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Research Article

Influence Surface Coefficients of Plates Resting on Pasternak Foundation

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ABSTRACT

In this study, internal force influence surface coefficients of plates resting on Pasternak foundation are obtained by the finite element method applying the classical Müller-Breslau Principle. The two-parameter elastic foundation is represented straightforwardly by the inclusion of the elastic bedding and shear parameter matrix terms of a 4-noded soil finite element to the corresponding stiffness matrix terms of the plate finite element used in the implementation. Using this approach, the influence surface coefficients are easily and directly obtained through the finite element analysis of the plate-foundation system using the loading vectors derived from the element matrices of the finite elements used. Internal force influence surface coefficients of plates with and without elastic foundation are given comparatively through numerical examples, verifying the values with another approach and the literature.

Introduction

An influence line or surface function of a particular point on a structure is the variation of any designated effect like displacements and internal forces due to a unit loading moving on the structure. These functions are of great importance when the structures are subjected to live loads since they are required when extremum displacement and internal force values at particular points of the structures due to live loads are to be determined.

Influence lines and surfaces can be obtained using three main approaches which are the equilibrium method, the Müller-Breslau principle and the adjoint method. The equilibrium method is the simplest method where the structure is analysed for different locations of the unit loading in order to obtain the influence functions. However, high computational cost arises in this method since the analyses are repeated many times. A more efficient approach, the Müller-Breslau principle [1-3] which is based on the principle of virtual work states that the influence function of any designated effect on a structure is proportional to the deformed shape of the structure obtained by applying a known displacement in the direction of the designated effect [4,5]. In the adjoint method proposed by [6,7], an adjoint variable vector for any designated effect function is calculated using the adjoint equations and influence line ordinates are obtained via the equilibrium equations. A remedy is proposed by [8] for the deficiency of the adjoint method which is the necessity of a correction to the adjoint variable vector for the designated effect function in the directions of the constrained degrees of freedom. In both of these methods, the analysis is performed only once which significantly reduces the computation time. In [9], finite element implementation of the three main approaches used for the determination of influence surfaces for internal forces is carried out.

Analysis of plates resting on elastic foundations are widely performed for the shallow reinforced concrete foundations [10] of various types of buildings. Extremum internal forces of these foundations due to live loads can be obtained via influence surface coefficients. In this paper, a a MATLAB code is written for the determination of internal force influence surface coefficients of plates resting on two-parameter elastic foundations using the approach proposed by [2] which is based on the finite element implementation of the classical Müller-Breslau Principle. According to this approach, the element loading matrices used to determine the influence surfaces of plates are obtained using the Betti's law and the governing equations of the finite element method. The elastic foundation is represented by adding the terms of elastic bedding and shear parameter matrices of a soil finite element derived by [11] to the plate element stiffness matrix terms corresponding to deflections.

Kirchhoff Plate Finite Element and Inclusion of Foundation Parameters

The finite element used in the implementation is a popular 4-noded non-conforming 12 degree of freedom rectangular plate finite element, Figure 1, developed by Melosh [12] and Zienkiewicz and Cheung [13,14], and its formulation is based on the well-known Kirchhoff plate theory.

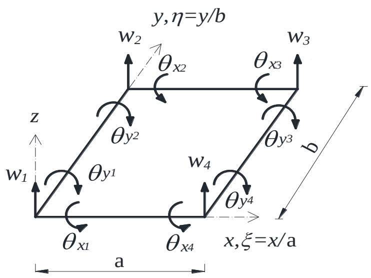

Figure 1. Plate bending element

The displacement vector and the rotations of the element are

$$\{d\} = \{w \quad \theta_x \quad \theta_y\}^T,$$
$$\theta_x = \frac{\partial w}{\partial y} \;, \qquad \theta_y = -\frac{\partial w}{\partial x} \tag{1}$$

The displacement function of the finite element is

$$w = \{1 \;\; \xi \;\; \eta \;\; \xi^2 \;\; \xi\eta \;\; \eta^2 \;\; \xi^3 \;\; \xi^2\eta \;\; \xi\eta^2 \;\; \eta^3 \;\; \xi^3\eta \;\; \xi\eta^3\} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{12} \end{Bmatrix} \tag{2}$$

where $c_1, c_2, \ldots, c_{12}$ are arbitrary constants. It can also be expressed in terms of nodal unknowns as

$$w = \sum_{i=1}^{12} N_i d_i \tag{3}$$

Here, N_i are C^1 continuous shape functions used to obtain the unknown nodal displacement components.

