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# On Polynomial Space Curves with Flc-frame

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Abstract. The first and second derivatives of a curve provide us fundamental information in the study of the behavior of curve near a point. However, if a curve is a polynomial space curve of degree $n$, we don't know much about the geometric meaning of the $n$-th derivative of the curve. There is no doubt that the Frenet frame is not suitable for this purpose because it is constructed by using first and second derivatives of a curve. On the other hand, in this paper by using a new frame called as Flc-frame we are able to give the geometric meaning of the $n$-th derivative of a curve. Moreover, we explore some basic concepts regarding polynomial space curves from point of view of Flc-frame in three dimensional Euclidean space.

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**Keywords:** Frenet frame, space curve, adapted frame, polynomial curve.

## 1. Introduction

As far as our knowledge the geometrical significance of the $n$-th order derivative of a curve does not seem to be discussed in literature. However, there is some paper which deals with geometrical interpretation of higher order derivatives of a curve. For instance, the geometrical significance of the third derivative of a curve is discussed in [11]. The third derivative is represented geometrically in terms of the quantity called aberrancy, which measures the asymmetry of a curve about its normal [12]. But, one can ask, what exactly prevented us from accomplishing geometrical significance of the $n$-th order derivative of a curve. The major difficulty arises from the fact that we can not write the $n$-th order derivative of a curve in term of Frenet frame. It is clear that the Frenet frame is not suitable for the investigation of the geometric interpretation of higher order derivatives of a curve. On the other hand by using the Flc-frame which is constructed by using the higher order derivatives of the curve, we are able to give geometric interpretation of $n$-th order derivatives of a curve.

Bishop [2] showed that apart from the Frenet frame, we can construct more frame along a space curve. His approach is based on rotating the Frenet frame by an angle [7, 14]

$$\theta = -\int \tau \left\|\alpha'(t)\right\| dt.$$

Despite the fact that Bishop frame is more suitable for applications [8], this frame is not an analitic frame [4]. Recently, Dede introduced a new frame along a polynomial space curve, called as Flc-frame. The computation of Flc-frame is easier than the both Frenet and Bishop frames. Moreover, the Flc-frame has less singular points than the Frenet frame. Therefore, the Flc-frame can be considered as an effective alternative to the RMF. Discussion of the Flc-frame and its

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application to the tube surfaces can be found in [3]. Morever, for some of the recent researchs about the Flc-frame, see [1, 9, 13].

Let $\alpha(t)$ be a polynomial space curve of degree $n$. The Flc-frame is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{D}_1 = \frac{\alpha' \wedge \alpha^{(n)}}{\left\|\alpha' \wedge \alpha^{(n)}\right\|}, \mathbf{D}_2 = \mathbf{D}_1 \wedge \mathbf{t}, \tag{1.1}$$

where the prime $'$ indicates the differentiation with respect to $t$ [3]. If the order of derivative exceeds three, we replaced prime by the superscript $(n)$, such as $\alpha^{''''} = \alpha^{(4)}$. The new vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ are called as binormal-like vector and normal-like vector, respectively.

The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{D}_2' \\ \mathbf{D}_1' \end{bmatrix} = v \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{D}_2 \\ \mathbf{D}_1 \end{bmatrix}, \tag{1.2}$$

where $\|\alpha'\| = v$.

We may define three new invariants of the curve by

$$d_1 = \frac{\langle \mathbf{t}', \mathbf{D}_2 \rangle}{v}, d_2 = \frac{\langle \mathbf{t}', \mathbf{D}_1 \rangle}{v}, d_3 = \frac{\left\langle \mathbf{D}_2', \mathbf{D}_1 \right\rangle}{v}. \tag{1.3}$$

**Theorem 1.1** ( [3]). *A polynomial space curve is a straight line if and only if the all of the curvatures vanish identically,* $d_1 = d_2 = d_3 = 0$ .

