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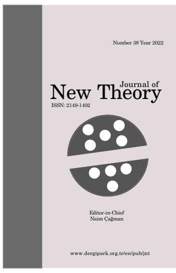

Geometry of Curves with Fractional Derivatives in Lorentz Plane

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Abstract − In this paper, the geometry of curves is discussed based on the Caputo fractional derivative in the Lorentz plane. Firstly, the tangent vector of a spacelike plane curve is defined in terms of the fractional derivative. Then, by considering a spacelike curve in the Lorentz plane, the arc length and fractional ordered frame of this curve are obtained. Later, the curvature and Frenet-Serret formulas are found for this fractional ordered frame. Finally, the relation between the fractional curvature and classical curvature of a spacelike plane curve is obtained. In the last part of the study, considering the timelike plane curve in the Lorentz plane, new results are obtained with the method in the previous section.

Keywords − *Caputo fractional derivative, tangent vectors, curvature, Frenet-Serret formulas*

Mathematics Subject Classification (2020) – 26A33, 53A04

1. Introduction

The fractional derivative was first established in the 17th century and with an adding number of studies, it has come the focus of attention for many researchers in numerous fields. Fractional analysis has lately become one of the important fields of study in differential geometry. While, in the classical sense, the differential and integral are determined by integer order, in fractional calculus the orders of the differential and integral are not necessarily integers but any real number. That is, fractional calculus is the generalization of ordinary differential and integral to arbitrary order. The difference between the fractional derivative from the integer derivative is that it is given by the integration of a function.

Many studies have been conducted on this subject, and it can be found in detail [1-4]. We can also say that a non-local fractional derivative of a function is related to history or a space-range interaction. Furthermore, fractional calculus has many applications to viscoelastic [5-11], analytical mechanics [12-14], and dynamical systems [15-19]. Fractional analysis has also started to be studied from a differential geometry perspective in recent studies. There are many types of fractional operators, but it is recommended to study the geometry of curves and surfaces mostly based on the Caputo fractional derivative [20]. However, the Caputo fractional derivative is not yet directly used to formulate the differential geometry of curves. Using the Caputo fractional derivative is more appropriate than other fractional derivative operators for formulating a geometric theory since the fractional derivative of the constant function is zero [21-25]. Based on the advantages of the Caputo fractional derivative, it is discussed in [22,24] as a quantification of Lagrangian mechanics and in the theory of gravity [21,23,26].

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In [27], the fractional geometry of curves in Euclidean 3-space is studied using the Caputo fractional derivative. Using the Caputo fractional derivative, the fractional geometry of curves in higher-dimensional Euclidean space is studied [28].

In this study, firstly, by considering a spacelike curve in the Lorentz plane, fractional ordered frame and Frenet-Serret formulas of this curve are obtained. Later, the relation between the fractional curvature and classical curvature of a spacelike plane curve is obtained. In the last part of this paper, considering the timelike plane curve in the Lorentz plane, new results are obtained with the method in the previous section.

2. Preliminaries

In general, the concepts of the Leibnitz rule and derivative of the composite function are needed when studying fractional differential geometry. However, within the scope of fractional analysis, these concepts are obtained with infinite series and are used in impact situations at the initial moment and after a long period [3,4].

Leibnitz's rule and derivative of the composite function can be given as follows for two functions *f(x)* and *g(x)* [29]:

$$(D_x^\alpha fg)(x) = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{d^i f}{dx^i} \left(D_x^{\alpha-i} g\right)(x) - \frac{f(0)g(0)}{\Gamma(1-\alpha)} x^{-\alpha}$$

and

$$(D_x^\alpha f)\big(g(x)\big) = \sum_{i=1}^{\infty} \binom{\alpha}{i} \frac{x^{i-\alpha}}{\Gamma(i-\alpha+1)} \frac{d^i f\big(g(x)\big)}{dx^i} + \frac{f\big(g(x)\big) - f\big(g(0)\big)}{\Gamma(1-\alpha)} x^{-\alpha} \tag{1}$$

