Research Article / Araştırma Makalesi GU J Sci, Part A, 9(3): 259-266 (2022) https://doi.org/10.54287/gujsa.1093536

Gazi University
Journal of Science
PART A: ENGINEERING AND INNOVATION
http://dergipark.org.tr/gujsa

A New Numerical Approach Using Chebyshev Third Kind Polynomial for Solving Integrodifferential Equations of Higher Order

Abdullahi Muhammed AYINDE[1], Adewale Adeyemi JAMES[2*], Ajimoti Adam ISHAQ[3], Taiye OYEDEPO[4]

[1]*Department of Mathematics, Modibbo Adama University, Yola, Nigeria*
[2]*Department of Mathematics & Statistics, American University of Nigeria, Yola, Nigeria*
[3]*Department of Physical Sciences, Al-Hikmah University, Ilorin, Nigeria*
[4]*Federal College of Dental Technology and Therapy, Enugu, Nigeria*

Keywords

Degree of Approximant

Exact Solution

Third Kind Chebyshev Polynomial

Trial Solution

Volterra-Fredholm Integrodifferential Equations

Abstract

There are several classifications of linear Integral Equations. Some of them include; Voltera Integral Equations, Fredholm Linear Integral Equations, Fredholm-Voltera Integrodifferential. In the past, solutions of higher-order Fredholm-Volterra Integrodifferential Equations [FVIE] have been presented. However, this work uses a computational techniques premised on the third kind Chebyshev polynomials method. The performance of the results for distinctive degrees of approximation (M) of the trial solution is cautiously studied and comparisons have been additionally made between the approximate/estimated and exact/definite solution at different intervals of the problems under consideration. Modelled Problems have been provided to illustrate the performance and relevance of the techniques. However, it turned out that as M increases, the outcomes received after every iteration get closer to the exact solution in all of the problems considered. The results of the experiments are therefore visible from the tables of errors and the graphical representation presented in this work.

Cite

Ayinde, A. M, James, A. A., Ishaq A. A., & Oyedepo, T. (2022). A New Numerical Approach Using Chebyshev Third Kind Polynomial for Solving Integrodifferential Equations of Higher Order. *GU J Sci, Part A, 9*(3), 259-266.

Author ID (ORCID Number)
A. M Ayinde, 0000-0002-2563-0952
A. A James, 0000-0003-2257-5596
A. A. Ishaq, 0000-0002-8931-5708
T. Oyedepo, 0000-0001-9063-8806

Article Process	
Submission Date	25.03.2022
Revision Date	07.07.2022
Accepted Date	29.08.2022
Published Date	27.09.2022

1. INTRODUCTION

Integrodifferential equations have been observed in several ways and on several occasions. These include; Biology models, Chemical Kinetics, Mechanics, glass-forming procedures, and so many other difficult areas like Dynamics, Economics, Electromagnetism, Astro-Physics, Modelling, and Nano-Hydrodynamics.

Worthy of note is the fact that many authors have also given numerous and analytical methods for solving Integrodifferential equations. Some examples include;

Eslahchi et al. (2012) combined the Adomian's decompositions technique with a Wavelet-Galerking approach to solving Integrodifferential Equations. To establish an approximate solution of higher-order linear Fredholm Integrodifferential equations, a realistic matrix technique can be used (Kurt & Sezer, 2008) which possess a constant coefficient beneath the initial boundary condition in phrases of Taylor polynomials, numerical solution of mixed linear Integrodifferential difference equations is considered using the Chebyshev collocation method.

This method is mainly dependent on Chebyshev expansion approach. The specified conditions and the mixed linear Integrodifferential difference equation are transformed into matrix equations, which equate to a system

*Corresponding Author, e-mail: adewale.james@aun.edu.ng

of linear algebraic equations, in this approach (Gulsu et al., 2010). A numerical solution of the system of linear Voltera Integrodifferential equations is proposed in (Rashidinia & Tahmasebi, 2012) where the Taylor series method was developed and modified to solve the system of linear Voltera Integrodifferential equations. Sezer and Gulsu (2005) explores the polynomial solution from the most generic linear Fredholm Integrodifferential difference equation using the Taylor matrix technique. Wazwaz (2011) also considered non-linear Voltera Integrodifferential equations, but with a comparative approach to solving Integrodifferential equations, using the differential numerical approach; Lapace transform-Adomian decomposition methods were specifically combined. (Rashed, 2004; Wazwaz, 2010) used the application Lagrange interpolation to compute the numerical solutions of integralodifferential equations. Yusufoglu (2007) solved Integrodifferential equations by hiring an efficient algorithm. Akgonullu et al. (2011) presented higher-order linear Fredholm Integrodifferential equations with variable coefficients in terms of Hermite polynomials. Taiwo and Fesojaye (2015) solved Fractional-order Integrodifferential equations by presenting perturbation Least-square Chebyshev method. A new numerical scheme for solving the Volterra-Integrodifferential equation system using Genocchi polynomials is presented in (Loh & Phang, 2018). Sakran (2019) constructs an algorithm for solving singularly perturbed Volterra integral type and integrodifferntial equations based on a finite expansion in Chebyshev polynomials of the third kind. Rabiei et al. (2019) investigated the numerical solution of Volterra integrodifferential equations using the General linear method; in the work, the order conditions of the proposed method are derived using B-series and rooted trees techniques. Lotfi and Alipanah (2020) describes the Legendre spectral element method for solving integrodifferential equations. Samaher (2021) proposes a reliable iterative method for resolving many types of Volterra-Fredholm integrodifferential equations, and the iterative method is used to obtain series solutions to the problems under consideration. Adebisi et al. (2021) employed the Galerkin method to solve Volterra integrodifferntial equations using Chebyshev polynomials as the basis function.