The stiffness matrix of the element is obtained using

$$K = \int_A B^T D B dA \tag{4}$$

Here, $[B] = [\partial][N]$ where

$$[\partial] = \begin{Bmatrix} -\dfrac{\partial^2}{\partial x^2} \\ -\dfrac{\partial^2}{\partial y^2} \\ -2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} \tag{5}$$

The elasticity matrix for the isotropic material is defined as

$$[D] = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{6}$$

where E is the Young's modulus, h is the plate thickness and ν is the Poisson's ratio.

$[C]$ and $[C_T]$ elastic bedding and shear parameter matrices to be included in the plate finite element stiffness matrix are derived by [11] where C_{ij} and $C_{T_{ij}}$ matrix terms are obtained using

$$C_{ij} = k_w \int_A w_i w_j dA \tag{7}$$

$$C_{T_{ij}} = k_p \int_A \frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x} + \frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y} dA \tag{8}$$

Here, k_w and k_p are the coefficient of subgrade reaction and the shear modulus of the foundation, respectively and the elastic bedding and shear parameter matrices are

$$[C] = \frac{k_w ab}{36} \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \tag{9}$$

and

$$[C_T] = \frac{k_p}{3} \begin{bmatrix} \alpha+\beta & \alpha/2-\beta & \beta/2-\alpha & -(\alpha+\beta)/2 \\ \alpha/2-\beta & \alpha+\beta & -(\alpha+\beta)/2 & \beta/2-\alpha \\ \beta/2-\alpha & -(\alpha+\beta)/2 & \alpha+\beta & \alpha/2-\beta \\ -(\alpha+\beta)/2 & \beta/2-\alpha & \alpha/2-\beta & \alpha+\beta \end{bmatrix} \tag{10}$$

where $\alpha = a/b$ and $\beta = b/a$

Influence Surface Coefficients for Stress Components

In a linear elastic 2D structure discretized into finite elements, displacement ordinates due to a unit loading in the direction of any displacement component give the influence surface coefficients of that displacement component according to the Betti's law. The influence coefficients of all nodal displacement components of an element can be obtained via

$$SU = Q \tag{11}$$

where S is the system stiffness matrix, U is a matrix consisting of influence coefficients of nodal displacement components and Q is the system loading matrix consisting of loading vectors with unit values in the directions of the nodal displacement components of the element in question while the rest of the vector elements are zero.

Using the governing equations of the finite element method, the influence coefficients of stress components at a particular point of an element can be obtained via

$$\sigma = DBU \tag{12}$$

Defining the element stress matrix as

$$G = DB \tag{13}$$

and substituting into Eq.(12), it yields

$$\sigma = GU \tag{14}$$

or

$$\sigma^T = UG^T \tag{15}$$

Multiplying both sides of Eq. (11) by G^T,

$$SUG^T = G^T \tag{16}$$

is obtained where G^T is taken as the element loading matrix. Substituting the element loading matrices into the global lading matrix Q and solving the linear simultaneous equations, σ^T is obtained giving the influence surface coefficients of stress components for any point in the system. If the influence surface coefficients of any stress component are to be calculated, the column of matrix G^T corresponding to the stress component in question is taken as the element loading vector which is denoted as r.

Computer Implementation

A MATLAB code is written to determine the influence surface coefficients for the stress components of any point on plates resting on Pasternak foundation. The procedure is as follows:

1. Stiffness matrix of the 12 degree-of-freedom plate finite element is constructed. $[C]$ and $[C_T]$ foundation parameter matrix terms are added to the relevant stiffness matrix terms of the element. For foundation extensions, stiffness matrix row and column terms of these elements corresponding to rotations are eliminated and foundation parameter matrix terms are assigned as the stiffness matrix terms corresponding to deflections.
2. System stiffness matrix is constructed.
3. The local coordinates of the node of which the influence surface coefficients are to be determined are defined for the connected 4 elements and the element stress vectors for the required stress component are obtained.
4. Average values of these stress vectors are assigned as the global loading vector terms.
5. Static analysis of the system is performed and the displacement vector is obtained where the deflection values correspond to the influence surface coefficients for the stress component of the node in question.

