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Solutions of Fractional Kinetic Equations using the $(p, q; \ell)$-Extended τ-Gauss Hypergeometric Function

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Abstract − The main objective of this paper is to use the newly proposed $(p, q; \ell)$-extended beta function to introduce the $(p, q; \ell)$-extended τ-Gauss hypergeometric and the $(p, q; \ell)$-extended τ-confluent hypergeometric functions with some of their properties, such as the Laplace-type and the Euler-type integral formulas. Another is to apply them to fractional kinetic equations that appear in astrophysics and physics using the Laplace transform method.

Keywords− *Beta function, hypergeometric function, fractional calculus, pochhammer symbol, integral transforms*

Mathematics Subject Classification (2020) – 33E99, 44A20

1. Introduction

Recently, the applications of the special functions of mathematics have developed significantly in such fields as fractional calculus, approximation theory, mathematical physics, engineering, science and technology [1-3]. One very interesting application area of special functions of mathematics is the extension of the standard kinetic equations by its integration [4]

$$\Lambda(t) - \Lambda_0 = -c^{\partial} \; {}_0D_t^{-\partial}\{\Lambda(t)\} \tag{1}$$

for any positive constant c, $\Lambda(t)$ represents the reaction rate, Λ_0 represents $\Lambda(t)$ at $t = 0$, and ${}_0D_t^{-\partial}$ is the Riemann-Liouville fractional integral operator defined by

$${}_0D_t^{-\partial}\{\Lambda(t)\} = \frac{1}{\Gamma(\partial)} \int_0^t (t-u)^{\partial-1} \Lambda(u) du, \; (Re(\partial) > 0, t > 0)$$

They [4] also give the following solution to equation (1):

$$\Lambda(t) = \Lambda_0 E_{\partial}\left(-c^{\partial} t^{\partial}\right), (\partial \in \mathbb{R}^+)$$

Extensions, generalizations and different forms of equation (1) have been studied by Saxena et al., [5, 6] using functions of Wiman and Prabhakar [7-9], Khan et al., [10] studied the following fractional kinetic equations:

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$$\Lambda(t)-\Lambda_0 t^{\aleph} R_{p,q}^{\tau}\left(a,b;c;\psi t^{\partial}\right)=-\sigma^{\partial}\ {}_0D_t^{-\partial}\{\Lambda(t)\} \tag{2}$$

and

$$\Lambda(t)-\Lambda_0 t^{\aleph} R_{p,q}^{\tau}\left(a,b;c;\psi t^{\partial}\right)=-\left\{\sum_{\omega\geq 1}\binom{\kappa}{\omega}\sigma^{\partial\omega}\ {}_0D_t^{-\partial\omega}\right\}\Lambda(t) \tag{3}$$

where $R_{p,q}^{\tau}(;)$ is the (p,q)-extended τ-Gauss hypergeometric function [11]

$$R_{p,q}^{\tau}\left(a,b;c;\psi t^{\partial}\right)=\sum_{\Bbbk\geq 0}(a)_{\Bbbk}\frac{B_{p,q}(b+\Bbbk\tau,c-b)}{B(b,c-b)}\frac{z^{\Bbbk}}{\Bbbk!}$$

for all $\min\{Re(p),Re(q)\}>0,\tau\geq 0,\ell\in\mathbb{R}^{+}\backslash\{1\},Re(a)>Re(b)>0,$ and $B_{p,q}(\wp,\Im)$ is the extended beta function defined by [12]

$$B_{p,q}(\wp,\Im)=\int_0^1 t^{\wp-1}(1-t)^{\Im-1}\exp\left(-\frac{p}{t}-\frac{q}{1-t}\right)dt$$

for all $\min\{Re(p),Re(q)\}>0,\ \min\{Re(\wp),Re(\Im)\}>0.$

Readers can refer to [13-20] for more generalizations and extensions of extended fractional kinetic equations.

The main objective of this paper is to introduce the new the $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions with some properties and their applications to fractional kinetic equations via the Laplace transforms methods. Furthermore, the resulting functions and equations can be reduced to well-known and perhaps new results. This paper is presented as follows: Section one is compressed with some preliminaries. In section 3, the $(p,q;\ell)$-extended τ-hypergeometric functions and some of their properties have been discussed. In section 4, the solution of the fractional kinetic equations contains the $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions. In section 5, include a conclusion.

