INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 15 NO. 1 PAGE 11–19 (2022)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1064158

Rulings on the Surfaces Having Null Axis and Null Profile Curve in Lorentz-Minkowski 3-Space

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(Communicated by Donghe Pei)

ABSTRACT

We define the curves family of the surfaces with null profile curve and null axis, and give some smooth functions in three dimensional Lorentz-Minkowski space $\mathbb{L}^3$. In addition, we compute the third Laplace-Beltrami operator of this type surfaces.

Keywords: *Lorentz-Minkowski space, rulings, surface, null axis, null profile curve, curvature.*
AMS Subject Classification (2020): *Primary: 53A35; Secondary: 53C42.*

1. Introduction

The rotational surfaces with the prescribed mean curvature was studied by Kenmotsu in [16]. Moreover, the helicoidal surfaces in $\mathbb{E}^3$ having the constant or prescribed mean curvature were introduced by Do Carmo and Dajczer [5], and also by Baikoussis and Koufogiorgos [1].

Beneki, Kaimakamis, and Papantoniou [3] obtained different types of helicoidal surfaces in 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ $(= \mathbb{E}^3_1)$. They [2] also studied the surfaces of revolution of the constant Gaussian curvature in $\mathbb{L}^3$. Hano and Nomizu [13] classified the spacelike rotational surfaces in $\mathbb{L}^3$ having the CMC and proved that the profile curve of a rotation surface with nonzero CMC in $\mathbb{L}^3$ can be described as the locus of focus when a quadratic curve is rolled along the axis of rotation. In addition, Sasahara [17] studied the spacelike helicoidal surfaces with CMC in $\mathbb{L}^3$. Dillen and Kühnel [8] proved that any ruled surface with a null ruling is a Weingarten surface whose mean curvature H and Gauss curvature K satisfy the equation $H^2 = K$.

Güler and Vanlı [11] worked Bour's theorem in Minkowski geometry. Güler [10] showed that a helicoidal surface and a rotational surface with light-like profile curve have an isometric relation by Bour's theorem in $\mathbb{L}^3$. He classified the space-like (resp., time-like) helicoidal (resp., rotational) surfaces with null (i.e., light-like) profile curve of (S,L), (T,L) and (L,L)-types. Some relations among the Laplace-Beltrami operator and curvatures of the helicoidal surfaces in Euclidean 3-space were shown by Güler, Yaylı, and Hacısalihoğlu in [12]. In addition, they gave Bour's theorem on Gauss map with some special examples.

We consider the null rulings surfaces having null axis in Lorentz-Minkowski 3-space $\mathbb{L}^3 = (\mathbb{R}^3, ds^2)$, where

$$ds^2 = dx^2 + dy^2 - dz^2 \tag{1.1}$$

is a Lorentzian metric with the pseudo-Euclidean coordinates x, y, z of type (2,1). In section 3, we solve the problems of finding explicitly the surfaces with *null axis* and *null profile curve* in $\mathbb{L}^3$ having the prescribed Gaussian curvature and the mean curvature given by smooth functions. In section 4, we study the third Laplace-Beltrami operator, and some differential geometric properties of the surfaces having null profile curve and null axis.

Received : *27–01–2022,* ***Accepted :*** *09–03–2022*
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2. Preliminaries

A vector (a, b, c) identifies with its transpose throughout the work. For an open interval I, let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{L}^3$ be a curve, and ℓ be a straight line in $\mathbb{L}^3$. A rotational surface is defined by a non-degenerate surface rotating a profile curve γ about a line ℓ (i.e., axis) in $\mathbb{L}^3$.

If the axis ℓ is light-like line, then we may suppose that ℓ is the line spanned by the vector $(0, 1, 1)$. Since the surface is non-degenerate, we may assume the null profile curve γ lies in the $\mathbb{L}^3$ and its parametrization given by

$$\gamma(u) = (h(u), f(u), g(u)), \quad u \in I, \tag{2.1}$$

where h, f, and g are the functions on I such that $h'^2 + f'^2 = g'^2$ for all u. It can be proved that the subgroup of the Lorentz group, which fixes the vector $(0, 1, 1)$, is given by the set of 3×3 orthogonal matrices

$$M(v) = \begin{pmatrix} 1 & -v & v \\ v & 1 - \frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1 + \frac{v^2}{2} \end{pmatrix}, \quad v \in \mathbb{R}. \tag{2.2}$$

Therefore, the surface is parametrized by

$$\mathbf{S}(u, v) = M(v) \cdot \gamma(u). \tag{2.3}$$

The above surface is called a rotational surface having null axis and null profile curve in $\mathbb{L}^3$.