**Theorem 1.2** ( [3]). *A polynomial space curve with the curvature $d_1 \neq 0$ is planar if and only if the curvatures $d_2$ and $d_3$ vanish identically, $d_2 = d_3 = 0$.*

The Darboux vector of a frame also known as angular velocity is a crucial information to understand the behaviour of the frame. The Darboux vector $\mathbf{d}_F = \tau\mathbf{t} + \kappa\mathbf{b}$ of Frenet frame describes the instantaneous rate of change of each of the vectors of Frenet frame at a given instant [5]. Therefore, the instantaneous angular speed satisfies $\|\mathbf{d}_F\| = \sqrt{\tau^2 + \kappa^2}$. The RMF is characterized by the fact that the Darboux vector $\mathbf{d}_{RMF}$ of RMF satisfies $\langle \mathbf{d}_{RMF}, \mathbf{t} \rangle = 0$, that is, the normal-plane vectors have no instantaneous rotation around the tangent vector [10].

## 2. Flc-frame Along a Polynomial Space Curve

In this chapter, we begin an investigation into the local theory of space curves by using the Flc-frame. Then, we obtain new formulas for calculating the three curvatures $d_1, d_2$ and $d_3$ of the curve.

**Theorem 2.1.** *Let $\alpha(t)$ be a polynomial space curve of degree $n$. The curvatures $d_1, d_2$ and $d_3$ of the curve can be computed as*

$$d_1 = \frac{\left\langle \alpha' \wedge \alpha'', \alpha' \wedge \alpha^{(n)} \right\rangle}{\|\alpha'\|^3 \left\|\alpha' \wedge \alpha^{(n)}\right\|}, d_2 = \frac{\det[\alpha'', \alpha', \alpha^{(n)}]}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|} \tag{2.1}$$

*and*

$$d_3 = \frac{\det[\alpha', \alpha'', \alpha^{(n)}] \left\langle \alpha', \alpha^{(n)} \right\rangle}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|^2}. \tag{2.2}$$

*Proof.* In order to find the curvature $d_1$, we first differentiate the unit tangent vector $\mathbf{t}$ in (1.1), then by substituting result in (1.3), we get

$$d_1 = \frac{\left\langle \alpha'', (\alpha' \wedge \alpha^{(n)}) \wedge \alpha' \right\rangle}{\|\alpha'\|^3 \left\|\alpha' \wedge \alpha^{(n)}\right\|}.$$

By using the vector triple product $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = -\mathbf{a} \langle \mathbf{b}, \mathbf{c} \rangle + \mathbf{b} \langle \mathbf{a}, \mathbf{c} \rangle$ gives

$$d_1 = \frac{\|\alpha'\|^2 \left\langle \alpha'', \alpha^{(n)} \right\rangle - \langle \alpha'', \alpha' \rangle \left\langle \alpha', \alpha^{(n)} \right\rangle}{\|\alpha'\|^3 \left\|\alpha' \wedge \alpha^{(n)}\right\|}. \tag{2.3}$$

Thus, from the Lagrange's identity, it follows that

$$d_1 = \frac{\left\langle \alpha' \wedge \alpha'', \alpha' \wedge \alpha^{(n)} \right\rangle}{\|\alpha'\|^3 \left\|\alpha' \wedge \alpha^{(n)}\right\|}.$$

Similar to the previous case, by using a direct computation one can obtain the curvature $d_2$ as follows

$$d_2 = \frac{\det[\alpha'', \alpha', \alpha^{(n)}]}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|}.$$

From (1.1) and (1.3), we have

$$d_3 = \frac{\left\langle \mathbf{D}_2', \mathbf{D}_1 \right\rangle}{\|\alpha'\|} = \frac{\left\langle \mathbf{D}_1' \wedge \mathbf{t}, \mathbf{D}_1 \right\rangle}{\|\alpha'\|}.$$

From which the curvature $d_3$ is given by

$$d_3 = \frac{\left\langle \mathbf{D}_1' \wedge \alpha', \alpha' \wedge \alpha^{(n)} \right\rangle}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|},$$

and therefore using Lagrange's identity, we can establish the following formula

$$d_3 = \frac{\left\langle \mathbf{D}_1', \alpha' \right\rangle \left\langle \alpha', \alpha^{(n)} \right\rangle - \left\langle \mathbf{D}_1', \alpha^{(n)} \right\rangle \langle \alpha', \alpha' \rangle}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|}. \tag{2.4}$$