Benchmark Examples

Example 1

A benchmark example taken from the literature [15,16] is solved in order to verify the plate-foundation model. The example is a simply supported homogeneous square plate resting on an isotropic Pasternak foundation and is subjected to a uniformly distributed load of $q_0=E/10^5$ kN/m^2. Length to height ratio of the plate is a/h =100 and the Poisson's ratio is υ=0.3. Dimensionless central deflections are obtained for constant coefficient of subgrade reaction and increasing shear moduli values and compared with the reference solutions as given in Table 1.

Note that K_w and K_p are dimensionless coefficient of subgrade reaction and shear modulus, respectively. $K_w = k_w a^4 / D$ and $K_p = k_p a^2 / D$ where $D = \frac{Eh^3}{12(1-\upsilon^2)}$.

It is observed that the deflections obtained in this study are very close to the reference values and decrease with increasing shear modulus as expected.

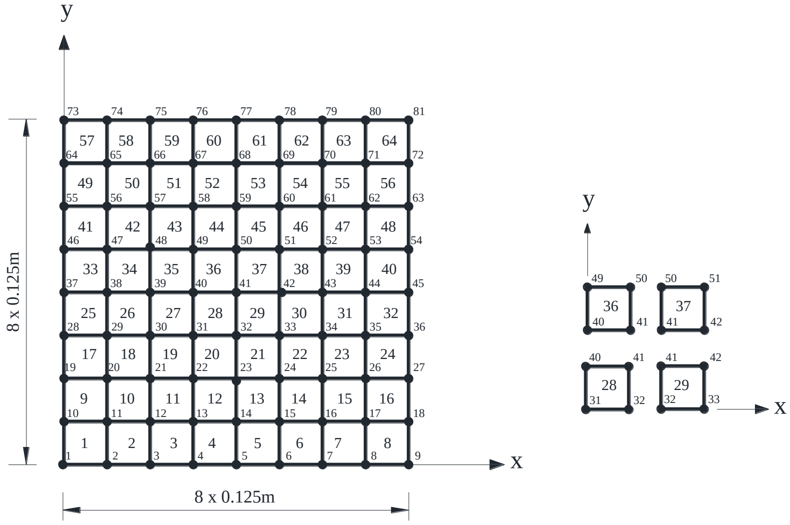

Figure 2. Finite element mesh of the plate

Table 1. Dimensionless central deflections of a uniformly loaded square plate resting on Pasternak foundation

K_w	K_p	Dimensionless central deflection ($w' = 10^3 D w / qa^4$)		
		present	[16]	[15]
1	1	3.8517	3.8546	3.8530
1	81	0.7637	0.7630	0.7630
1	625	0.1154	0.1153	0.1150

Example 2

The finite element mesh of a simply supported square plate solved by [2] is given in Figure 2. The thickness of the plate is t=1m, the Young's modulus is E=10.92 kN/m^2 and the Poisson's ratio is υ=0.3. First, bending moment (M_x) influence surface coefficients for the central node (node 41) are obtained using the present approach and compared with the reference solution, [2] and the values obtained by SAP2000 software package in order to verify the present implementation.

The loading vectors which are the column vectors of the element stress matrices corresponding to M_x (1st columns) belong to the connecting elements of node 41 (elements 28, 29, 36 and 37). ¼ of each element loading vector is used since the average M_x stress is taken into account and these vectors are given in Table 2 also verified with the values provided by [2]. Substituting these element loading vectors into the system loading vector and solving the plate, the influence surface coefficients for the central node (node 41) are obtained at once.

In SAP2000, the system model is solved for a unit vertical load applied to the selected nodes separately and M_x moment values at the central node corresponding to M_x influence surface coefficients are obtained. This operation is time consuming since the analysis is repeated for each location of the unit load.

It is observed that the coefficients for the selected nodes are very close to the reference values, Table 3. The slight differences may be due to the different finite elements used in the implementation. Twisting moment (M_{xy}) influence surface coefficients for the midpoint of the plate at some selected nodes are also obtained in addition to M_x coefficients and checked with the values obtained by SAP2000, Table 4. Note that the loading vectors for M_{xy} are the column vectors of the connecting element stress matrices which correspond to M_{xy} (3rd columns).