We say that a surface in $\mathbb{L}^3$ is a *time-like surface* if $EG - F^2 < 0$, where E, F, G are the coefficients of the first fundamental form of the surface.

For the surfaces around a light-like axis the corresponding equations are explored only for some particular functional forms of the Gaussian or mean curvatures and are presented in Section 3.

Hence, by using the given smooth function $K = K(u)$ we can find a two-parameter family of curves $\delta(u) = \delta(u, K(u), c_1, c_2)$ defined on a neighborhood of $u_0 \in I$. By applying a rotational motion on the curve δ, we get a two-parameter family of surfaces in $\mathbb{L}^3$ having the Gaussian curvature $K = K(u)$. In the similar method, we can construct a two-parameter family of surfaces in $\mathbb{L}^3$ having the mean curvature $H = H(u)$. This study constitutes a generalization of [1, 4, 6, 14]. For more details concerning surfaces in $\mathbb{E}^3$, see [7, 9, 18].

3. Rulings on the time-like surfaces having null axis, null profile curve

In this section, we study the case of surfaces in $\mathbb{L}^3$ obtained by a motion of a C^2-curve $\delta = \delta(u)$ around a null axis, let us assume the axis defined by the origin and the vector $(0, 1, 1)$. As it was mentioned earlier in (2.3), the parametric equation of the rotational surfaces is given by

$$\mathbf{S}(u, v) = \begin{pmatrix} -f(u)v + g(u)v + h(u) \\ \left(1 - \frac{1}{2}v^2\right) f(u) + \frac{1}{2}v^2 g(u) + vh(u) \\ -\frac{1}{2}v^2 f(u) + \left(1 + \frac{1}{2}v^2\right) g(u) + vh(u) \end{pmatrix}. \tag{3.1}$$

Without losing the generality, let $f(u) = \varphi(u) - u$ and $g(u) = \varphi(u) + u$. Then, equation (3.1) reduces to

$$\mathbf{S}(u, v) = \begin{pmatrix} h(u) + 2uv \\ \varphi(u) - u + uv^2 + vh(u) \\ \varphi(u) + u + uv^2 + vh(u) \end{pmatrix}. \tag{3.2}$$

Theorem 3.1. *Let $\gamma(u) = (h(u), \varphi(u) - u, \varphi(u) + u)$, $u \in I \subset \mathbb{R}$ be a null profile curve of the rotational surface immersed in $\mathbb{L}^3$ given by (3.2). Then, the Gaussian and mean curvatures at point $(h(u), \varphi(u) - u, \varphi(u) + u)$ are the functions of the same variable u, i.e., $K = K(u)$, $H = H(u)$. Given constants, $c_1, c_2 \in \mathbb{R}$ and a smooth function $K = K(u)$ (resp. $H = H(u)$), $u \in I$, we define the family of curves $\delta(u) \equiv \delta(u, K(u); c_1, c_2)$ (resp. $\delta(u) \equiv \delta(u, H(u); c_1, c_2)$). Hence, the differential equation of the surfaces immersed in $\mathbb{L}^3$ is (3.3) (or (3.4)), the solution of which is given for some particular functional forms of the Gaussian (or mean) curvature.*

Proof. Since the profile curve is null in $\mathbb{L}^3$, then $h'^2(u) + (\varphi(u) - u)'^2 = (\varphi(u) + u)'^2$, and then $h'^2 = 4\varphi'$. The fundamental quantities of the first order of the rotational surface with null axis and null profile curve are $E = h'^2(u) - 4\varphi'(u) = 0$, $F = 2uh'(u) - 2h(u)$, $G = 4u^2$. So the discriminant of the corresponding first fundamental form is $\det I = -4\left(uh'(u) - h(u)\right)^2 < 0$. Hence, the surface is a kind of *time-like rotational surface*. The fundamental quantities of the second order of the surface are $L = \frac{4u\varphi''(u)-2h(u)h''(u)}{\sqrt{|\det I|}}$, $M = \frac{4uh'(u)-4h(u)}{\sqrt{|\det I|}}$, $N = \frac{8u^2}{\sqrt{|\det I|}}$. The Gaussian curvature and the mean curvature of the surface are given by as follows, respectively,

$$K(u) = \frac{2u^3\varphi''(u) - u^2h(u)h''(u) - \left(-uh'(u) + h(u)\right)^2}{|\det I|^2}, \tag{3.3}$$

and

$$H(u) = \frac{-2u^3\varphi''(u) + u^2h(u)h''(u) + 2u^2h'^2(u) - 4uh(u)h'(u) + 2h^2(u)}{|\det I|^{3/2}}. \tag{3.4}$$