This different form of the integer derivative presents a challenge for deriving geometric concepts such as the curvature of a curve and the unit tangent vector. So, a certain simplification of the infinite series is used to construct the geometric theory of the derivative. With this simplification, most fundamental terms are removed from the infinite series, which retain the properties of the fractional derivative. Hence, with $t = g(x)$, the following equality is achieved [30]:

$$(D_x^\alpha f)\big(g(x)\big) = \frac{\alpha x^{1-\alpha}}{\Gamma(2-\alpha)} \frac{df}{dt} \frac{dg}{dx} \tag{2}$$

This simplification formula is obtained by taking only the $i = 1$ term of the infinite series in equation (1). This formula gives a partial effect of the fractional derivative and is expressed by the ordinary derivative. After this simplification, the construction of the fractional geometric theory based on the direct Caputo derivative can be expected using the simplified Leibnitz rule and the derivative of the composite function. In other words, using the Caputo derivative researchers have an advantage when studying the differential geometry of curves and surfaces, especially since it is ineffective on a constant function. Throughout the study, the derivative formula given by (2) is discussed.

Now, we will talk about some basic concepts in the Lorentz plane that we will use in the following sections. More detailed information on the following topics can be found in [31].

The Lorentz plane L^2 is the Euclidean plane R^2 with metric given by $g = -dx_1^2 + dx_2^2$ where (x_1, x_2) is a rectangular coordinate system of L^2. It is known that a vector $v \in L^2 \setminus \{0\}$ can be spacelike if $g(v,v) > 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$. The null (lightlike) curves in L^2 are lines, which curvature is identically zero.

Therefore, in this study, we will only deal with spacelike and timelike plane curves. The norm of any vector v in them is given by $\|v\| = \sqrt{g(v,v)|}$. Two vectors v and w are said to be orthogonal if $g(v,w) = 0$. An arbitrary curve $\gamma(s)$ in L^2 , can locally be spacelike or timelike if all of its velocity vectors $\dot{\gamma}(s)$ are spacelike, respectively timelike. A spacelike or timelike curve γ is parameterized by the arc-length parameter s if $g(\dot{\gamma}(s), \dot{\gamma}(s)) = \pm 1$.

The curvature κ and the Frenet formulas of the spacelike curve γ can be given as follows:

$$\kappa = -g\ (\dot{t}, n)$$

and

$$\begin{aligned} \dot{t} &= \kappa n \\ \dot{n} &= \kappa t \end{aligned}$$

where $\dot{\gamma} = t,\ t$ and n are the tangent, and unit normal vector of a spacelike curve γ, respectively. If γ is a spacelike curve in Lorentz plane, then $g(t,t) = 1$ and $g(n,n) = -1$. Moreover, the curvature k and the Frenet formulas of the timelike curve β can be given as follows:

$$k = g\ (\dot{v_1}, v_2)$$

and

$$\begin{aligned} \dot{v_1} &= \kappa v_2 \\ \dot{v_2} &= \kappa v_1 \end{aligned}$$

where $\dot{\beta} = v_1,\ v_1$ and v_2 are the tangent, and unit normal vector of a spacelike curve β, respectively. If β is a timelike curve in Lorentz plane, then $g(v_1, v_1) = -1$ and $g(v_2, v_2) = 1$.

3. Geometry of Spacelike Curves with Fractional Derivative

In this section, the geometry of spacelike curves is discussed based on the Caputo fractional derivative in the Lorentz plane.

Let us consider a smooth spacelike curve γ in the 2-dimensional L^2 space is given by

$$\gamma : I \subset \mathbb{R} \to L^2, \gamma(t) = \big(x(t), y(t)\big)$$

where t is an arbitrary parameter. From the definition of the length σ of a spacelike curve γ, we can write

$$\sigma = \int_0^t \sqrt{|-\dot{x}^2 + \dot{y}^2|}\, dt, t \in I \tag{3}$$

where $\dot{x}$ and $\dot{y}$ denote the ordinary derivatives of x and y concerning t, respectively. The above formula is arclength of the spacelike curve for the tangent vector: $t(\sigma) = \left(\frac{dx}{d\sigma}, \frac{dy}{d\sigma}\right)$.

