2}) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right],\\ &\text{for } n>3\end{aligned} \tag{19}$$

and

$$\begin{aligned}\|P(Rz) - P(\sigma z)\| \leq{}& \frac{R^3-\sigma^3}{2}[M_\alpha + M_{\alpha+\pi}] - |a_1|\left(\frac{R^3-\sigma^3-3(R-\sigma)}{6}\right)\\ &- |a_2|\left[\frac{(R-1)^3-(\sigma-1)^3}{3}\right],\ \text{for } n=3,\end{aligned} \tag{20}$$

where M_α is defined by (6) and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha+\pi$. The result is the best possible and equality in (19) and (20) holds for $p(z) = z^n + re^{i\alpha}$, $-1 \leq r \leq 1$.

Proof. Let $n > 3$. Since $p(z)$ is a polynomial of degree $n > 3$, therefore $p'(z)$ is of degree $n \geq 3$, applying inequality (17) of Lemma 2 we obtain for all $v \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(ve^{i\theta})| \leq v^{n-1}\|p'\| - 2\frac{(v^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{v^{n-1}-1}{n-1} - \frac{v^{n-3}-1}{n-3}\right].$$

Using inequality (5) we get,

$$|p'(ve^{i\theta})| \leq \frac{nv^{n-1}}{2}(M_\alpha + M_{\alpha+\pi}) - 2\frac{(v^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{v^{n-1}-1}{n-1} - \frac{v^{n-3}-1}{n-3}\right].$$

For each θ, $0 \leq \theta < 2\pi$ and $R > \sigma \geq 1$, it follows that

$$\begin{aligned}
|p(Re^{i\theta}) - p(\sigma e^{i\theta})| &= \left|\int_\sigma^R e^{i\theta}p'(ve^{i\theta})dv\right| \\
&\leq \int_\sigma^R |p'(ve^{i\theta})|dv \\
&\leq \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)}{2}\int_\sigma^R v^{n-1}dv - \frac{2|a_1|}{n+1}\int_\sigma^R (v^{n-1}-1)dv \\
&\quad - 2|a_2|\int_\sigma^R \left(\frac{v^{n-1}-1}{n-1} - \frac{v^{n-3}-1}{n-3}\right)dv \\
&= \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)}{2}\frac{(R^n - \sigma^n)}{n} - \frac{2|a_1|}{n+1}\left(\frac{R^n - \sigma^n}{n} - (R-\sigma)\right) \\
&\quad - 2|a_1|\left[\frac{(R^n-\sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2}-\sigma^{n-2}) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right],
\end{aligned}$$

equivalently

$$\begin{aligned}
\|p(Rz) - p(\sigma z)\| \leq \frac{R^n - \sigma^n}{2}[M_\alpha + M_{\alpha+\pi}] - \frac{2|a_1|}{n+1}\left(\frac{R^n - \sigma^n}{n} - (R-\sigma)\right) \\
- 2|a_2|\left[\frac{(R^n-\sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2}-\sigma^{n-2}) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right].
\end{aligned}$$

This is the desired result for $n > 3$. Furthermore the case for $n = 3$ follows on the same lines but instead of using inequality (17) of Lemma 2 we use inequality (18) of the same Lemma. $\square$

Theorem 2. *If $p(z)$ is a polynomial of degree n such that $p(1) = 0$, then for $0 < w \le n$ and $\alpha = 0$*

$$\left|p\left(1-\frac{w}{n}\right)\right| \le \frac{1}{2}\left[1-\left(1-\frac{w}{n}\right)^n\right]\{M_0+M_\pi\} - \frac{2|a_{n-1}|}{n+1}\left(\frac{1-(1-w/n)^n}{n} - \frac{w}{n}(1-w/n)^{n-1}\right) - 2|a_{n-2}|\chi(w,n), \text{ for } n>3 \tag{21}$$