The work of (Shah & Singh, 2015) prompted us to study the linear Integrodifferential equations. In the work, the basis function used for the class of initial value problems was the Homotopy Analysis Method this triggered a study of the work, and it was again applied to Integrodifferential equations (Linear Case). We taken into consideration a standard Higherer-order linear Voltera, and Fredholm Integrodifferential equation of the form;

$$\mathrm{B_{01}}\varphi^{\mathrm{m}}(\mathrm{z}) + \mathrm{B}\varphi^{\mathrm{m}-1}(\mathrm{z}) + \cdots + \mathrm{B_{m-1}}\varphi'(\mathrm{z}) + \mathrm{B_m}\varphi(\mathrm{z}) + +\lambda \int_{\mathrm{h(z)}}^{\mathrm{i(z)}} \mathrm{K(z,s)}\varphi(\mathrm{s})\mathrm{dt} = \mathrm{f(z)} \tag{1}$$

subject to the conditions

$$\varphi(\mathrm{p}) = \mathrm{P}\ \&\ \varphi(\mathrm{q}) = \mathrm{Q} \tag{2}$$

where B'^s are real constants; i, h are finite constants; $K(z,s)$ and $f(z)$ are specified given real-valued functions; φ are unknown constants to be determined. We then solved these problems by assuming an approximate solution given by Equation (4) below.

2. BASIC DEFINITION

This section contains basic definition that are essential to the research work in this paper.

2.1. Integrodifferential Equations (Wazwaz, 2010)

Integrodifferential equations (IDEs) are equations in which the unknown function $\varphi(z)$ is written with the integral sign and also has an ordinary derivative$\varphi^{(k)}$. The following is a typical Integrodifferential equation:

$$\varphi^{(k)}(z) = f(z) + \lambda \int_{h(z)}^{i(z)} K(z,s)\varphi(s)ds \tag{3}$$

$i(z)$ and $h(z)$ are integration limits that can be constants, variables, or blended. is a free parameter, $f(z)$ is a specified function, and $K(z,s)$ stands for kernel.

If the limit $\varphi(z)$ is substituted by a variable of integration z, we have the Volterra Integrodifferential equation, and if the limit of integration is constants, we have the Fredholm Integrodifferential equation.

2.2. Collocation Method

A method of evaluating an approximate solution in a suitable collection of functions, sometimes referred to as a trial solution or basis function.

2.3. Exact Solution

If a solution may be expressed in a closed form, it's known as an exact solution. Examples are polynomials, exponential functions, trigonometric functions, or an aggregate of or extra of these standard functions.

2.4. Approximate Solution

An approximate solution denoted by $\varphi_M(z)$ is given in the form

$$\varphi_M(z) = \sum_{i=0}^{M} b_j \zeta_m(z) \tag{4}$$

where $b_j (j \geq 0)$ are to be determined.

2.5. Chebyshev Polynomials of Third Kind (Loh & Phang, 2018)

The Chebyshev polynomial of the third kind in [-1, 1] of degree m is represented by $V_m(z)$, where:

$$V_m(z) = \cos \frac{\left(m + \frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)}, \qquad \text{where } z = \cos\vartheta \tag{5}$$

This elegance of Chebyshev polynomials satisfied the subsequent recurrence relation given by

$$V_0(z) = 1, \qquad V_1(z) = 2z - 1, V_m(z) = 2zV_{m-1}(z) - V_{m-2}(z), \qquad m = 2, 3, \cdots \tag{6}$$

The Chebyshev polynomial of the third kind in $[\alpha, \beta]$ of degree, m is represented by $V_m^*(z)$, where:

$$V_m^*(z) = \cos \frac{\left(m + \frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)}, \cos\vartheta = \frac{2z - (\alpha + \beta)}{\beta - \alpha}, \qquad \vartheta \in [0, \pi] \tag{7}$$

3. THE RESEARCH METHODOLOGY

Equation (1) was solved using the third kind of Chebyshev polynomials and the standard collocation method.