Table 2. Loading vectors for M_x influence surface coefficients for node 41.

freedom	r^{28}	r^{29}	r^{36}	r^{37}
1	0.0	-115.2	-384.0	499.2
2	0.0	-4.8	0.0	9.6
3	0.0	0.0	16.0	-32.0
4	-384.0	499.2	0.0	-115.2
5	0.0	-9.6	0.0	4.8
6	16.0	-32.0	0.0	0.0
7	499.2	-384.0	-115.2	0.0
8	-9.6	0.0	4.8	0.0
9	32.0	-16.0	0.0	0.0
10	-115.2	0.0	499.2	-384.0
11	-4.8	0.0	9.6	0.0
12	0.0	0.0	32.0	-16.0

Table 3. Nodal Influence Surface Coefficients for M_x

Node Number	Influence surface coefficient for M_x		
	present	[11]	SAP2000
11	0.011273	0.010770	0.011536
21	0.046316	0.044470	0.047332
31	0.121609	0.116450	0.124347
39	0.058019	0.057770	0.058426
41	0.366410	0.346090	0.375326

Then, a two-parameter elastic foundation is added to the square plate and M_x influence surface coefficients for node 41 are obtained for constant coefficient of subgrade reaction and increasing shear modulus and given comparatively in Table 5. It is seen that the influence surface coefficients at the selected nodes decrease as the shear modulus increases.

In order to verify the results, the system model is also created in SAP2000 using the procedure given in [17]. The two-parameter elastic foundation is modelled using "shell" element where the section type is selected as "Plane-Strain" and the thickness of the section is assigned to a unit value. The elastic moduli and Poisson's ratios in all directions are set to zero values and the in-plane shear moduli (G_{13} and G_{23}) are assigned to the shear moduli of the foundation. Thus, shear stresses in the thickness direction and vertical end forces occur only. The first parameter of the Pasternak foundation is represented by "area springs" assigned to the surface of the foundation. The deflections of the plate and the foundation nodes are equalised in order to provide the connection between the plate and the foundation, [17]. It is seen that the results obtained are close to each other, Table 6.

Table 4. Nodal influence surface coefficients for M_{xy}

Node Number	Influence surface coefficient for M_{xy}	
	present	SAP2000
11	-0.005105	-0.005262
17	0.005105	0.005262
21	-0.016096	-0.016459
25	0.016096	0.016459
31	-0.026572	-0.026646
33	0.026572	0.026646
39	0.000000	0.000000
41	0.000000	0.000000
49	0.026572	0.026646
51	-0.026572	-0.026646
57	0.016096	0.016459
61	-0.016096	-0.016459
65	0.005105	0.005262
71	-0.005105	-0.005262

Table 5. Nodal M_x Influence Surface Coefficients for different soil parameters

Node Number	Influence surface coefficients for M_x		
	No Foundation	K_w=1 K_p=1	K_w=1 K_p=625
11	0.011273	0.005873	0.000004
21	0.046316	0.026425	0.000054
31	0.121609	0.082964	0.001179
39	0.058019	0.029468	0.001283
41	0.366410	0.314923	0.019719

Table 6. Nodal M_x Influence Surface Coefficients for different soil parameters

Node Number	Influence surface coefficients for M_x			
	K_w=1 K_p=1		K_w=1 K_p=625	
	present	SAP2000	present	SAP2000
11	0.005873	0.006043	0.000004	0.000002
21	0.026425	0.027149	0.000054	0.000061
31	0.082964	0.085161	0.001179	0.001192
39	0.029468	0.029610	0.001283	0.001439
41	0.314923	0.322896	0.019719	0.019542

Example 3

Plan and vertical section views of a rectangular plate resting on a two-parameter elastic foundation solved by [11] is given in Figure 3. The foundation has extensions in all directions and no boundary conditions are assigned to the plate edges which is a more realistic approach. First, the system is solved under the given distributed load for different soil parameters and midpoint deflections are compared with the values given in reference [11] in order to verify the present model. It is observed that the deflection values are very close to each other as given in Table 7. It is also seen that the results are very close to those obtained in an earlier study where the finite difference method is used in the system analysis, [18].