Case 1. We consider Eq. $K(u) = 0$. Then, (3.3) reduces to

$$2u^3\varphi''(u) - u^2h(u)h''(u) - \left(-uh'(u) + h(u)\right)^2 = 0,\ a \in \mathbb{R}. \tag{3.5}$$

Suppose that

$$\varphi' = k \ \Rightarrow\ \varphi'' = k\frac{dk}{d\varphi}. \tag{3.6}$$

Then, Eq. (3.5) reduces to $2u^3k\frac{dk}{d\varphi} - u^2h(u)h''(u) - \left(-uh'(u) + h(u)\right)^2 = 0$. So, we have $k\frac{dk}{d\varphi} = \frac{1}{2u}h(u)h''(u) + \frac{1}{2u^3}\left(-uh'(u) + h(u)\right)^2$. The solution of this equation is given by $k = -\frac{h^2(u)}{4u^2} + \frac{h(u)h'(u)}{2u} + c_1,\ c_1 \in \mathbb{R}$. From the Eq. (3.6), we have $\frac{d\varphi}{du} = -\frac{h^2(u)}{4u^2} + \frac{h(u)h'(u)}{2u} + c_1$. Finally, we get the following

$$\varphi(u) = \frac{h^2(u)}{4u} + c_1u + c_2,\ \ c_2 \in \mathbb{R}. \tag{3.7}$$

If $c_2 = 0$, then we have

$$\varphi(u) = \frac{1}{4u}\left(h^2 + 4c_1u^2\right). \tag{3.8}$$

Therefore, we can define the one-parameter family of curves

$$\delta(u) \equiv \delta(K(u), h(u), c_1) = \left(h(u), \frac{h^2(u) + 4c_1u^2}{4u} - u, \frac{h^2(u) + 4c_1u^2}{4u} + u\right). \tag{3.9}$$

The surfaces is given by

$$\mathbf{S}(u,v) = \left(h + 2uv, \frac{1}{4u}\left(h^2 + 4c_1u^2\right) - u + vh + uv^2,\ \frac{1}{4u}\left(h^2 + 4c_1u^2\right) + u + vh + uv^2\right). \tag{3.10}$$

If $c_1 = 0$, $c_2 \neq 0$, then

$$\varphi(u) = \frac{1}{4u}h^2 + c_2. \tag{3.11}$$

Therefore, we can define the two-parameter family of curves as follows

$$\begin{aligned}\delta(u) &\equiv \delta(K(u), h(u); c_1, c_2) \\ &= \left(h(u), \frac{1}{4u}\left(h^2(u) + 4c_1u^2 + 4c_2u\right) - u, \frac{1}{4u}\left(h^2(u) + 4c_1u^2 + 4c_2u\right) + u\right).\end{aligned} \tag{3.12}$$

Consequently, the equation of these surfaces is given by

$$\mathbf{S}(u,v) = \left(h + 2uv, \frac{1}{4u}\left(h^2 + 4c_1u^2 + 4c_2u\right) - u + hv + uv^2, \frac{1}{4u}\left(h^2 + 4c_1u^2 + 4c_2u\right) + u + hv + uv^2\right). \tag{3.13}$$

From the above analysis, we deduce that given the function $K(u) = 0$, we determine a one or two-parameter family of curves by (3.9) or (3.12), respectively, and define the Eqs. (3.10) and (3.13) of the surfaces in $\mathbb{L}^3$.

Case 2. (a) We consider the Gaussian curvature in the equation (3.3). The Eq. (3.3) takes the form

$$K(u) = \frac{2u^3\varphi'' - u^2hh'' - \left(-uh' + h\right)^2}{\left[-16u^2\varphi' + 8uhh' - 4h^2\right]^2} = \frac{-u^2hh'' - \left(-uh' + h\right)^2}{\left[-16u^2m + 8uhh' - 4h^2\right]^2}$$

satisfied by the function $\varphi(u) = mu + n$, $m \neq 0$. Therefore, we have $f(u) = (m-1)u + n$ and $g(u) = (m+1)u + n$.