and

$$\left|p\left(1-\frac{w}{n}\right)\right| \le \frac{1}{2}\left[1-\left(1-\frac{w}{n}\right)^3\right]\{M_0+M_\pi\} - \frac{|a_{n-1}|}{6}\left(1-\left(1-\frac{w}{3}\right)^3 - w\left(1-\frac{w}{3}\right)^2\right) - \frac{|a_{n-2}|}{3}\left(\frac{w}{3}\right)^3, \text{ for } n=3, \tag{22}$$

where

$$\chi(w,n) = \left[\frac{1-(1-w/n)^n - w(1-w/n)^{n-1}}{n(n-1)} - \frac{(1-w/n)^2-(1-w/n)^n-(w-2w/n)(1-w/n)^{n-1}}{(n-2)(n-3)}\right]$$

and M_0 is defined by (6). The result is the best possible and equality in (21) holds for $p(z) = z^n - 1$.

Proof. **Case I, n > 3**: If $t(z) = z^n\overline{p(\frac{1}{\bar z})}$, then $|t(z)| = |p(z)|$ for $|z| = 1$ and by the hypothesis we have $t(1) = \overline{p(1)} = 0$. On using inequality (19) of Theorem 1 to the polynomial $t(z)$ for $\alpha = 0$ and $\sigma = 1$, we get for $R > 1$

$$|t(R)| \le \frac{R^n-1}{2}[M_0+M_\pi] - \frac{2|a_{n-1}|}{n+1}\left(\frac{R^n-\sigma^n}{n} - (R-\sigma)\right) - 2|a_{n-2}|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2}-\sigma^{n-2})-(n-2)(R-\sigma)}{(n-2)(n-3)}\right].$$

This gives for $R > 1$

$$|t(1/R)| \le \frac{1}{2}(1-R^{-n})[M_0+M_\pi] - \frac{2|a_{n-1}|}{n+1}\left(\frac{1-R^{-n}}{n} - (R^{1-n}-R^{-n})\right) - 2|a_{n-2}|\left[\frac{(1-R^{-n})-n(R^{1-n}-R^{-n})}{n(n-1)} - \frac{(R^{-2}-R^{-n})-(n-2)(R^{1-n}-R^{-n})}{(n-2)(n-3)}\right].$$

Since $0 < w \leq n$, so that $(1-w/n)^{-1} > 1$ and therefore, in particular, replace R by $(1-w/n)^{-1} > 1$ and after simplification we have,

$$\begin{aligned}\left|p\left(1-\frac{w}{n}\right)\right| \leq \frac{1}{2}\left[1-\left(1-\frac{w}{n}\right)^n\right]\{M_0+M_\pi\} \\ -\frac{2|a_{n-1}|}{n+1}\left(\frac{1-(1-w/n)^n}{n}-\frac{w}{n}(1-w/n)^{n-1}\right) \\ -2|a_{n-2}|\chi(w,n),\end{aligned}$$

where

$$\begin{aligned}\chi(w,n) = \Bigg[&\frac{1-(1-w/n)^n - w(1-w/n)^{n-1}}{n(n-1)} \\ &-\frac{(1-w/n)^2-(1-w/n)^n-(w-2w/n)(1-w/n)^{n-1}}{(n-2)(n-3)}\Bigg].\end{aligned}$$

Case II, n = 3: This can be established identically as above by using inequality (20) of Theorem 1. ☐

Now we present the refinement of inequality (12). Here we are able to prove

Theorem 3. *If $p(z)$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \geq 1$ and $m = \min_{|z|=1}|p(z)|$, then for all real α and $R > \sigma \geq 1$*

$$\begin{aligned}\|p(Rz)-p(\sigma z)\| \leq \frac{R^n-\sigma^n}{2}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n}-(R-\sigma)\right) \\ -2|a_2|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2}-\sigma^{n-2})-(n-2)(R-\sigma)}{(n-2)(n-3)}\right], \\ \text{if } n>3\end{aligned} \tag{23}$$

and

$$\begin{aligned}\|p(Rz)-p(\sigma z)\| \leq \frac{R^3-\sigma^3}{2}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}} - |a_1|\left(\frac{(R^3-\sigma^3)-3(R-\sigma)}{6}\right) \\ -|a_2|\left[\frac{(R-1)^2-(\sigma-1)^3}{3}\right], \text{ if } n=3,\end{aligned} \tag{24}$$