2. Preliminaries

In this paper, the extended fractional kinetic equations will be studied by using the following $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions:

Definition 2.1. The new $(p,q;\ell)$-extended τ-Gauss hypergeometric function is

$$R_{p,q}^{\tau;\phi,\varphi}(a,b;c;z;\ell)=\sum_{\Bbbk\geq 0}(a)_{\Bbbk}\frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau,c-b;\ell)}{B(b,c-b)}\frac{z^{\Bbbk}}{\Bbbk!} \tag{4}$$

for all $\min\{Re(p),Re(q)\}>0,\ \min\{Re(\phi),Re(\varphi)\}>0,\tau\geq 0,\ell\in\mathbb{R}^{+}\backslash\{1\},Re(a)>Re(b)>0.$

Definition 2.2. The new $(p,q;\ell)$-extended $(p,q;\ell)$-confluent hypergeometric function is

$$\Phi_{p,q}^{\tau;\phi,\varphi}(b;c;z;\ell)=\sum_{\Bbbk\geq 0}\frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau,c-b;\ell)}{B(b,c-b)}\frac{z^{\Bbbk}}{\Bbbk!} \tag{5}$$

for all $\min\{Re(p),Re(q)\}>0,\ \min\{Re(\phi),Re(\varphi)\}>0,\tau\geq 0,\ell\in\mathbb{R}^{+}\backslash\{1\},Re(a)>Re(b)>0,$ and $B_{p,q}^{\phi,\varphi}(\wp,\Im;\ell)$ is the extended beta function proposed in [21]

$$B_{p,q}^{\phi,\varphi}(\wp,\Im;\ell)=\int_0^1 t^{\wp-1}(1-t)^{\Im-1}\ell^{\left(-\frac{p}{t^{\phi}}-\frac{q}{(1-t)^{\varphi}}\right)}dt \tag{6}$$

for all $\min\{Re(p),Re(q)\}>0,\ \min\{Re(\wp),Re(\Im)\}>0,\tau\geq 0,\ell\in\mathbb{R}^{+}\backslash\{1\},\ \min\{Re(\wp),Re(\Im)\}>0.$

3. The $(p,q;\ell)$-Extended τ- Hypergeometric Functions

In this section, the integral representation of the $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions are established in the following theorem:

Theorem 3.1. The following Laplace-type integral formula holds:

$$R_{p,q}^{\tau;\phi,\varphi}(a,b;c;z;\ell)=\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\exp(-t)\,\Phi_{p,q}^{\tau;\phi,\varphi}(b;c;z;\ell)dt$$

for all $\min\{Re(p),Re(q)\}>0,\ \min\{Re(\phi),Re(\varphi)\}>0,\tau\geq 0,\ell\in\mathbb{R}^+\backslash\{1\},Re(a)>0,Re(z)<1.$

Proof. Consider equation (4) and expansion of the pochhammer notation in [22]

$$(a)_{\Bbbk}=\frac{1}{\Gamma(a)}\int_0^\infty t^{a+\Bbbk-1}\exp(-t)\,dt$$

gives

$$R_{p,q}^{\tau;\phi,\varphi}(a,b;c;z;\ell)=\sum_{\Bbbk\geq 0}\left\{\frac{1}{\Gamma(a)}\int_0^\infty t^{a+\Bbbk-1}\exp(-t)\,dt\right\}\frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau,c-b;\ell)}{B(b,c-b)}\frac{z^{\Bbbk}}{\Bbbk!}$$

As a result of changing the order of integration and summation,

$$\begin{aligned}R_{p,q}^{\tau;\phi,\varphi}(a,b;c;z;\ell) &= \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\exp(-t)\left\{\sum_{\Bbbk\geq 0}\frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau,c-b;\ell)}{B(b,c-b)}\frac{(tz)^{\Bbbk}}{\Bbbk!}\right\}dt\\ &= \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\exp(-t)\,\Phi_{p,q}^{\tau;\phi,\varphi}(b;c;z;\ell)dt\end{aligned}$$

Theorem 3.2. The following Euler-type equality holds:

$$R_{p,q}^{\tau;\phi,\varphi}(a,b;c;z;\ell)=\frac{1}{B(b,c-b)}\int_0^1 t^{a-1}(1-t)^{c-b-1}(1-t^\tau z)^{-a}\ell^{\left(-\frac{p}{t^\phi}-\frac{q}{(1-t)^\varphi}\right)}dt$$

for all $\min\{Re(p),Re(q)\}>0,\ \min\{Re(\phi),Re(\varphi)\}>0,\ \tau\geq 0,\ \ell\in\mathbb{R}^+\backslash\{1\},\ Re(b)>Re(b)>0,$ and $|\arg(1-z)|<\pi.$