Hence, given the function $K = K(u)$ by (3.3), we have the following

$$\mathbf{S}(u,v) = (h(u) + 2uv, (m-1)u + n + h(u)v + uv^2, (m+1)u + n + h(u)v + uv^2). \tag{3.14}$$

(b) We consider again the Gaussian curvature in (3.3). The Eq. (3.3) takes the form

$$K(u) = \frac{4u^3c_1 - u^2hh'' - (-uh' + h)^2}{|\det I|^2}. \tag{3.15}$$

This Eq. is satisfied by $\varphi(u) = c_1u^2 + c_2u + c_3$, $c_i \in \mathbb{R}$, $i = 1, 2, 3$, and therefore $f(u) = c_1u^2 + (c_2 - 1)u + c_3$ and $g(u) = c_1u^2 + (c_2 + 1)u + c_3$.

Consequently, given the function $K = K(u)$ by (3.15), there is the following family of surfaces

$$\mathbf{S}(u,v) = (h(u) + 2uv, c_1u^2 + c_2u + c_3 - u + h(u)v + uv^2, c_1u^2 + c_2u + c_3 + u + h(u)v + uv^2).$$

(c) We regard to the equation

$$2u^3\varphi''(u) - u^2h(u)h''(u) - (-uh'(u) + h(u))^2 = d,\ d \in \mathbb{R}\setminus\{0\}.$$

So, we have

$$\varphi'' = \left[u^2hh'' + (-uh' + h)^2 + d\right]/2u^3.$$

Integrating this differential equation, we deduce the following implicit equation

$$\varphi(u) = \frac{h^2(u)}{4u} + \frac{d}{64u} + d_1u + d_2, \tag{3.16}$$

where $d_1, d_2 \in \mathbb{R}$.

We can find the curvature $K = K(u)$ from (3.3) of the following surfaces

$$\begin{aligned}\mathbf{S}(u,v) = (h(u) + 2uv, &\frac{h^2(u)}{4u} + \frac{d}{64u} + d_1u + d_2 - u + h(u)v + uv^2,\\ &\frac{h^2(u)}{4u} + \frac{d}{64u} + d_1u + d_2 + u + h(u)v + uv^2).\end{aligned} \tag{3.17}$$

Case 3. It is understood that we are interested in the functions $f = f(u)$ and $g = g(u)$ such that $f'(u) = \varphi'(u) - 1 \neq 0$ and $g'(u) = \varphi'(u) + 1 \neq 0$, $f''(u) = g''(u) = \varphi''(u)$ for every $u \in \mathbb{R}\setminus\{0\}$.

Therefore, considering $u = u(\varphi)$. Eq. (3.3) can be written as

$$K(u(\varphi)) = \frac{2u^3\varphi'' - u^2hh'' - (-uh' + h)^2}{[-16u^2\varphi' + 8uhh' - 4h^2]^2}.$$

Let $\varphi' = t$. Then, this equation reduces to

$$0 \;=\; 2u^3t\frac{dt}{d\varphi} - 2^8u^4Kt^2 + 2^7u^2\left[8uhh' - 4h^2\right]^2Kt - 2^4\left[2uhh' - h^2\right]K - u^2hh'' - (-uh' + h)^2$$

which is a Ricatti differential equation and, as it is known, we can not get its general solution if we do not know some particular solutions.

Case 4. Consider the mean curvature given by (3.4) of the surface (3.2) in $\mathbb{L}^3$.

The problem now is to finding the solution of this equation in $f = f(u)$ and $g = g(u)$, where the function $H = H(u)$ is the smooth function. We can find the minimal surfaces, since we may give the solution of the equation

$$u^2hh'' - 2u^3\varphi'' + 2u^2h'^2 - 4uhh' + 2h^2 = 0. \tag{3.18}$$

The solution of this equation is

$$\varphi(u) = \frac{h^2}{2u} + c_1u + c_2,\quad c_1 \in \mathbb{R}^+, c_2 \in \mathbb{R}.$$

Consequently, for every $\varphi = \varphi(u)$ satisfying Eq. (3.11), there is a minimal surface in $\mathbb{L}^3$ whose parametric representation is given by (3.2).