3.1. The Standard Collocation Method Employs a Third-Order Chebyshev Polynomial Basis

The standard collocation method can be used to solve the well-known problem provided in equation (1), subject to the conditions given in equation (2). This is accomplished by assuming a form trail solution.

$$\varphi_m(z) = \sum_{i=0}^{M} b_j V_j^*(z) \tag{8}$$

where b_j, $j = 0,1$ M are undefined constants and $V_j^*(z)(j \geq 0)$ are 1/3-order Chebyshev polynomials described in equations (5-7). In most instances, a larger M, produces a better approximate solution, and b_j is the specialized coordinate referred to as the degree of freedom. Thus, differentiating equation (8) with respect to mth-times as functions of z, to obtain the following equations

$$\left.\begin{aligned} \varphi'_m(z) &= \sum_{i=0}^{M} b_j V_j^{*'}(z) \\ \varphi''_m(z) &= \sum_{j=0}^{M} b_j V_j^{*''}(z) \\ &\vdots \\ \varphi^{(m)}(z) &= \sum_{j=0}^{M} b_j V_j^{*(m)}(z) \end{aligned}\right\} \tag{9}$$

As a result of putting Equations (8-9) into Equation (1), we get

$$\begin{aligned} &B_{01}\sum_{j=0}^{M} b_j V_j^{*(m)}(z)\, B_{11}\sum_{j=0}^{M} b_j V_j^{*(m-1)}(z)\, B_{21}\sum_{j=0}^{M} b_j V_j^{*(m-2)}(z) \\ &B_{m1}\sum_{j=0}^{M} b_j V_j^{*}(z) + \lambda\int_{h(z)}^{i(z)} K(z,t)\left(\sum_{j=0}^{M} b_j V_j^{*}(t)\right)dt = f(z) \end{aligned} \tag{10}$$

The integral part of Equation (10) is evauated to produce

$$\begin{aligned} &B_{01}\sum_{i=0}^{M} b_j V_j^{*(m)}(z) + B_{11}\sum_{j=0}^{M} b_j V_j^{*(m-1)}(z) + B_{21}\sum_{j=0}^{M} b_j V_j^{*(m-2)}(z) + \\ &+ B_{m1}\sum_{j=0}^{M} b_j V_j^{*}(z) + \lambda G(z) = f(z) \end{aligned} \tag{11}$$

and $G(z) = \int_{h(z)}^{i(z)} K(z,t)\left(\sum_{i=0}^{M} b_j V_j^{*}(t)\right)dt$

We collocate the resulting equation after simplification at the point $z = z_k$

$$\begin{aligned} &B_{01}\sum_{i=0}^{M} b_j V_j^{*(m)}(z_k) + B_{11}\sum_{j=0}^{M} b_j V_j^{*(m-1)}(z_k) + B_{21}\sum_{j=0}^{M} b_j V_j^{*(m-2)}(z_k) + \\ &\cdots B_{m1}\sum_{j=0}^{M} b_j V_j^{*}(z_k) + \lambda G(z_k) = f(z_k) \end{aligned} \tag{12}$$

where

$$z_k = \alpha + \frac{(\beta-\alpha)k}{M};\ k = 1, 2, \dots, M-1 \tag{13}$$

Equation (12) is then transformed into a matrix as

$$B\underline{z} = \underline{d}z_k \tag{14}$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & & b_{1,m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2,m} \\ b_{31} & b_{32} & b_{33} & & b_{3,m} \\ & \vdots & & \ddots & \vdots \\ b_{m,1} & b_{m,2} & b_{m,3} & \cdots & b_{m,m} \end{pmatrix} \tag{15}$$

$$\underline{z} = (z_1, z_2, z_3, \cdots, z_m)^T \tag{16}$$

$$\underline{d} = (f(d_1), f(d_2), f(d_3) \ldots, f(d_m))^T \tag{17}$$

Consequently, Equation (12) yields a (M-1) algebraic linear system of equations in (M+1) unknown constants, and the specified conditions in Equation (12) yield m additional equations (2). We now have an algebraic linear system of equations with (M+1) variables. These equations are then solved using Maple 18 software to provide (M+1) unknown constants b_j $(j \geq 0)$, which are then used to approximate the solution given by Equation (8).

4. PROBLEMS AND RESULTS

With third-kind Chebyshev as the basis functions, the standard collocation approximation approach on higherer-order integrodifferential equations was demonstrated. At different intervals of the problems under consideration, the results obtained by the exact solution were compared with the approximate solution.