Then, M_x influence surface coefficients for the midpoint at some selected nodes shown in Fig. 4 are obtained for the given soil parameters and verified with the coefficients obtained by SAP2000, Table 8.

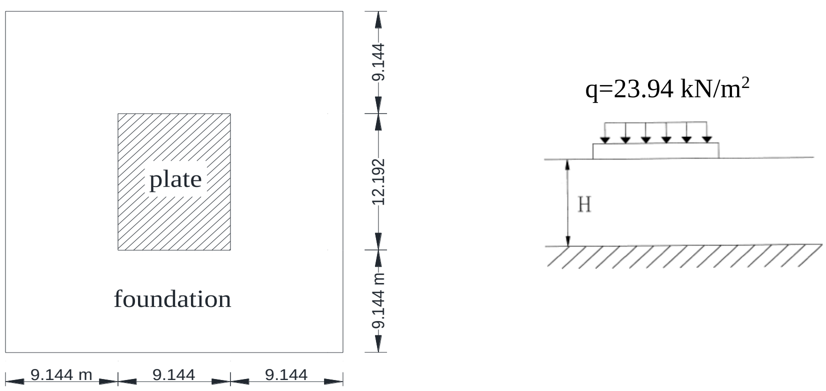

Figure 3. Plan and vertical section of the plate-foundation system

Table 7. Central deflections for different soil parameters

k_w	k_p	Central deflection (m)	
		present	[11]
27192	26826	0.000878	0.000853
13757	50410	0.001500	0.001526
9377	70586	0.001900	0.001893
5964	104664	0.002200	0.002212

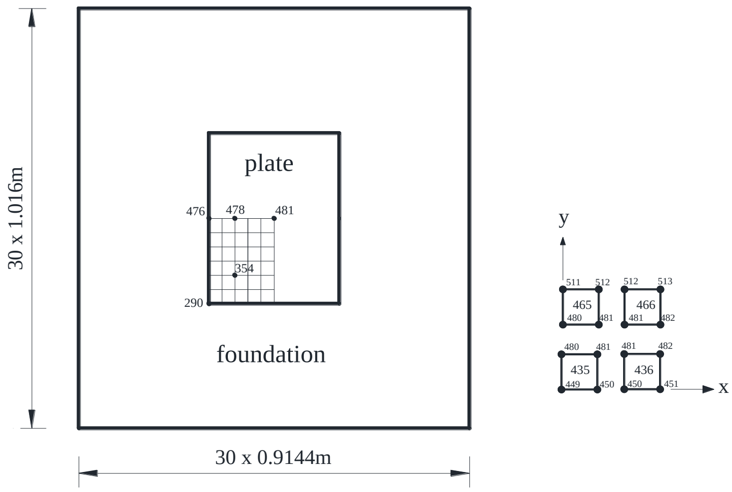

Figure 4. Finite element mesh of the plate-foundation system

Table 8. M_x Influence Surface Coefficients for different soil parameters

Node Number	Influence surface coefficients for M_x							
	k_w=27192 k_p=26826		k_w=13757 k_p=50410		k_w=9377 k_p=70586		k_w=5964 k_p=104664	
	present	SAP2000	present	SAP2000	present	SAP2000	present	SAP2000
290	-0.000002	-0.000003	-0.000032	-0.000025	-0.000042	-0.000031	-0.000036	-0.000020
354	-0.000059	-0.000063	-0.000110	-0.000091	-0.000075	-0.000052	-0.000021	-0.000035
476	-0.000106	-0.000196	-0.000486	-0.000506	-0.000500	-0.000503	-0.000403	-0.000328
478	-0.002412	-0.002304	-0.001977	-0.002064	-0.001532	-0.001684	-0.001122	-0.001275
481	0.122743	0.111566	0.101862	0.094453	0.088089	0.081619	0.071632	0.064130