For example, if we get

$$\varphi(u) = \frac{h^2(u)}{4u} + c_1 u + c_2, \quad u \in \mathbb{R} \backslash \{0\},$$

then we can define the curve $(c_1 = 1, c_2 = 0)$

$$\delta(u) \equiv \delta(u, H(u), h) = \left(h, \frac{1}{4u}h^2, \frac{1}{4u}h^2 + 2u \right).$$

Hence, there is a time-like minimal surface as follows

$$\mathbf{S}(u, v) = (h + 2uv, \frac{1}{4u}h^2 + vh + uv^2, \frac{1}{4u}h^2 + 2u + vh + uv^2).$$

If $\varphi(u) = e^u$, then Eq. (3.4) takes form

$$\frac{-2e^{4u} - ue^{3u} - 4u^2 e^u - u^2 h'^2}{|\det I|^{3/2}} = \frac{-2u^3 \varphi'' + u^2 hh'' + 2u^2 h'^2 - 4uhh' + 2h^2}{|\det I|^{3/2}}. \tag{3.19}$$

Consequently, the function $H = H(u)$ given by (3.19), there is a surface in $\mathbb{L}^3$, as follows

$$\mathbf{S}(u, v) = (h + 2uv, e^u - u + vh + uv^2, e^u + u + vh + uv^2).$$

□

Next, we examine the third Laplace-Beltrami operator of the surfaces in Eq. (3.2).

4. The third Laplace-Beltrami operator

In this section, we focus the third Laplace-Beltrami operator of the surfaces having null axis and null profile curve.

We assume that the vector $\ell = (0, 1, 1)$ is a null axis, profile curve $\gamma(u) = (h(u), \varphi(u) - u, \varphi(u) + u)$ is a null curve, and $u \in \mathbb{R} \setminus \{0\}$.

Now, let $x = x(u^1, u^2)$ be a surface in $\mathbb{L}^3$ defined in domain $D \subset \mathbb{R}^2$. The same for the functions ϕ, ψ. Let $n = n(u^1, u^2)$ be the normal vector of the surface. We write

$$g_{ij} = \langle x_i, x_j \rangle, \; b_{ij} = \langle x_{ij}, n \rangle, \; e_{ij} = \langle n_i, n_j \rangle. \tag{4.1}$$

The equations of Weingarten are

$$\begin{aligned} x_i &= b_{ij} e^{jr} n_r = -g_{ij} b^{jr} n_r, \\ n_i &= -e_{ij} b^{jr} x_r = -b_{ij} g^{jr} x_r, \end{aligned}$$

where $x_i = \frac{\partial x}{\partial u^i}$. Then, the first parameter Beltrami is defined by

$$grad^{III}(\phi, \psi) = e_i^{ik} \phi \psi_k.$$

By using following expressions

$$\begin{aligned} grad^{III}(\phi) &= grad^{III}(\phi, \phi) = e^{ik} \phi_i \phi_k, \\ grad^{III} \phi &= grad^{III}(\phi, n) = e^{ik} \phi_i n_k, \end{aligned}$$

the second parameter Beltrami is defined by

$$\Delta^{III} \phi = -e^{ik} grad_k^{III} \phi_i.$$

Using the last relation, we get the expression the third Laplace-Beltrami operator of the function ϕ. So, we have the third fundamental form (see [15] for details) as follows

$$\Delta^{III} \phi = -\frac{\sqrt{|\det I|}}{\det II} \left[\frac{\partial}{\partial u} \left(\frac{Z\phi_u - Y\phi_v}{\sqrt{|\det I|} \det II} \right) - \frac{\partial}{\partial v} \left(\frac{Y\phi_u - X\phi_v}{\sqrt{|\det I|} \det II} \right) \right], \tag{4.2}$$

where the coefficients of the first (resp., second, and third) fundamental form of the function ϕ is E, F, G (resp., $L, M, N,$ and X, Y, Z), $\det I = EG - F^2$, $\det II = LN - M^2$, $X = EM^2 - 2FLM + GL^2$, $Y = EMN - FLN + GLM - FM^2$, $Z = GM^2 - 2FNM + EN^2$.

Let $\mathbf{F}(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$ be a vector function defined on the domain D, then we set $\Delta\mathbf{F}(u,v) = (\Delta f_1(u,v), \Delta f_2(u,v), \Delta f_3(u,v))$.

We can now state and prove the following theorems.

Theorem 4.1. *The surface $\mathbf{S}(u,v) = (\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3)$ in the Eq. (3.2) with null profile curve and null axis which is spanned by null vector $(0,1,1)$ in $\mathbb{L}^3$ has the third Laplace-Beltrami operator as follows*