Proof. Rewritten equation (4) in term of $(p,q;\ell)$-extended beta function in (6), yields

$$R_{p,q}^{\tau;\phi,\varphi}(a,b;c;z;\ell)=\sum_{\Bbbk\geq 0}\frac{(a)_{\Bbbk}}{B(b,c-b)}\left\{\int_0^1 t^{b+\Bbbk\tau-1}(1-t)^{c-b-1}(1-t^\tau z)^{-a}\ell^{\left(-\frac{p}{t^\phi}-\frac{q}{(1-t)^\varphi}\right)}dt\right\}\frac{z^{\Bbbk}}{\Bbbk!}$$

Changing the order of integration and summation will result in

$$\begin{aligned}R_{p,q}^{\tau;\phi,\varphi}(a,b;c;z;\ell) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\ell^{\left(-\frac{p}{t^\phi}-\frac{q}{(1-t)^\varphi}\right)}\left\{\sum_{\Bbbk\geq 0}(a)_{\Bbbk}\frac{(tz)^{\Bbbk}}{\Bbbk!}\right\}dt\\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{a-1}(1-t)^{c-b-1}(1-t^\tau z)^{-a}\ell^{\left(-\frac{p}{t^\phi}-\frac{q}{(1-t)^\varphi}\right)}dt\end{aligned}$$

Considering equation (5), the following corollary can be obtained:

Corollary 3.1. The following result is also holds true:

$$\Phi_{p,q}^{\tau;\phi,\varphi}(b;c;z;\ell)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp(t^\tau z)\,\ell^{\left(-\frac{p}{t^\phi}-\frac{q}{(1-t)^\varphi}\right)}dt$$

for all $\min\{Re(p),Re(q)\}>0,\ \min\{Re(\phi),Re(\varphi)\}>0,\ \tau\geq 0,\ \ell\in\mathbb{R}^+\backslash\{1\},\ Re(c)>Re(b)>0.$

4. Extended Fractional Kinetic Equations Solutions

In this section, the applications of $(p,q;\ell)$-extended τ-Gauss hypergeometric and τ-confluent hypergeometric functions to extended fractional kinetic equations are established using the Laplace transform method in the following theorem:

Theorem 4.1. The extended fractional kinetic equation

$$\Lambda(t)-\Lambda_0 t^{\aleph} R_{p,q}^{\tau;\phi,\varphi}\left(a,b;c;\psi t^{\partial};\ell\right)=-\sigma^{\partial}\ {}_0D_t^{-\partial}\{\Lambda(t)\} \tag{7}$$

for all $\aleph,\partial,\sigma\in\mathbb{R}^+,\psi\in\mathbb{C}$ with $\delta\neq\psi,\tau\in\mathbb{R}_0^+;\min\{Re(p),Re(q)\}>0$, $\min\{Re(\phi),Re(\varphi)\}>0$, $\tau\geq 0$, $\ell\in\mathbb{R}^+\backslash\{1\}$, $Re(c)>Re(b)>0$.

$$\Lambda(t)=\Lambda_0 t^{\aleph-1}\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\left(\psi t^{\partial}\right)^{\mathbb{k}}}{\mathbb{k}!}\Gamma(\partial\mathbb{k}+\aleph)\mathrm{E}_{\partial,\partial\mathbb{k}+\aleph}\left(-\sigma^{\partial}t^{\partial}\right)$$

is the solution.

Proof. Applying the Laplace transform [23] to equation (7), gives

$$\mathcal{L}\{\Lambda(t);s\}-\Lambda_0\mathcal{L}\left\{t^{\aleph}R_{p,q}^{\tau;\phi,\varphi}\left(a,b;c;\psi t^{\partial};\ell\right);s\right\}=-\sigma^{\partial}\mathcal{L}\left\{{}_0D_t^{-\partial}\{\Lambda(t)\};s\right\}$$