Proof. Since $p(z)$ has all its zeros in $|z| \geq 1$ and $m = \min_{|z|=1}|p(z)|$, therefore by Lemma 3 with $k=1$, we have for $|z|=1$

$$(|p'(z)|+mn)^2 \leq |np(z)-zp'(z)|^2.$$

. This in conjunction with Lemma 4 gives

$$\begin{aligned}|p'(z)|^2 + (|p'(z)| + mn)^2 &\le |p'(z)|^2 + |np(z) - zp'(z)|^2 \\ &\le \frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].\end{aligned}$$

Since we have $(|p'(z)| + mn)^2 = |p'(z)|^2 + (mn)^2 + 2mn|p'(z)|$.
This gives

$$(|p'(z)| + mn)^2 \ge |p'(z)|^2 + (mn)^2.$$

Therefore, we have

$$\|p'\| \le \frac{n}{2}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}}. \tag{25}$$

Applying inequality (17) of Lemma 2 with $R = s \ge 1$ to the polynomial $p'(z)$ which is of degree $n-1$, we obtain for $n > 3$

$$|p'(se^{i\theta})| \le s^{n-1}\|p'\| - \frac{2(s^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right].$$

With the help of inequality (25), we obtain for $n > 3$

$$|p'(se^{i\theta})| \le \frac{ns^{n-1}}{2}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}} - \frac{2(s^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right].$$

Now for each $0 \le \theta < 2\pi$ and $R > \sigma \ge 1$, we have

$$\begin{aligned}|p(Re^{i\theta}) - p(\sigma e^{i\theta})| &= \left|\int_\sigma^R e^{i\theta}p'(se^{i\theta})ds\right| \\ &\le \int_\sigma^R |p'(se^{i\theta})|ds \\ &\le \frac{n}{2}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}}\int_\sigma^R s^{n-1}ds - \frac{2|a_1|}{n+1}\int_\sigma^R (s^{n-1}-1)ds \\ &- 2|a_2|\int_\sigma^R\left(\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right)ds \\ &= \frac{R^n-\sigma^n}{2}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n} - (R-\sigma)\right) \\ &- 2|a_2|\left[\frac{(R^n-\sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^n-\sigma^n) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right],\end{aligned}$$

which implies

$$\begin{aligned}\|p(Rz)-p(\sigma z)\| \leq{}& \frac{R^n-\sigma^n}{2}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}}-\frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n}-(R-\sigma)\right)\\ &-2|a_2|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)}-\frac{(R^n-\sigma^n)-(n-2)(R-\sigma)}{(n-2)(n-3)}\right].\end{aligned}$$

This proves the result in case $n>3$. For the case $n=3$, the result follows from similar lines but instead of using inequality (17) of Lemma 2, we use inequality (18) of the same Lemma and this proves the theorem completely. □

As a refinement of inequality (13), we prove the following result.

Theorem 4. *If $p(z)$ is a polynomial of degree $n\geq 3$ having all its zeros in $|z|\geq k\geq 1$ and $m=\min_{|z|=k}|p(z)|$, then for all real α and $R>\sigma\geq 1$*

$$\begin{aligned}\|p(Rz)-p(\sigma z)\| \leq{}& \frac{R^n-\sigma^n}{\sqrt{2(1+k^2)}}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}}-\frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n}-(R-\sigma)\right)\\ &-2|a_2|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)}-\frac{(R^n-\sigma^n)-(n-2)(R-\sigma)}{(n-2)(n-3)}\right],\\ &\textit{if } n>3\end{aligned}\tag{26}$$