Let us now investigate the effect of the fractional derivative on the curvature of a spacelike curve. Since curvature is generally related to the change of the tangent vector of a spacelike curve, take a fractional tangent vector:

$$t^{(\alpha)}(\sigma) = \left(\frac{d^\alpha x(\sigma)}{d\sigma^\alpha}, \frac{d^\alpha y(\sigma)}{d\sigma^\alpha}\right) \tag{4}$$

Considering the infinite series given in (1), from the fractional derivative of the composite function, we can write $\left\|t^{(\alpha)}(\sigma)\right\| \neq 1$. This means that the classical arclength given by (3) cannot be used in the geometry of curves with fractional derivatives. To define a fractional unit tangent vector, it is necessary to consider the fractional derivative of the composite function in a simpler form. Therefore, instead of the formula (1), only the first term of the summation is considered in the fractional derivative of the composite function. Thus, both the effect of fractional derivative and first-order derivative are obtained. In this case, we can write

$$\frac{d^\alpha \gamma\big(t(s)\big)}{ds^\alpha} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d\gamma}{dt} \frac{dt}{ds} \tag{5}$$

Throughout this study, the theory of curves in the Lorentz plane are examined by considering this simple version of the derivative of the composite function. Using equation (5), we can give the following transformation:

$$s = \left[\frac{\alpha^2}{\Gamma(2-\alpha)} \sigma\right]^{\frac{1}{\alpha}} \tag{6}$$

where α denotes the order of the fractional derivative and $0 < \alpha \leq 1$. For the parameter s given by (6), we write

$$\frac{ds}{dt} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \sqrt{|-\dot{x}^2 + \dot{y}^2|} \tag{7}$$

Since $s > 0$ and $0 < \alpha \leq 1$ in (7), $\frac{ds}{dt}$ is positive. So that parameter t becomes a function dependent on s: $t = t(s)$. In this case, the spacelike curve γ can be written depending on the parameter s and is denoted by $\gamma(s) = \big(x(s), y(s)\big)$.

Now let us define the tangent vector of a given spacelike curve using the parameter s and the Caputo fractional derivative:

$$t^{(\alpha)}(s) \equiv \frac{d^\alpha \gamma(s)}{ds^\alpha} = \left(\frac{d^\alpha x(s)}{ds^\alpha}, \frac{d^\alpha y(s)}{ds^\alpha}\right) \tag{8}$$

Considering Equation (5), the norm of the tangent vector of the spacelike curve is

$$\left\|t^{(\alpha)}(s)\right\| = \sqrt{\left|-\left(\frac{d^\alpha x}{ds^\alpha}\right)^2 + \left(\frac{d^\alpha y}{ds^\alpha}\right)^2\right|} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{dt}{ds} \sqrt{|-\dot{x}^2 + \dot{y}^2|} = 1 \tag{9}$$

Then, using (5), a unit vector of the spacelike curve γ orthogonal to t can be defined as follows:

$$n^{(\alpha)}(s) \equiv \left(\frac{d^\alpha y}{ds^\alpha}, \frac{d^\alpha x}{ds^\alpha}\right) \tag{10}$$

So, we can give the following theorem.

Theorem 3.1. Let γ be a spacelike curve in the Lorentz plane that satisfies the condition (5) and has the parameter s given by (6). Then $t^{(\alpha)}(s)$ and $n^{(\alpha)}(s)$ given by (8) and (10) are the unit tangent vector and unit normal vector of the spacelike curve γ, respectively, and s is the arclength.

In the following, we is constructed the geometry of a spacelike curve with the fractional $\left(t^{(\alpha)}(s), n^{(\alpha)}(s)\right)$ Frenet-Serret frame using the Caputo derivative.