Consider equation (4) and the Laplace transform of the Riemann-Liouville fractional integral [24]

$$\mathcal{L}\left\{{}_0D_t^{-\partial}\{\Lambda(t)\};s\right\}=-s^{\partial}\mathcal{L}\{\Lambda(t)\}$$

yields

$$\mathcal{L}\{\Lambda(t);s\}-\Lambda_0\left[\int_0^{\infty}\exp(-st)\left\{\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\left(\psi t^{\partial}\right)^{\mathbb{k}}}{\mathbb{k}!}\right\}dt\right]=-\sigma^{\partial}s^{\partial}\mathcal{L}\{\Lambda(t)\}$$

When integration and summation are changed, it leads to

$$\mathcal{L}\{\Lambda(t);s\}=\Lambda_0\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\psi^{\mathbb{k}}}{\mathbb{k}!}\left\{\int_0^{\infty}\exp(-st)t^{\partial\mathbb{k}+\aleph-1}dt\right\}\left\{\frac{1}{1+(\sigma s^{-1})^{\partial}}\right\}$$

Using result [25]

$$\int_0^{\infty}\exp(-st)t^{\aleph}dt=\frac{\Gamma(\aleph+1)}{s^{\aleph+1}}\ ,(Re(\aleph)>-1)$$

gives

$$\mathcal{L}\{\Lambda(t);s\}=\Lambda_0\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\psi^{\mathbb{k}}}{\mathbb{k}!}\frac{\Gamma(\partial\mathbb{k}+\aleph)}{s^{\partial\mathbb{k}+\aleph}}\left\{\frac{1}{1+(\sigma s^{-1})^{\partial}}\right\}$$

Applying the geometric series expansion [26]

$$\frac{1}{1+(\sigma s^{-1})^{\partial}}=\sum_{\xi\geq 0}(-1)^{\xi}\sigma^{\sigma\xi}\,s^{-\sigma\xi}$$

leads to

$$\mathcal{L}\{\Lambda(t);s\}\ =\ \Lambda_0\sum_{\mathbb{k}\geq 0}(a)_{\mathbb{k}}\frac{B_{p,q}^{\phi,\varphi}(b+\mathbb{k}\tau,c-b;\ell)}{B(b,c-b)}\frac{\psi^{\mathbb{k}}}{\mathbb{k}!}\frac{\Gamma(\partial\mathbb{k}+\aleph)}{s^{\partial\mathbb{k}+\aleph}}\sum_{\xi\geq 0}(-1)^{\xi}\sigma^{\sigma\xi}\,s^{-\sigma\xi}$$

$$= \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \sum_{\xi\geq 0} (-1)^{\xi} \sigma^{\sigma\xi}\, s^{-(\sigma\xi+\partial\Bbbk+\aleph)}$$

Using the inverse Laplace transform and the result in [25]

$$\mathcal{L}^{-1}\{s^{-\partial}\} = \frac{t^{\partial-1}}{\Gamma(\partial)}$$

one may obtain

$$\Lambda(t) = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \sum_{\xi\geq 0} \frac{(-1)^{\xi}\sigma^{\partial\xi}}{\Gamma(\sigma\xi+\partial\Bbbk+\aleph)} t^{\partial\xi+\partial\Bbbk+\aleph-1}$$

$$= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \sum_{\xi\geq 0} \frac{\left(-\sigma^{\partial} t^{\partial}\right)^{\xi}\sigma^{\sigma\xi}}{\Gamma(\sigma\xi+\partial\Bbbk+\aleph)}$$

$$= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}\left(-\sigma^{\partial} t^{\partial}\right)$$

Theorem 4.2. The extended fractional kinetic equation

$$\Lambda(t) - \Lambda_0 t^{\aleph-1} R_{p,q}^{\tau;\phi,\varphi}\left(a,b;c;\psi t^{\partial};\ell\right) = -\left\{\sum_{\omega\geq 1} \binom{\kappa}{\omega} \sigma^{\partial\omega}\ {}_0D_t^{-\partial\omega}\right\} \Lambda(t) \tag{8}$$

for all $\aleph, \partial, \sigma \in \mathbb{R}^+, \psi, \kappa \in \mathbb{C}$ with $\delta \neq \psi, \tau \in \mathbb{R}_0^+$; $\min\{Re(p), Re(q)\} > 0$, $\min\{Re(\phi), Re(\varphi)\} > 0$, $\tau \geq 0$, $\ell \in \mathbb{R}^+\backslash\{1\}$, $Re(c) > Re(b) > 0$.

$$\Lambda(t) = \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}^{\kappa}\left(-\sigma^{\partial} t^{\partial}\right)$$

is the solution.