and

$$\begin{aligned}\|p(Rz)-p(\sigma z)\| \leq{}& \frac{R^3-\sigma^3}{\sqrt{2(1+k^2)}}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}}-|a_1|\left(\frac{(R^3-\sigma^3)-3(R-\sigma)}{6}\right)\\ &-|a_2|\left[\frac{(R-1)^2-(\sigma-1)^3}{3}\right],\ \textit{if } n=3,\end{aligned}\tag{27}$$

where M_α is defined by (6).

Proof. The proof of this theorem follows easily on using arguments similar to that used in the proof of Theorem 3 but instead of using inequality (25) we use Lemma 5. We omit the details. □

Next we establish the upper bound estimate for $\|p(Rz)-p(\xi z)\|$ and thereby prove the following improvement of inequality (15).

Theorem 5. *Let $p(z)$ be a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, $k \leq 1$, then for all real α and $R > \xi \geq 1$*

$$\begin{aligned}\|p(Rz) - p(\xi z)\| \leq & \frac{R^n - \xi^n}{\sqrt{2(1+k^{2n})}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n - \xi^n}{n} - (R-\xi)\right) \\ & - 2|a_2|\left[\frac{(R^n - \xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2} - \xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)}\right], \\ & \text{for } n > 3 \end{aligned} \tag{28}$$

and

$$\begin{aligned}\|p(Rz) - p(\xi z)\| \leq & \frac{R^3 - \xi^3}{\sqrt{2(1+k^6)}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - |a_1|\left(\frac{(R^3 - \xi^3) - 3(R-\xi)}{6}\right) \\ & - |a_2|\left[\frac{(R-1)^3 - (\xi-1)^3}{3}\right], \text{ for } n = 3. \end{aligned} \tag{29}$$

Proof. Let $n > 3$. Since $p(z)$ is a polynomial of degree $n > 3$, it follows that $p'(z)$ is a polynomial of degree $n \geq 3$. Hence applying inequality (17) of Lemma 2 to the polynomial $p'(z)$ with $k = s \geq 1$, we have for $n > 3$

$$|p'(se^{i\theta})| \leq s^{n-1}\|p'\| - 2\frac{(s^n - 1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1} - 1}{n-1} - \frac{s^{n-3} - 1}{n-3}\right]$$

This gives with the help of Lemma 1,

$$|p'(se^{i\theta})| \leq s^{n-1}\left[\frac{n}{2^{\frac{1}{2}}(1+k^{2n})^{\frac{1}{2}}}[M_\alpha^2 + M_{\alpha+\pi}^2]^{\frac{1}{2}}\right] - 2\frac{(s^n-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right].$$

Hence for each θ, $0 \le \theta < 2\pi$ and $R > \xi \ge 1$

$$\begin{aligned}
|p(Re^{i\theta}) - p(\xi e^{i\theta})| &= \left| \int_{\xi}^{R} e^{i\theta} p'(se^{i\theta}) ds \right| \\
&\le \int_{\xi}^{R} |p'(se^{i\theta})| ds \\
&\le \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^{2n})^{\frac{1}{2}}} \int_{\xi}^{R} s^{n-1} ds - \frac{2|a_1|}{n+1} \int_{\xi}^{R} (s^{n-1} - 1) ds \\
&- 2|a_2| \int_{\xi}^{R} \left(\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3} \right) ds \\
&= \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^{2n})^{\frac{1}{2}}} \frac{(R^n - \xi^n)}{n} - \frac{2|a_1|}{n+1} \left(\frac{R^n - \xi^n}{n} - (R - \xi) \right) \\
&- 2|a_1| \left[\frac{(R^n - \xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2} - \xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)} \right].
\end{aligned}$$