Let us take a smooth spacelike curve $\gamma(s) = \big(x(s), y(s)\big)$ given by the arclength parameter (6) in the Lorentz plane. Based on the fractional frenet frame, let us define the frenet-serret formulas and the curvature of the spacelike curve γ. From Theorem 3.1, the tangent vector $t^{(\alpha)}(s)$ of the spacelike curve provides $g(t^{(\alpha)}(s), t^{(\alpha)}(s)) = 1$. If we take the derivative of both sides of this equation concerning s, we get

$$g\left(t^{(\alpha)}(s), \frac{dt^{(\alpha)}(s)}{ds}\right) = 0 \tag{11}$$

which means that $\frac{dt^{(\alpha)}(s)}{ds}$ can be expressed with the normal vector $n^{(\alpha)}(s)$:

$$\frac{dt^{(\alpha)}(s)}{ds} = \kappa^{(\alpha)}(s)n^{(\alpha)}(s) \tag{12}$$

where $\kappa^{(\alpha)}(s)$ is the fractional curvature of the spacelike curve γ. Then the norm of the normal vector $n^{(\alpha)}(s)$ is also equal to one, $\left\| n^{(\alpha)}(s) \right\| = 1$. Then we write $g(n^{(\alpha)}(s), n^{(\alpha)}(s)) = -1$. If we take the derivative of both sides of this last equation concerning s, we have

$$g\left(n^{(\alpha)}(s), \frac{dn^{(\alpha)}(s)}{ds}\right) = 0 \tag{13}$$

From (13), $\frac{dn^{(\alpha)}(s)}{ds}$ can be given using a certain arclength function $\lambda^{(\alpha)}$:

$$\frac{dn^{(\alpha)}(s)}{ds} = \lambda^{(\alpha)}(s)t^{(\alpha)}(s) \tag{14}$$

Considering the orthogonality relation $g\big(t^{(\alpha)}(s), n^{(\alpha)}(s)\big) = 0$, taking the derivative of both sides of this relation concerning s, we get the following expression:

$$t^{(\alpha)}(s)\frac{dn^{(\alpha)}}{ds} + \frac{dt^{(\alpha)}}{ds}n^{(\alpha)}(s) = 0 \tag{15}$$

If the equations (12) and (14) are substituted in the expression (15), we get $\lambda^{(\alpha)} = \kappa^{(\alpha)}$. Thus, the following theorem is obtained.

Theorem 3.2. Let γ be a spacelike curve in the Lorentz plane that satisfies the condition (5) and has the parameter s given by (6). Let us consider $\left(t^{(\alpha)}(s), n^{(\alpha)}(s)\right)$ as the fractional frame of this spacelike curve γ. Then the Frenet-Serret formulas for γ can be given as

$$\frac{dt^{(\alpha)}(s)}{ds} = \kappa^{(\alpha)}(s)n^{(\alpha)}(s) \tag{16}$$

$$\frac{dn^{(\alpha)}(s)}{ds} = \kappa^{(\alpha)}(s)t^{(\alpha)}(s) \tag{17}$$

Let us now investigate the relationship between the fractional curvature and classical curvature of a given spacelike curve γ. Considering (8) and (16), we can write

$$\frac{d}{ds}\left(\frac{d^{\,\alpha}\gamma(s)}{ds^{\,\alpha}}\right) = \kappa^{(\alpha)}(s)n^{(\alpha)}(s) \tag{18}$$

If the normal vector $n^{(\alpha)}$ is applied to both sides of (18), we can write the fractional curvature as

$$\kappa^{(\alpha)}(s) = -g\left(n^{(\alpha)}(s), \frac{d}{ds}\left(\frac{d^{\,\alpha}\gamma(s)}{ds^{\,\alpha}}\right)\right) \tag{19}$$