Proof. Applying the Laplace transform [23] to equation (8), gives

$$\mathcal{L}\{\Lambda(t);s\} - \Lambda_0 \mathcal{L}\left\{t^{\aleph-1} R_{p,q}^{\tau;\phi,\varphi}\left(a,b;c;\psi t^{\partial};\ell\right);s\right\} = -\sum_{\omega\geq 1} \binom{\kappa}{\omega} \sigma^{\partial\omega} \mathcal{L}\left\{{}_0D_t^{-\partial\omega}\Lambda(t);s\right\}$$

Consider equation (4) and the Laplace transform of the Riemann-Liouville fractional integral [24]

$$\mathcal{L}\left\{{}_0D_t^{-\partial}\{\Lambda(t)\};s\right\} = -s^{\partial}\mathcal{L}\{\Lambda(t);s\},$$

yields

$$\mathcal{L}\{\Lambda(t);s\} - \Lambda_0 \left[\int_0^{\infty} \exp(-st) \left\{t^{\aleph} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!}\right\} dt\right] = \sum_{\omega\geq 1} \binom{\kappa}{\omega} \sigma^{\partial\omega} s^{\partial} \mathcal{L}\{\Lambda(t);s\}$$

By reordering integral and summation, we get

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \left\{\int_0^{\infty} \exp(-st) t^{\partial\Bbbk+\aleph-1} dt\right\} \left\{\frac{1}{\sum_{\omega\geq 1}\binom{\kappa}{\omega}(\sigma s^{-1})^{\partial\omega}}\right\}$$

Using result [25]

$$\int_0^{\infty} \exp(-st) t^{\aleph} dt = \frac{\Gamma(\aleph+1)}{s^{\aleph+1}}\ ,\ (Re(\aleph) > -1)$$

gives

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \frac{\Gamma(\partial\Bbbk+\aleph)}{s^{\partial\Bbbk+\aleph}} \left\{\frac{1}{\sum_{\omega\geq 1}\binom{\kappa}{\omega}(\sigma s^{-1})^{\partial\omega}}\right\}$$

Applying the geometric series expansion in [27]

$$\sum_{\omega\geq 1}\binom{\kappa}{\omega}\sigma^{\partial} = (1+z)^{\kappa}, (\kappa\in\mathbb{C}, |z|<1)$$

leads to

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \frac{\Gamma(\partial\Bbbk+\aleph)}{s^{\partial\Bbbk+\aleph}} \left(1+\sigma^{\partial}s^{\partial}\right)^{\kappa}$$

Can be rewritten using [27]

$$(1-z)^{\kappa} = \sum_{\omega\geq 0}\frac{(\kappa)_{\omega}}{\omega!}z^{\omega}, (\kappa\in\mathbb{C}, |z|<1)$$

so that

$$\mathcal{L}\{\Lambda(t);s\} = \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \frac{\Gamma(\partial\Bbbk+\aleph)}{s^{\partial\Bbbk+\aleph}} \left\{\sum_{\xi\geq 0}\frac{(-1)^{\xi}(\kappa)_{\xi}}{\xi!}\sigma^{\partial\xi}s^{-(\sigma\xi+\partial\Bbbk+\aleph)}\right\}$$

Using the inverse Laplace transform and the result in [25]

$$\mathcal{L}^{-1}\{s^{-\partial}\} = \frac{t^{\partial-1}}{\Gamma(\partial)}$$

The following can be obtained:

$$\begin{aligned}\Lambda(t) &= \Lambda_0 \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \left\{\sum_{\xi\geq 0}\frac{(-1)^{\xi}\sigma^{\partial\xi}(\kappa)_{\xi}}{\Gamma(\partial\xi+\partial\Bbbk+\aleph)}\frac{t^{\,\partial\xi+\partial\Bbbk+\aleph-1}}{\xi!}\right\}\\ &= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\psi^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \left\{\sum_{\xi\geq 0}\frac{(-1)^{\xi}\sigma^{\partial\xi}(\kappa)_{\xi}}{\Gamma(\partial\xi+\partial\Bbbk+\aleph)}\frac{\left(-\sigma^{\partial}t^{\partial}\right)^{\xi}}{\xi!}\right\}\\ &= \Lambda_0 t^{\aleph-1} \sum_{\Bbbk\geq 0} (a)_{\Bbbk} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}^{\kappa}\left(-\sigma^{\partial}t^{\partial}\right)\end{aligned}$$