This implies,

$$\begin{aligned}
\|p(Rz) - p(\xi z)\| \le \frac{R^n - \xi^n}{\sqrt{2}(1+k^{2n})^{\frac{1}{2}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1} \left(\frac{R^n - \xi^n}{n} - (R - \xi) \right) \\
- 2|a_2| \left[\frac{(R^n - \xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2} - \xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)} \right].
\end{aligned}$$

This is the desired result for the case $n > 3$. For $n = 3$, using inequality (18) of Lemma 2 with $k = s \ge 1$ to the polynomial $p'(z)$ we obtain

$$|p'(se^{i\theta})| \le s^2 \|p'\| - \frac{(s-1)}{2} [(s+1)|a_1| + (s-1)|a_2|].$$

As before, again this gives with the help of Lemma 1 that

$$|p'(se^{i\theta})| \le s^2 \frac{3}{\sqrt{2}(1+k^6)^{\frac{1}{2}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \frac{(s-1)}{2} [(s+1)|a_1| + (s-1)|a_2|].$$

Now for each θ, $0 \le \theta < 2\pi$ and $R > \xi \ge 1$

$$
\begin{aligned}
|p(Re^{i\theta}) - p(\xi e^{i\theta})| &\le \int_{\xi}^{R} |p'(se^{i\theta})| ds \\
&\le \int_{\xi}^{R} \left[\frac{3(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^6)^{\frac{1}{2}}} s^2 - \frac{s^2-1}{2}|a_1| - (s-1)^2|a_2| \right] ds \\
&= \frac{3(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^6)^{\frac{1}{2}}} \frac{R^3-\xi^3}{3} - \frac{1}{2}\left[\frac{R^3-\xi^3}{3} - (R-\xi)\right]|a_1| \\
&\quad - \left[\frac{(R-1)^3 - (\xi-1)^3}{3}\right]|a_2|,
\end{aligned}
$$

i.e,

$$
\begin{aligned}
\|p(Rz) - p(\xi z)\| \le \frac{R^3-\xi^3}{2^{\frac{1}{2}}(1+k^6)^{\frac{1}{2}}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - |a_1| \left(\frac{(R^3-\xi^3) - 3(R-\xi)}{6} \right) \\
- |a_2| \left[\frac{(R-1)^3 - (\xi-1)^3}{3} \right].
\end{aligned}
$$

This proves the theorem for the case $n = 3$. □

Finally we present the refinement and generalization for the upper bound of inequality (16). More precisely we prove the following result.

Theorem 6. *Let $p(z)$ be a polynomial of degree $n \ge 3$ which has no zeros in $|z| < k$, $k \le 1$ and $m = \min\limits_{|z|=k} |p(z)|$ then for all real α and $R > \xi \ge 1$*

$$
\begin{aligned}
\|p(Rz) - p(\xi z)\| \le \frac{R^n - \xi^n}{\sqrt{2(1+k^{2n})}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\xi^n}{n} - (R-\xi) \right) \\
- 2|a_2| \left[\frac{(R^n-\xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2} - \xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)} \right], \\
\text{if } n > 3
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
\|p(Rz) - p(\xi z)\| \le \frac{R^3 - \xi^3}{\sqrt{2(1+k^6)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - |a_1| \left(\frac{(R^3-\xi^3) - 3(R-\xi)}{6} \right) \\
- |a_2| \left[\frac{(R-1)^3 - (\xi-1)^3}{3} \right], \ \text{if } n = 3,
\end{aligned}
\tag{31}
$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain maximum at the same point on* $|z| = 1$*, where* $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$. *The result is best possible and equality in (30) holds for* $p(z) = z^n + k^n$.