$$\begin{aligned}
\Delta^{III}\mathbf{S}_1 \;=\;& -\frac{T^3}{2W^3}\{[-\frac{u^3}{2}(\frac{1}{2}h+uv)Thh''' + (\frac{1}{2}h+uv)u^4T\varphi''' \\
&+\frac{1}{2}u^4(\frac{1}{2}h+uv)hh''^2 - [u^2(\frac{1}{2}h+uv)\varphi'' + ((uv+\frac{1}{4}h)h' \\
&-\frac{1}{2}vh)T]u^3h'' + [(-\frac{1}{2}u^4h' + u^3h + u^4v)\varphi'' \\
&+(\frac{1}{4}uh' + uv + \frac{1}{4}h)T^2]]T + \frac{1}{2}(-\frac{1}{2}u^2hh'' - \frac{1}{2}T^2 + u^3\varphi'')T^2\},
\end{aligned}$$

$$\begin{aligned}
\Delta^{III}\mathbf{S}_2 \;=\;& \frac{T^2}{2W^3}\{-\frac{1}{4}(-\frac{1}{2}uhh' + u^2\varphi' + \frac{1}{2}h^2 + uvh \\
&+(-u^4 + u^4v^2)T^2hh''' + \frac{1}{2}(-\frac{1}{2}uhh' + u^2\varphi' + \frac{1}{2}h^2 + uvh \\
&-u^2 + u^2v^2)T^2u^3\varphi''' - \frac{1}{8}u^5h^3h''^3 + \frac{1}{4}(3u^3h\varphi'' + (-\frac{5}{2}uhh' \\
&+u^2\varphi' + \frac{5}{2}h^2 + uvh - u^2 + u^2v^2)T)u^3hh''^2 - \frac{1}{2}(3u^6h\varphi''^2 \\
&+(-\frac{9}{2}uhh' + u^2\varphi' + \frac{9}{2}h^2 + uvh - u^2 + u^2v^2)Tu^3\varphi'' \\
&+(\frac{3}{4}u^2hh'^2 + (u^2\varphi' - \frac{7}{4}h^2 + uvh - u^2 + u^2v^2)uh' - \frac{1}{2}h(u^2\varphi' \\
&-2h^2 + uvh - u^2 + u^2v^2))T^2)uh'' + u^8\varphi''^3 - 2u^5T^2\varphi''^2 \\
&+\frac{1}{2}(\frac{5}{2}u^2h'^2 - \frac{11}{2}uhh' + u^2\varphi' + 3h^2 + uvh - u^2 + u^2v^2)u^2T^2\varphi'' \\
&+\frac{1}{2}(-\frac{1}{2}uh'^2 + uv^2 + u\varphi' - u + \frac{1}{2}hh' + vh)T^4)\},
\end{aligned}$$

$$\begin{aligned}
\Delta^{III}\mathbf{S}_3 \;=\;& \frac{T^2}{2W^3}\{-\frac{1}{4}(-\frac{1}{2}uhh' + u^2\varphi' + \frac{1}{2}h^2 + uvh \\
&+(u^4 + u^4v^2)T^2hh''' + \frac{1}{2}(-\frac{1}{2}uhh' + u^2\varphi' + \frac{1}{2}h^2 + uvh \\
&+u^2 + u^2v^2)T^2u^3\varphi''' - \frac{1}{8}u^5h^3h''^3 + \frac{1}{4}(3u^3h\varphi'' + (-\frac{5}{2}uhh' \\
&+u^2\varphi' + \frac{5}{2}h^2 + uvh + u^2 + u^2v^2)T)u^3hh''^2 - \frac{1}{2}(3u^6h\varphi''^2 \\
&+(-\frac{9}{2}uhh' + u^2\varphi' + \frac{9}{2}h^2 + uvh + u^2 + u^2v^2)Tu^3\varphi'' \\
&+(\frac{3}{4}u^2hh'^2 + (u^2\varphi' - \frac{7}{4}h^2 + uvh + u^2 + u^2v^2)uh' - \frac{1}{2}h(u^2\varphi' \\
&-2h^2 + uvh + u^2 + u^2v^2))T^2)uh'' + u^8\varphi''^3 - 2u^5T^2\varphi''^2 \\
&+\frac{1}{2}(\frac{5}{2}u^2h'^2 - \frac{11}{2}uhh' + u^2\varphi' + 3h^2 + uvh + u^2 + u^2v^2)u^2T^2\varphi'' \\
&+\frac{1}{2}(-\frac{1}{2}uh'^2 + uv^2 + u\varphi' + u + \frac{1}{2}hh' + vh)T^4)\},
\end{aligned}$$

and $T = uh' - h$, $W = -\frac{1}{2}T^2 - \frac{1}{2}u^2hh'' + u^3\varphi''$.