Considering the normal vector $n^{(\alpha)}(s)$ in (19), $\kappa^{(\alpha)}(s)$ according to the fractional derivative is written as

$$\kappa^{(\alpha)}(s) = \frac{d^{\,\alpha}y}{ds^{\,\alpha}}\frac{d}{ds}\left(\frac{d^{\,\alpha}x}{ds^{\,\alpha}}\right) - \frac{d^{\,\alpha}x}{ds^{\,\alpha}}\frac{d}{ds}\left(\frac{d^{\,\alpha}y}{ds^{\,\alpha}}\right) \tag{20}$$

If a curve is given by an arbitrary parameter t and not by the arc length s, then we must calculate the fractional curvature according to an arbitrary parameter t. Then let us calculate the fractional curvature according to an arbitrary parameter t. From the expression (5) for the composite function $t = t(s)$, we can write

$$\frac{d}{ds}\left(\frac{d^{\alpha}x}{ds^{\alpha}}\right)=\frac{\alpha s^{-\alpha}\dot{x}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds}+s\frac{d^2t}{ds^2}\right)+\frac{\alpha s^{1-\alpha}\ddot{x}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \tag{21}$$

$$\frac{d}{ds}\left(\frac{d^{\alpha}y}{ds^{\alpha}}\right)=\frac{\alpha s^{-\alpha}\dot{y}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds}+s\frac{d^2t}{ds^2}\right)+\frac{\alpha s^{1-\alpha}\ddot{y}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \tag{22}$$

where $\ddot{x}=\frac{d^2x}{dt^2}$ and $\ddot{y}=\frac{d^2y}{dt^2}$. If the expressions (21) and (22) are written instead in (20), the fractional-order curvature is

$$\kappa^{(\alpha)}(t)=\left\{\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right\}^2(-\dot{x}\ddot{y}+\ddot{x}\dot{y})\left(\frac{dt}{ds}\right)^3 \tag{23}$$

Moreover, from (7), (23) it can be rewritten by

$$\kappa^{(\alpha)}(t)=\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}\kappa(t) \tag{24}$$

$\kappa(t)$ in this last equation is the classical curvature and

$$\kappa(t)=\frac{-\dot{x}\ddot{y}+\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \tag{25}$$

Thus, using the arclength definition given by (6), we can give the following theorem.

Theorem 3.3. The fractional curvature of a spacelike plane curve given as $\gamma(t)=\big(x(t),y(t)\big)$ is

$$\kappa^{(\alpha)}(t)=\left\{\frac{\beta(2-\alpha)}{\alpha}\right\}^{\frac{1}{\alpha}}\left[\alpha\int_0^t\sqrt{|-\dot{x}^2+\dot{y}^2|}dt\right]^{1-\frac{1}{\alpha}}\kappa(t) \tag{26}$$

where t is an arbitrary parameter.

The part of $\frac{1}{s^{1-\alpha}}$ in (24) characterizes the effects of the fractional derivative given by the fractional tangent vector (8). The effect of the fractional derivative is strong at the start but becomes less effective over a longer period. This property of the effect influences the change of fractional curvature.

4. Geometry of Timelike Curves with Fractional Derivative

In this section, the geometry of timelike curves is discussed based on the Caputo fractional derivative in the Lorentz plane.

Let us consider a smooth timelike curve β in the 2-dimensional L^2 space is given by

$$\beta: I\subset\mathbb{R}\to L^2,\qquad \beta(t)=\big(\beta_1(t),\beta_2(t)\big)$$

where t is an arbitrary parameter. From the definition of the length σ of a timelike curve β, we can write

$$\sigma=\int_0^t\sqrt{\left|-(\dot{\beta}_1)^2+(\dot{\beta}_2)^2\right|}\,dt,\ t\in I \tag{27}$$

where $\dot{\beta}_1$ and $\dot{\beta}_2$ denote the ordinary derivatives of β_1 and β_2 concerning *t*, respectively. The above formula is arclength of the timelike curve for the tangent vector: $v_1(\sigma)=\left(\frac{dx}{d\sigma},\frac{dy}{d\sigma}\right)$.