On the other hand, by differentiating $\mathbf{D}_1$ in (1.1), we have

$$\mathbf{D}_1' = \frac{(\alpha'' \wedge \alpha^{(n)} + \alpha' \wedge \alpha^{(n+1)})}{\left\|\alpha' \wedge \alpha^{(n)}\right\|} - \frac{\left\|\alpha' \wedge \alpha^{(n)}\right\|' (\alpha' \wedge \alpha^{(n)})}{\left\|\alpha' \wedge \alpha^{(n)}\right\|^2}. \tag{2.5}$$

Note that the $n+1$-th derivative of the curve vanishes therefore by substituting (2.5) into (2.4), we get

$$d_3 = \frac{\det[\alpha', \alpha'', \alpha^{(n)}] \left\langle \alpha', \alpha^{(n)} \right\rangle}{\|\alpha'\|^2 \left\|\alpha' \wedge \alpha^{(n)}\right\|^2}.$$

□

Thus, we state the following fundamental corollary.

**Corollary 2.2.** *The curvatures $d_1, d_2$ and $d_3$ of the Flc-frame can be computed directly from the parametric curve.*

**Theorem 2.3.** *The new curvatures $d_1, d_2$ and $d_3$ of the curve does not depend on the parameter representation of the curve. Thus, the curvatures form a complete system of differential invariants.*

*Proof.* Let $\beta(t)$ be a space curve of degree $n$ with curvature $d_1$ and let $\alpha(s) = \beta(\phi(t))$ be an another parametrization of the same curve with curvature $\widetilde{d_1}$. Without loss of generality, we can choose $\phi(t) = s^m$. Since the degree of the curve $\alpha$ is $n+m$, we need to calculate the (n+m)-th derivative.

The first three derivatives of $\alpha$ are evaluated as follows:

$$\alpha' = \beta'(\phi)\phi', \tag{2.6}$$

$$\alpha'' = \beta''(\phi)\phi'^2 + \beta'(\phi)\phi'', \tag{2.7}$$

$$\alpha''' = \beta'''(\phi)\phi'^3 + 3\beta''(\phi)\phi'\phi'' + \beta'(\phi)\phi'''.$$

Similarly, n-th derivative of the curve $\alpha$ is obtained by

$$\alpha^{(n)} = \beta^{(n)}(\phi)\phi'^n + \frac{n(n-1)}{2}\beta^{(n-1)}(\phi)\phi'^{n-2}\phi'' \ldots + \beta'(\phi)\phi^{(n)}.$$

Differentiating again $\alpha^{(n)}$, then using $\beta^{(n+1)} = (0,0,0)$, we have

$$\alpha^{(n+1)} = n\beta^{(n)}(\phi)\phi'^{n-1}\phi' + \frac{(n+1)n}{2}\beta^{(n)}(\phi)\phi'^{n-1}\phi'' + \ldots + \beta'(\phi)\phi^{(n+1)}.$$

Similarly differentiating up to order m+n and using $\phi^{(n+m)} = 0$ gives

$$\alpha^{(n+m)} = \delta\beta^{(n)}(\phi), \tag{2.8}$$

where

$$\delta = ([n(n-1)...(n-m+1)]\phi'^{n-m}\phi'^{m} + n\phi'^{n-1}\phi^{(m)} + ...+).$$

Substituting (2.6), (2.7) and (2.8) into (2.1) gives

$$\widetilde{d_1} = \left.\frac{\phi'^4\delta\left\langle \beta'(\phi)\wedge\beta''(\phi), \beta'(\phi)\wedge\beta^{(n)}(\phi)\right\rangle}{\phi'^4\delta\left\|\beta'(\phi)\right\|\left\|\beta'(\phi)\wedge\beta''(\phi), \beta'(\phi)\wedge\beta^{(n)}(\phi)\right\|}\right|_t,$$

which implies that

$$\widetilde{d_1} = \left.\frac{\left\langle \beta'\wedge\beta'', \beta'\wedge\beta^{(n)}\right\rangle}{\left\|\beta'\right\|\left\|\beta'\wedge\beta'', \beta'\wedge\beta^{(n)}\right\|}\right|_{\phi(t)} = d_1\circ\phi(t).$$