Proof. We obtain the third fundamental quantities as follows

$$X=\frac{4[-2u^{3}\varphi''+2\left(uh'-h\right)^{2}+u^{2}hh'](2u\varphi''+hh'')}{\left(uh'-h\right)^{2}}, Y=8\left(uh'-h\right), Z=16u^{2}.$$

By using this quantities of the surface (3.2) on (4.2), we can easily compute as follows

$$\Delta^{III}\mathbf{S}(u,v)=\begin{pmatrix}\Delta^{III}\mathbf{S}_1\\ \Delta^{III}\mathbf{S}_2\\ \Delta^{III}\mathbf{S}_3\end{pmatrix}=\begin{pmatrix}\Delta^{III}(h(u)+2uv)\\ \Delta^{III}(\varphi(u)-u+h(u)v+uv^2)\\ \Delta^{III}(\varphi(u)+u+h(u)v+uv^2)\end{pmatrix}.$$

□

In the case $h'^2=4\varphi'$ we have the following:

Remark 4.1. In the surface $\mathbf{S}(u,v)$, if we take the linear function $h(u)=c_1u+c_2$, where $c_1\neq 0$, $c_2=0$, then we get $\det I=0$. So, the *surface* $\mathbf{S}(u,v)$ is a light-like surface. We have to choose the $h(u)\neq c_1u$ in $\mathbb{L}^3$. And if we take the linear function $\varphi(u)=c_1u+c_2$, where $c_1=1, c_2=0$ then we get $h(u)=2u$ and we have $\det I=0$. Therefore, we also obtain the indefinite *surface*.

Corollary 4.1. *Choosing $h(u)=2u^2$, we have $\varphi(u)=4/3u^3$, then we get*

$$\Delta^{III}\mathbf{S}(u,v)=(0,0,0),$$

where $u>0$. That is, the time-like surface $\mathbf{S}(u,v)$ with null cubic profile curve is an time-like III-minimal surface in $\mathbb{L}^3$.

Corollary 4.2. *If $\varphi(u)=c_1u^2+c_2u+c_3$, $c_1=c_2=1, c_3=0$ then the time-like surface $\mathbf{S}(u,v)$ with null axis and null profile curve has the function $h(u)=\mp\frac{2}{3}(2u+1)^{3/2}$, where $u\in\mathbb{R}-\{0\}$ in $\mathbb{L}^3$.*

Proposition 4.1. *A surface with null profile curve and null axis in Eq. (3.2) immersed in $\mathbb{L}^3$, with mean curvature $H=H(u)$, given by the relation (3.4), supplies the following relation*

$$\Delta^{III}\mathbf{S}(u,v)=H\cdot N+A$$

where $\det I=EG-F^2$, $N=(N_1,N_2,N_3)$ is the unit normal vector field,

$$\begin{aligned}
H\cdot N_1 &= \frac{2}{\left|\det I\right|^2}[-2(u^4v+u^3h)\varphi''+(u^2h^2+2u^3vh)h''\\
&\quad +2(u^2h+2u^3v)h'^2-4(uh^2h'+2u^2vh)h'+2(h+2uv)h^2],
\end{aligned}$$

$$\begin{aligned}
H\cdot N_2 &= \frac{2}{\left|\det I\right|^2}[(-2u^4\varphi'+u^3hh'+2u^3vh+2u^4v^2-2u^4)\varphi''\\
&\quad +(u^3hh''+2u^3h'^2-4hu^2h'+2uh^2)\varphi'\\
&\quad +(-\frac{1}{2}u^2h^2h'-u^2vh^2-u^3v^2h+u^3h)h''\\
&\quad +(-u^2hh'+2uh^2-2u^2vh-2u^3v^2+2u^3)h'^2\\
&\quad +(-h^3+4uvh^2+4u^2v^2h-4u^2h)h'-2(vh^3+uv^2-u)h^2],
\end{aligned}$$

$$\begin{aligned}
H\cdot N_3 &= \frac{2}{\left|\det I\right|^2}[(-2u^4\varphi'+u^3hh'+2u^3vh+2u^4v^2+2u^4)\varphi''\\
&\quad +(u^3hh''+2u^3h'^2-4u^2hh'+2uh^2)\varphi'\\
&\quad +(-\frac{1}{2}u^2h^2h'-u^2vh^2-u^3v^2h-u^3h)h''\\
&\quad +(-u^2hh'+2uh^2-2u^2vh-2u^3v^2-2u^3)h'^2\\
&\quad +(-h^3+4uvh^2+4u^2v^2h+4u^2h)h'-2(vh+uv^2+u)h^2],
\end{aligned}$$

$$A_1=\Delta^{III}\mathbf{S}_1-H\cdot N_1,\ A_2=\Delta^{III}\mathbf{S}_2-H\cdot N_2,\ A_3=\Delta^{III}\mathbf{S}_3-H\cdot N_3,$$

and $A(u,v)=(A_1(u,v),A_2(u,v),A_3(u,v))$.