Let us now investigate the effect of the fractional derivative on the curvature of a timelike curve. Since curvature is generally related to the change of the tangent vector of a timelike curve, let's define a fractional tangent vector:

$$v_1{}^{(\alpha)}(\sigma) = \left(\frac{d^\alpha \beta_1(\sigma)}{d\sigma^\alpha}, \frac{d^\alpha \beta_2(\sigma)}{d\sigma^\alpha}\right) \tag{28}$$

Considering the infinite series given in (1), from the fractional derivative of the composite function, we can write $\left\|v_1{}^{(\alpha)}(\sigma)\right\| \neq 1$. This means that the classical arclength given by (27) cannot be used in the geometry of curves with fractional derivatives. To define a fractional unit tangent vector, it is necessary to consider the fractional derivative of the composite function in a simpler form. Therefore, instead of the formula (1), only the first term of the summation is considered in the fractional derivative of the composite function. Thus, both the effect of fractional derivative and first-order derivative are obtained. In this case, we can write

$$\frac{d^\alpha \beta(t(s))}{ds^\alpha} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d\beta}{dt} \frac{dt}{ds} \tag{29}$$

Throughout this study, the theory of timelike curves in the Lorentz plane are examined by considering this simple version of the derivative of the composite function. Using equation (29), we can give the following transformation:

$$s = \left[\frac{\alpha^2}{\Gamma(2-\alpha)} \sigma\right]^{\frac{1}{\alpha}} \tag{30}$$

where α denotes the order of the fractional derivative and $0 < \alpha \leq 1$. For the parameter s given by (30), we write

$$\frac{ds}{dt} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \sqrt{\left|-(\dot{\beta}_1)^2 + (\dot{\beta}_2)^2\right|} \tag{31}$$

Since $s > 0$ and $0 < \alpha \leq 1$ in (31), $\frac{ds}{dt}$ is positive. So that parameter t becomes a function dependent on s: $t = t(s)$. In this case, the timelike curve β can be written depending on the parameter s and is denoted by $\beta(s) = \big(\beta_1(s), \beta_2(s)\big)$.

Now let us define the tangent vector of a given timelike curve using the parameter s and the Caputo fractional derivative:

$$v_1{}^{(\alpha)}(s) \equiv \frac{d^\alpha \beta(s)}{ds^\alpha} = \left(\frac{d^\alpha \beta_1(s)}{d\sigma^\alpha}, \frac{d^\alpha \beta_2(s)}{d\sigma^\alpha}\right) \tag{32}$$

Considering Equation (29), the norm of the tangent vector of the timelike curve is

$$\left\|v_1{}^{(\alpha)}(s)\right\| = \sqrt{\left|-\left(\frac{d^\alpha \beta_1}{ds^\alpha}\right)^2 + \left(\frac{d^\alpha \beta_2}{ds^\alpha}\right)^2\right|} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{dt}{ds} \sqrt{\left|-(\dot{\beta}_1)^2 + (\dot{\beta}_2)^2\right|} = 1 \tag{33}$$

Then using (29), a unit vector of the timelike curve β orthogonal to $\boldsymbol{v_1}$ can be defined as follows:

$$v_2{}^{(\alpha)}(s) \equiv \left(\frac{d^\alpha \beta_2}{ds^\alpha}, \frac{d^\alpha \beta_1}{ds^\alpha}\right) \tag{34}$$

So, we can give the following theorem.

Theorem 4.1. Let β be a timelike curve in the Lorentz plane that satisfies the condition (29) and has the parameter s given by (30). Then $v_1{}^{(\alpha)}(s)$ and $v_2{}^{(\alpha)}(s)$ given by (32) and (34) are the unit tangent vector and unit normal vector of the timelike curve β, respectively, and s is the arclength.

In the following, we is constructed the geometry of a timelike curve with the fractional $\left(v_1{}^{(\alpha)}(s), v_2{}^{(\alpha)}(s)\right)$ Frenet-Serret frame using the Caputo derivative.