4.1. Problem 1 (Akgonullu et al., 2011)

Here, we looked at the Fredholm Integrodifferential equation of second order.

$$\varphi''(z) = e^z - \frac{4}{3}z + \int_0^1 zt\varphi(t)dt. \tag{18}$$

with initial conditions

$$\varphi(0) = 1, \qquad \varphi'(0) = 2 \tag{19}$$

The exact solution is as follows

$$\varphi(z) = z + e^z \tag{20}$$

4.2. Problem 2 (Wazwaz, 2011)

Here, we considered the second-order linear Volterra Integrodifferential equation

$$\varphi''(z) = 2 - 2\mathrm{z}\sin z - \int_0^z (z-t)\varphi(t)dt. \tag{21}$$

with initial conditions

$$\varphi(0) = 0, \qquad \varphi'(0) = 0 \tag{22}$$

The exact solution is given as

$$\varphi(z) = z\, sin\, z \tag{23}$$

4.3. Problem 3 (Wazwaz, 2011)

Here, we considered the second-order linear Volterra Integrodifferential equation

$$\varphi^{(\prime v)}(z) = -1 + z - \int_0^z (z-t)\varphi(t)dt. \tag{24}$$

with initial conditions

$$\varphi(0) = -1, \quad \varphi'(0) = 1, \quad \varphi''(0) = 1, \quad \varphi'''(0) = 1 \tag{25}$$

The exact solution is

$$\varphi(z) = \sin x - \cos x \tag{26}$$

Note: We defined absolute error as follows:

$$Absolute\ Error = |\varphi(z) - \varphi_M(z)| \tag{27}$$

where, $\varphi(z)$ stands for the exact solution and $\varphi_M(z)$ stands for the approximate solution obtained for the various *M values*.

4.4. Tables of Errors and Approximate for the Problems

Table 1. Table of Error and Approximate for Problem 1

Z	φ (Exact)	φ (Approximate) For Case M = 5	φ (Approximate) For Case M = 10	Absolute (Error) For M = 5	Absolute (Error) For M = 10
0.0	1.0000000000000	1.0000000020000	1.000004792000	2.00 e-09	4.79e-06
0.2	1.4214027581602	1.4213780016409	1.421407783585	2.48 e-05	5.03e-06
0.4	1.8918246976413	1.8917670852221	1.891831437531	5.76 e-05	6.74e-06
0.6	2.4221188003905	2.4220279027664	2.422126707764	9.09 e-05	7.91e-06
0.8	3.0255409284925	3.0254159909159	3.025548507593	1.25 e-04	7.58e-06
1.0	3.7182818284590	3.7181120074800	3.718293781960	1.61 e-04	1.11e-05

Table 2. Table of Error and Approximate for Problem 2

Z	φ (Exact)	φ (Approximate) For Case M = 5	φ (Approximate) For Case M = 10	Absolute (Error) For M = 5	Absolute (Error) For M = 10
0.0	0.000000000000	-5.000000000e-11	-1.13494000e-07	5.00e-11	1.13e-10
0.2	0.039733866159	0.0396474110875	0.039734121852	8.65e-05	2.56e-07
0.4	0.155767336923	0.1555676200122	0.155767558521	1.91e-04	2.22e-07
0.6	0.338785484037	0.2394579599488	0.338785316341	9.93e-04	1.68e-07
0.8	0.573884872711	0.5734637964333	0.573884334512	4.21e-04	5.38e-07
1.0	0.841470984808	0.7045186581451	0.841470029934	5.53e-04	9.55e-07

Table 3. Table of Error and Approximate for Problem 3

Z	φ (Exact)	φ (Approximate) For Case M = 6	φ (Approximate) For Case M = 10	Absolute (Error) For M = 6	Absolute (Error) For M = 10
0.0	-1.000000000000	-0.999999999900	1.000000006000	1.00e-10	6.00e-09
0.2	-0.781397247046	-0.781397490149	-0.781397249066	2.43e-07	2.10e-09
0.4	-0.531642651694	-0.531645288439	-0.531642645501	2.64e-05	6.20e-09
0.6	-0.260693141515	-0.260702948221	-0.260693148269	9.81e-05	6.80e-09
0.8	0.020649381552	0.020625391964	0.020649386322	2.31e-05	4.77e-09
1.0	0.841470984808	0.7045186581451	0.841470029934	5.53e-04	9.55e-07

5. CONCLUSION

In terms of error and approximate solutions, Table 1-3 provide the numerical solutions for the Fredholm-Volterra Integrodifferential equations computed using the third kind of Chebyshev polynomial basis function. In all of the problems solved, the approximate solution is much closer to the precise solution when evaluated at an equally spaced interior point.

However, as shown in the tables of errors, the obtained results provide a good approximation to the precise solution for varying degrees of M, in other words, as M increases, the obtained results provide a good approximation to the exact solution with a few iterations. As a result, we conclude that the method was realistic and effective under the given circumstances.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

AVAILABILITY OF DATA AND MATERIAL

Not applicable

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