The proof of the invariance of the curvatures $d_2$ and $d_3$ are similar. □

For example, let us consider the curve given by

$$\beta(t) = (t, t^2, t^3).$$

The curveture $d_1$ is obtained by

$$d_1 = \frac{12t^3 + 6t}{\sqrt{4t^2+1}(9t^4+4t^2+1)^{3/2}}.$$

For $\phi(t) = s^2$, we have another curve given by

$$\alpha(s) = (s^2, s^4, s^6).$$

Now, $\alpha(s)$ is a curve of degree six. Then, the curveture $\widetilde{d_1}$ is obtained by

$$\widetilde{d_1} = \frac{12s^6 + 6s^2}{\sqrt{4s^4+1}(9s^8+4s^4+1)^{3/2}}.$$

It is easy to see that $\widetilde{d_1} = d_1\circ\phi(t)$.

**Corollary 2.4.** *Let $\alpha(t)$ be a polynomial space curve of degree $n$. Apart from the Frenet frame, there are $(n-2)$ type frames which can be obtained by using this method. For example, we can obtain a new binormal vector by using the first and third derivatives of the curve. But, there is just one frame(Flc-frame) which is invariant under the reparameterization of the curve.*

**Corollary 2.5.** *If the degree of polynomial space curve is two, then the Flc-frame coincides with the Frenet frame with curvatures $d_1 = \kappa, d_2 = 0$ and $d_3 = \tau = 0$.*

**Theorem 2.6.** *Let $\alpha(t)$ be a polynomial space curve of degree $n$. The relation between curvatures $d_1, d_2$ and $d_3$ of the Flc-frame is obtained by*

$$-\frac{(\frac{d_3}{d_2})'}{1+(\frac{d_3}{d_2})^2} = vd_1.$$

*Proof.* Let $\psi$ be an Euclidean angle between the normal-like vector $\mathbf{D}_2$ and the $n$-th derivative of the curve $\alpha^{(n)}$, shown in Figure 1, then we have

$$\left\langle \alpha^{(n)}, \mathbf{D}_2\right\rangle = \left\|\alpha^{(n)}\right\|\cos\psi. \tag{2.9}$$

From (1.1) we see that, the vectors $\alpha^n$ and $\mathbf{D}_1$ are orthogonal and so

$$\left\langle \alpha^{(n)}, \mathbf{t}\right\rangle = \left\|\alpha^{(n)}\right\|\sin\psi, \tag{2.10}$$

which implies that

$$\left\langle \alpha^{(n)}, \alpha'\right\rangle = v\left\|\alpha^{n}\right\|\sin\psi. \tag{2.11}$$

Combining (2.1), (2.2) and (2.10), we have

$$\frac{d_3}{d_2} = -\frac{\left\langle \alpha', \alpha^{(n)}\right\rangle}{\left\|\alpha'\wedge\alpha^{(n)}\right\|} = -\tan\psi. \tag{2.12}$$

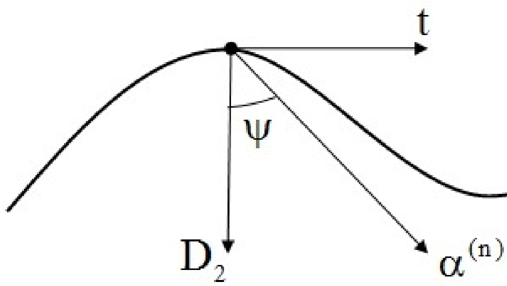



FIGURE 1. The angle between the vectors $\mathbf{D}_2$ and $\alpha^{(n)}$.