Let us take a smooth timelike curve $\beta(s) = \big(\beta_1(s), \beta_2(s)\big)$ given by the arclength parameter (30) in the Lorentz plane. Based on the fractional frenet frame, let us define the frenet-serret formulas and the curvature of the timelike curve β. From Theorem 4.1, the tangent vector ${v_1}^{(\alpha)}(s)$ of the timelike curve provides $g({v_1}^{(\alpha)}(s), {v_1}^{(\alpha)}(s)) = -1$. If we take the derivative of both sides of this equation concerning s, we get

$$ {v_1}^{(\alpha)}(s)\frac{d{v_1}^{(\alpha)}(s)}{ds} = 0 \tag{35} $$

which means that $\frac{d{v_1}^{(\alpha)}(s)}{ds}$ can be expressed with the normal vector ${v_2}^{(\alpha)}(s)$:

$$ \frac{d{v_1}^{(\alpha)}(s)}{ds} = k^{(\alpha)}(s){v_2}^{(\alpha)}(s) \tag{36} $$

where $k^{(\alpha)}(s)$ is the fractional curvature of the timelike curve β. Then the norm of the normal vector ${v_2}^{(\alpha)}(s)$ is also equal to one, $\left\|{v_2}^{(\alpha)}(s)\right\| = 1$. Then we write $g({v_2}^{(\alpha)}(s), {v_2}^{(\alpha)}(s)) = 1$. If we take the derivative of both sides of this last equation concerning s, we have

$$ g\left({v_2}^{(\alpha)}(s), \frac{d{v_2}^{(\alpha)}(s)}{ds}\right) = 0 \tag{37} $$

From (37), $\frac{d{v_2}^{(\alpha)}(s)}{ds}$ can be given using a certain arclength function $\mu^{(\alpha)}$:

$$ \frac{d{v_2}^{(\alpha)}(s)}{ds} = \mu^{(\alpha)}(s){v_1}^{(\alpha)}(s) \tag{38} $$

Considering the orthogonality relation $g({v_1}^{(\alpha)}(s), {v_2}^{(\alpha)}(s)) = 0$, taking the derivative of both sides of this relation concerning s, we get the following expression:

$$ {v_1}^{(\alpha)}(s)\frac{d{v_2}^{(\alpha)}}{ds} + \frac{d{v_1}^{(\alpha)}}{ds}{v_2}^{(\alpha)}(s) = 0 \tag{39} $$

If the equations (36) and (38) are substituted in the expression (39), we get $\mu^{(\alpha)} = k^{(\alpha)}$. Thus, the following theorem is obtained.

Theorem 4.2. Let β be a timelike curve in the Lorentz plane that satisfies the condition (5) and has the parameter s given by (30). Let us consider $\left({v_1}^{(\alpha)}(s), {v_2}^{(\alpha)}(s)\right)$ as the fractional frame of this timelike curve β. Then the Frenet-Serret formulas for β can be given as

$$ \frac{d{v_1}^{(\alpha)}(s)}{ds} = k^{(\alpha)}(s){v_2}^{(\alpha)}(s) \tag{40} $$

$$ \frac{d{v_2}^{(\alpha)}(s)}{ds} = k^{(\alpha)}(s){v_1}^{(\alpha)}(s) \tag{41} $$

Let us now investigate the relationship between the fractional curvature and classical curvature of a given timelike curve β. Considering (32) and (40), we can write

$$ \frac{d}{ds}\left(\frac{d^{\,\alpha}\beta(s)}{ds^{\,\alpha}}\right) = k^{(\alpha)}(s){v_2}^{(\alpha)}(s) \tag{42} $$

If the normal vector ${v_2}^{(\alpha)}$ is applied to both sides of (42), we can write the fractional curvature as

$$ k^{(\alpha)}(s) = g\left({v_2}^{(\alpha)}(s), \frac{d}{ds}\left(\frac{d^{\,\alpha}\beta(s)}{ds^{\,\alpha}}\right)\right) \tag{43} $$