Conclusions

In this paper, internal force influence surface coefficients required for the extremum internal force values due to live loads are obtained for the plates resting on Pasternak foundation. It is demonstrated that the two-parameter elastic foundation can be accounted for by adding the elastic bedding and shear parameter matrices of a soil element to the stiffness matrix terms of the plate element corresponding to the deflection freedom which is a straightforward procedure. The approach for the determination of influence surface coefficients of plates proposed in the literature is adapted to the plates on two-parameter foundations. Compared with the classical techniques, the influence surface coefficients are easily and directly obtained with this approach through the finite element analysis of the plate-foundation system using the loading vectors derived from the element matrices obtained by the governing equations and the Betti's law. No modifications to the input data of the systems discretised by different types of finite elements are required since the loading vectors are derived from the element matrices of the finite elements used in the implementation. Besides, the values within the elements can easily be obtained by using the nodal values of the influence surface coefficients and the element shape functions. The obtained results show good agreement with the reference values and it is demonstrated that the aforementioned approach is also very suitable for the plates on two-parameter elastic foundations since the plate-foundation system is solved only once to obtain the influence surface coefficients for the stress component at a particular point.

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References

[1] A. Cifuentes, M. Paz, "A note on the determination of influence lines and surfaces using finite elements", *Finite Elements in Analysis and Design*, vol. 7, pp. 299-305, 1991.

[2] E. Orakdöğen, K. Girgin, "Direct determination of influence lines and surfaces by F.E.M.", *Structural Engineering and Mechanics*, vol. 20, no. 3, pp. 279-292, 2005.

[3] W. Shen, "The generalized Müller-Breslau principle for higher-order elements", *Computers & Structures*, vol. 44, pp. 207-212, 1992.

[4] A. Ghali, A.M. Neville, *Structural Analysis*, Chapman and Hall, London, 1978.

[5] T.M. Charlton, "The principle of virtual work in relation to Müller-Breslau's principle", *International Journal of Mechanical Sciences*, vol. 22, no. 2, pp. 523-525, 1980.

[6] A.D. Belegundu, "Interpreting adjoint equations in structural optimization", *Journal of Structural Engineering, ASCE*, vol. 112, no. 8, pp. 1971-1976, 1986.

[7] A.D. Belegundu, "The adjoint method for determining influence lines", *Computers & Structures*, vol. 29, no. 2, pp. 345-350, 1988.

[8] A.M. Memari, H.H. West, "Computation of bridge design forces from influence surfaces", *Computers & Structures*, vol. 38, pp. 547-556, 1991.

[9] A. Albuquerque, V.G. Haach, R.R. Paccola, "Dependency of modeling parameters for the construction of influence surfaces by the finite element method", *Engineering with Computers*, vol. 34, pp. 143-154, 2018.

[10] A. Trupia, A. Saygun, "Shallow reinforced concrete foundations", *Nobel Publication, Ankara (in Turkish)*, 2009.

[11] M. Çelik, A. Saygun, "A method for the analysis of plates on a two-parameter foundation", *International Journal of Solids and Structures*, vol. 36, pp. 2891-2915, 1999.

[12] R.J. Melosh, "Structural analysis of solids", *Journal of Structural Engineering, ASCE*, vol. 4, pp. 205-223, 1963.

[13] O.C. Zienkiewicz, Y.K. Cheung, "The finite element method for analysis of elastic isotropic and orthotropic slabs", *Proceedings of the Institution of Civil Engineers*, vol. 28, pp. 471-88, 1964.

[14] O.C. Zienkiewicz, Y.K. Cheung, "*Finite element procedures in the solution of plate and shell problems in stress analysis*", Chapter 8. John Wiley & Sons, Chichester, 1965.

[15] K.Y. Lam, C.M. Wang, X.Q. He, "Canonical exact solutions for Levy-plates on two parameter foundation using Green's functions", *Engineering Structures*, vol. 22, no. 4, pp. 364-378, 2000.

[16] Z.Y. Huang, C.F. Lü, W.Q. Chen, "Benchmark solutions for functionally graded thick plates resting on Winkler–Pasternak elastic foundations", *Composite Structures*, vol. 85, pp. 95-104, 2008.

[17] Ü.H. Çalık Karaköse, "FE Analysis of FGM Plates on Arbitrarily Orthotropic Pasternak Foundations for Membrane Effects", *Teknik Dergi*, vol. 33, no. 2, pp. 11799-11822, 2022.

[18] C.V.G. Vallabhan, W.T. Straughan, Y.C. Das, "Refined model for analysis of plates on elastic foundation", *Journal of Engineering Mechanics,* vol. 117, no.12, pp. 2830-2844, 1991.