Proof. Since $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, therefore,

$$|q'(z)| = |np(z) - zp'(z)| \text{ for } |z| = 1.$$

By hypothesis $|p'(z)|$ and $|q'(z)|$ attain maximum at the same point on $|z| = 1$. If we consider

$$\max_{|z|=1} |p'(z)| = |p(e^{i\alpha})|,\ 0 \le \alpha < 2\pi$$

then it is clear that,

$$\max_{|z|=1} |q'(z)| = |q(e^{i\alpha})|,\ 0 \le \alpha < 2\pi.$$

Since $p(z)$ does not vanish in $|z| < k, k \le 1$ and $m = \min_{|z|=k} |p(z)|$. Therefore by Lemma 6 and by using above maximum values of $|p'(z)|$ and $|q'(z)|$, we get

$$(k^n|p'(e^{i\alpha})| + nm)^2 \le |q'(e^{i\alpha})|^2.$$

This gives with the help of Lemma 4

$$|p'(e^{i\alpha})|^2 + (k^n|p'(e^{i\alpha})| + nm)^2 \le |p'(e^{i\alpha})|^2 + |q'(e^{i\alpha})|^2$$
$$\frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].$$

Since

$$(k^n|p'(e^{i\alpha})| + nm)^2 \ge k^{2n}|p'(e^{i\alpha})|^2 + n^2m^2.$$

Consequently,

$$|p'(e^{i\alpha})|^2 + k^{2n}|p'(e^{i\alpha})|^2 + n^2m^2 \le \frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].$$

Equivalently,

$$|p'(e^{i\alpha})|^2 \le \frac{n^2}{2(1+k^2)}[M_\lambda^2 + M_{\lambda+\pi}^2 - 2m^2]$$

and therefore, we have

$$\|p'\| \le \frac{n}{\sqrt{2(1+k^{2n})}}[M_\lambda^2 + M_{\lambda+\pi}^2 - 2m^2]^{\frac{1}{2}}. \tag{32}$$

Since $p(z)$ is a polynomial of degree $n > 3$, it follows that $p'(z)$ is a polynomial of degree $n \ge 3$. Hence applying inequality (17) of Lemma 2 to the polynomial $p'(z)$ with $k = s \ge 1$, we have for $n > 3$

$$|p'(se^{i\theta})| \le s^{n-1}\|p'\| - 2\frac{(s^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right],$$

This in conjunction with (32) gives,

$$|p'(se^{i\theta})| \leq s^{n-1}\left[\frac{n}{\sqrt{2(1+k^{2n})}}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}}\right] - 2\frac{(s^{n-1}-1)}{n+1}|a_1|$$
$$- 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right].$$

Hence for each θ, $0 \leq \theta < 2\pi$ and $R > \xi \geq 1$

$$\begin{aligned}
|p(Re^{i\theta}) - p(\xi e^{i\theta})| &= \left|\int_\xi^R e^{i\theta}p'(se^{i\theta})ds\right| \\
&\leq \int_\xi^R |p'(se^{i\theta})|ds \\
&\leq \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}}{\sqrt{2(1+k^{2n})}}\int_\xi^R s^{n-1}ds - \frac{2|a_1|}{n+1}\int_\xi^R (s^{n-1}-1)ds \\
&- 2|a_2|\int_\xi^R \left(\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right)ds \\
&= \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}}{\sqrt{2(1+k^{2n})}}\frac{(R^n - \xi^n)}{n} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\xi^n}{n} - (R-\xi)\right) \\
&- 2|a_1|\left[\frac{(R^n-\xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2}-\xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)}\right].
\end{aligned}$$

This implies,

$$\|p(Rz) - p(\xi z)\| \leq \frac{R^n - \xi^n}{\sqrt{2(1+k^{2n})}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\xi^n}{n} - (R-\xi)\right)$$
$$- 2|a_2|\left[\frac{(R^n-\xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2}-\xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)}\right].$$

This proves inequality (30). For the proof of inequality (31), we use inequality (18) of Lemma 2 rather than inequality (17) of the same Lemma. □

Author Contribution Statements The authors have made substantial contributions to the analysis and interpretation of the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments The authors are extremely thankful to the referees and editors for their valuable suggestions which made this paper much improved.

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