Proof. The unit normal vector field of the two-parameter family of surfaces in Eq. (3.2) immersed in $\mathbb{L}^3$ with mean curvature $H = H(u)$, given by (3.4), is given by

$$N = \frac{-2}{\sqrt{|\det I|}} \begin{pmatrix} h + 2uv \\ \left(-u\varphi' + \frac{1}{2}hh'\right) - u + uv^2 + vh \\ \left(-u\varphi' + \frac{1}{2}hh'\right) + u + uv^2 + vh \end{pmatrix}, \tag{4.3}$$

where $\det I = -4\left(uh'(u) - h(u)\right)^2$, $u \neq h/h'$. Using $\Delta^{III}\mathbf{S}$ in the Theorem 2, Eqs. (3.2) and (4.3), then we can see $A(u,v)$. □

Proposition 4.2. *We get the relation between the surface in the Eq.(3.2) and the unit normal of its as follows*

$$\mathbf{S}(u,v) = (h - uh')\,N - (0, \Phi - \varphi, \Phi - \varphi),$$

where $\Phi = -u\varphi' + \frac{1}{2}hh'$, $h \neq c_1 u$.

Proposition 4.3. *A surface with null profile curve and null axis immersed in $\mathbb{L}^3$, with Gaussian curvature $K = K(u)$, given by the relation (3.3), supplies the following relation*

$$\Delta^{III}\mathbf{S} = K(u)\cdot N + B,$$

where $\det I = -4\left(uh'(u) - h(u)\right)^2$, *$N$ is the unit normal vector field,*

$$\begin{aligned} K\cdot N_1 &= \frac{2}{|\det I|^{5/2}}[2(u^3h + 4u^4v)\varphi'' - (2u^3vh + u^2h^2)h'' \\ &\quad -2(u^2h + 2u^3v)h'^2 + 4(2u^2v + uh)hh' - 2h^3 - 4uvh^2], \end{aligned}$$

$$\begin{aligned} K\cdot N_2 &= \frac{2}{|\det I|^{5/2}}[u^3(hh' + 2vh - 2u\varphi' + 2uv^2 - 2u)\varphi'' \\ &\quad +u(u^2hh'' + 2u^2h'^2 - 4uhh' + 2h^2)\varphi' - \frac{1}{2}u^2h^2h'h'' \\ &\quad +u^2(-vh - uv^2 + u)hh'' \\ &\quad +(u^2hh' - 2u^2vh + 2uh^2 - 2u^3v^2 + 2u^3)h'^2 \\ &\quad +(-h^2 + 4uvh + 4u^2v^2 - 4u^2)hh' \\ &\quad +2(-vh - uv^2 + u)h^2], \end{aligned}$$

$$\begin{aligned} K\cdot N_3 &= \frac{2}{|\det I|^{5/2}}[u^3(hh' + 2vh - 2u\varphi' + 2uv^2 + 2u)\varphi'' \\ &\quad +u(u^2hh'' + 2u^2h'^2 - 4uhh' + 2h^2)\varphi' - \frac{1}{2}u^2h^2h'h'' \\ &\quad +u^2(-vh - uv^2 - u)hh'' \\ &\quad +(-u^2hh' - 2hu^2v + 2uh^2 - 2u^3v^2 - 2u^3)h'^2 \\ &\quad +(-h^2 + 4uvh + 4u^2v^2 + 4u^2)hh' \\ &\quad +2(-vh - uv^2 - u)h^2], \end{aligned}$$

$$B_1 = \Delta^{III}\mathbf{S}_1 - K\cdot N_1,\ B_2 = \Delta^{III}\mathbf{S}_2 - K\cdot N_2,\ B_3 = \Delta^{III}\mathbf{S}_3 - K\cdot N_3,$$

and $B(u,v) = (B_1(u,v), B_2(u,v), B_3(u,v))$.

Proof. The unit normal vector field of the two-parameter family of surfaces in Eq. (3.2) immersed in $\mathbb{L}^3$ with mean curvature $K = K(u)$, given by (3.3), is in the Eq. (4.3). Using $\Delta^{III}\mathbf{S}$ in the Theorem 2, Eqs. (3.2) and (4.3), then we get $B(u,v)$. □

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