Considering equations (5), (7), and (8), the following corollaries can be obtained:

Corollary 4.1. The extended fractional kinetic equation

$$\Lambda(t) - \Lambda_0 t^{\aleph}\Phi_{p,q}^{\tau;\phi,\varphi}\left(b;c;\psi t^{\partial};\ell\right) = -\sigma^{\partial}\ {}_0D_t^{-\partial}\{\Lambda(t)\}$$

for all $\aleph,\partial,\sigma\in\mathbb{R}^{+}, \psi\in\mathbb{C}$ with $\delta\neq\psi, \tau\in\mathbb{R}_0^{+}; \min\{Re(p),Re(q)\}>0$, $\min\{Re(\phi),Re(\varphi)\}>0$, $\tau\geq 0$, $\ell\in\mathbb{R}^{+}\backslash\{1\}$, $Re(c)>Re(b)>0$.

$$\Lambda(t) = \Lambda_0 \sum_{\Bbbk\geq 0} \frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau, c-b;\ell)}{B(b,c-b)} \frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!} \Gamma(\partial\Bbbk+\aleph) \mathrm{E}_{\partial,\partial\Bbbk+\aleph}\left(-\sigma^{\partial}t^{\partial}\right)$$

is the solution.

Corollary 4.2. The extended fractional kinetic equation

$$\Lambda(t)-\Lambda_0 t^{\aleph-1}\Phi_{p,q}^{\tau;\phi,\varphi}\left(b;c;\psi t^{\partial};\ell\right)=-\left\{\sum_{\omega\geq 1}\binom{\kappa}{\omega}\sigma^{\partial\omega}\ {}_0D_t^{-\partial\omega}\right\}\Lambda(t)$$

for all $\aleph,\partial,\sigma\in\mathbb{R}^+,\psi,\kappa\in\mathbb{C}$ with $\delta\neq\psi,\tau\in\mathbb{R}_0^+$; $\min\{Re(p),Re(q)\}>0$, $\min\{Re(\phi),Re(\varphi)\}>0$, $\tau\geq 0$, $\ell\in\mathbb{R}^+\backslash\{1\}$, $Re(c)>Re(b)>0$.

$$\Lambda(t)=\Lambda_0 t^{\aleph-1}\sum_{\Bbbk\geq 0}\frac{B_{p,q}^{\phi,\varphi}(b+\Bbbk\tau,c-b;\ell)}{B(b,c-b)}\frac{\left(\psi t^{\partial}\right)^{\Bbbk}}{\Bbbk!}\Gamma(\partial\Bbbk+\aleph)\mathrm{E}_{\partial,\partial\Bbbk+\aleph}^{\kappa}\left(-\sigma^{\partial}t^{\partial}\right)$$

is the solution.

5. Conclusion

The new $(p,q;\ell)$-extended τ-Gauss hypergeometric and $(p,q;\ell)$-extended τ-confluent hypergeometric functions are defined by using the $(p,q;\ell)$-extended beta function in [21] with some of their properties such as integral formulas and their application to the solutions of extended fractional kinetic equations. If the parameters of these newly established functions and equations are appropriately substituted, a number of works already established in the literature are obtained, for example: if $\ell=e$ and $\phi=\varphi=1$, then the results of Khan et al., [10] and Parmar et al., [11]; by setting $\ell=e,\ \phi=\varphi=1$,and $\tau=1$, the extended Gauss hypergeometric and confluent hypergeometric functions presented by Choi et al., [12] will be obtained; by setting $\ell=e,\ \phi=\varphi=1$ and $p=q=0$, the proposed results will be returned to Virchenko et al., [28] and Virchenko [29]; the substituting $\ell=e,\ \phi=\varphi=1,\tau=1$and $p=q$ leads to the results of Chaudhry et al., [30, 31]; finally, by taking $\ell=e,\ \phi=\varphi=1,\tau=1$, and $p=q=1$, the results under discussion will naturally return to the classical results. The extended kinetic equations are expected to have potential applications in nuclear energy, nuclear physics, astrophysics and other related fields. Furthermore, the functions under discussion can be used to study fractional integrals and derivatives such as the Riemann-Liouville, Caputo, Eydilyi-kober, Saigo, Merichev-Saigo-Maide and the Caputo-type Merichev-Saigo-Maide.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

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