Considering the normal vector ${v_2}^{(\alpha)}(s)$ in (43), $k^{(\alpha)}(s)$ according to the fractional derivative is written as

$$k^{(\alpha)}(s) = -\frac{d^{\alpha}\beta_2}{ds^{\alpha}}\frac{d}{ds}\left(\frac{d^{\alpha}\beta_1}{ds^{\alpha}}\right) + \frac{d^{\alpha}\beta_1}{ds^{\alpha}}\frac{d}{ds}\left(\frac{d^{\alpha}\beta_2}{ds^{\alpha}}\right) \tag{44}$$

If a curve is given by an arbitrary parameter t and not by the arc length s, then we must calculate the fractional curvature according to an arbitrary parameter t. Then let us calculate the fractional curvature according to an arbitrary parameter t. From the expression (29) for the composite function $t = t(s)$, we can write

$$\frac{d}{ds}\left(\frac{d^{\alpha}\beta_1}{ds^{\alpha}}\right) = \frac{\alpha s^{-\alpha}\dot{\beta_1}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds} + s\frac{d^2t}{ds^2}\right) + \frac{\alpha s^{1-\alpha}\ddot{\beta_1}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \tag{45}$$

$$\frac{d}{ds}\left(\frac{d^{\alpha}\beta_2}{ds^{\alpha}}\right) = \frac{\alpha s^{-\alpha}\dot{\beta_2}}{\Gamma(2-\alpha)}\left((1-\alpha)\frac{dt}{ds} + s\frac{d^2t}{ds^2}\right) + \frac{\alpha s^{1-\alpha}\ddot{\beta_2}}{\Gamma(2-\alpha)}\left(\frac{dt}{ds}\right)^2 \tag{46}$$

where $\ddot{\beta_1} = \frac{d^2\beta_1}{dt^2}$ and $\ddot{\beta_2} = \frac{d^2\beta_2}{dt^2}$. If the expressions (45) and (46) are written instead in (44), the fractional-order curvature is

$$k^{(\alpha)}(t) = \left\{\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right\}^2 \left(\dot{\beta_1}\ddot{\beta_2} - \ddot{\beta_1}\dot{\beta_2}\right)\left(\frac{dt}{ds}\right)^3 \tag{47}$$

Moreover, from (31), (47) it can be rewritten by

$$k^{(\alpha)}(t) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}k(t) \tag{48}$$

$k(t)$ in this last equation is the classical curvature and

$$k(t) = \frac{-\dot{\beta_1}\ddot{\beta_2} + \ddot{\beta_1}\dot{\beta_2}}{\left(\left|-(\dot{\beta_1})^2 + (\dot{\beta_2})^2\right|\right)^{\frac{3}{2}}} \tag{49}$$

Thus, using the arclength definition given by (30), we can give the following theorem.

Theorem 4.3. The fractional curvature of a timelike plane curve given as $\beta(t) = \big(x(t), y(t)\big)$ is

$$k^{(\alpha)}(t) = \left\{\frac{\Gamma(2-\alpha)}{\alpha}\right\}^{\frac{1}{\alpha}}\left[\alpha\int_0^t \sqrt{\left|-(\dot{\beta_1})^2 + (\dot{\beta_2})^2\right|dt}\right]^{1-\frac{1}{\alpha}} k(t) \tag{50}$$

where t is an arbitrary parameter.

The part of $\frac{1}{s^{1-\alpha}}$ in (48) characterizes the effects of the fractional derivative given by the fractional tangent vector (32). The effect of the fractional derivative is strong at the start but becomes less effective over a longer period. This property of the effect influences the change of fractional curvature.

5. Conclusion

In this paper, firstly, the tangent vector of a spacelike (timelike) curve in the Lorentz plane are defined in terms of the fractional derivative. Then, by considering a spacelike (timelike) curve in the Lorentz plane, the arc length and fractional ordered frame of this curve are obtained. Later, the Caputo fractional derivative is considered and the relations between the standard curvature and fractional curvature of the spacelike (timelike) curves in the Lorentz plane are obtained. It has been observed that these relations geometrically overlap with the results obtained using the derivative in the classical sense.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interests

The author declares no conflict of interest.

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