It follows that,

$$\psi = \arctan(-\frac{d_3}{d_2}),$$

and differentiating the above equation yields

$$d\psi = -\frac{(\frac{d_3}{d_2})'}{1 + (\frac{d_3}{d_2})^2}. \tag{2.13}$$

On the other hand by using $\alpha' = v\mathbf{t}$ and $\alpha'' = v'\mathbf{t}+v^2(d_1\mathbf{D}_2 + d_2\mathbf{D}_1)$, we have

$$\langle \alpha'', \alpha' \rangle = v'v. \tag{2.14}$$

Differentiating (2.11) yields

$$\left\langle \alpha'', \alpha^{(n)} \right\rangle = v' \|\alpha^n\| \sin\psi + v \|\alpha^n\| \cos\psi d\psi. \tag{2.15}$$

Substituting (2.10), (2.14) and (2.15) into (2.3) gives

$$d_1 = \frac{d\psi}{v}. \tag{2.16}$$

Combining (2.13) and (2.16), we have the desired formula. □

**Theorem 2.7.** *Let $\alpha(t)$ be a polynomial space curve of degree $n$. The $n$-th derivative of curve can be written in term of the basis $\{\mathbf{t}, \mathbf{D}_1, \mathbf{D}_2\}$ in the following form*

$$\alpha^{(n)}(t) = \left\|\alpha^{(n)}\right\| (\sin\psi\mathbf{t} + \cos\psi\mathbf{D}_2),$$

*where $\psi = \int d_1 v dt$.*

*Proof.* By using (2.9) and (2.10) the $n$-th derivative of the curve can be written as

$$\alpha^{(n)}(t) = \|\alpha^n\| (\sin\psi\mathbf{t} + \cos\psi\mathbf{D}_2.) \tag{2.17}$$

By differentiating (2.17) with respect to $t$ gives

$$\mathbf{t}\cos\psi(d\psi - vd_1) + \mathbf{D}_2\sin\psi(vd_1 - d\psi) + \mathbf{D}_1(\sin\psi vd_2 + \cos\psi vd_3) = 0.$$

Since the vectors $\mathbf{t}, \mathbf{D}_2$ and $\mathbf{D}_1$ are linearly independent, the above equation is satisfied if and only if

$$\begin{cases} d\psi - vd_1 &= 0 \\ v\sin\psi d_2 + v\cos\psi d_3 &= 0. \end{cases} \tag{2.18}$$

From (2.12), we have $\sin\psi d_2 + \cos\psi d_3 = 0$, therefore the solution of the equation (2.18) is $d\psi - vd_1 = 0$, that is

$$\psi = \int d_1 v dt.$$

In addition, the tangential $\alpha_T^{(n)}$ and normal $\alpha_N^{(n)}$ components of the $n$-th derivative of the curve $\alpha^{(n)}$ can be written as

$$\alpha^{(n)}(t) = \alpha_T^{(n)}(t)\mathbf{t}+\alpha_N^{(n)}(t)\mathbf{D}_2,$$

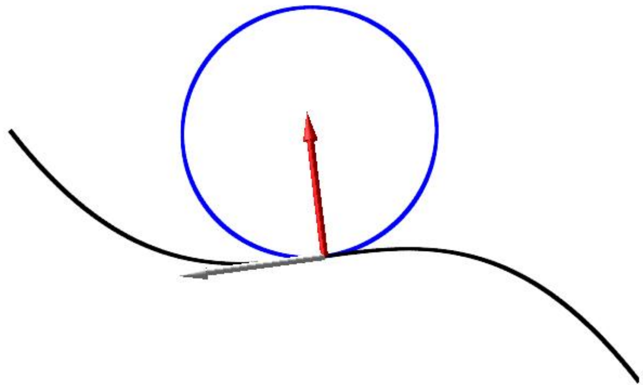


FIGURE 2. The circle (blue) has the same tangent (grey) and normal-like vector (red) with the curve (black).

where

$$\alpha_T^{(n)}(t) = \frac{\left\|\alpha' \wedge \alpha^{(n)}\right\|}{\|\alpha'\|}, \alpha_N^{(n)}(t) = \frac{\left\langle \alpha', \alpha^{(n)}\right\rangle}{\|\alpha'\|}.$$

Note that, a osculating-like plane of a curve $\alpha(t)$ at point $t_0$ is the plane spanned by $\mathbf{t}(t_0)$ and $\mathbf{D}_2(t_0)$ with normal vector $\mathbf{D}_1(t_0)$. The equation of osculating-like plane is given by

$$\langle X - \alpha(t_0), \mathbf{D}_1(t_0)\rangle = 0.$$

It follows that

$$\left[X - \alpha(t_0), \alpha'(t_0), \alpha^{(n)}(t_0)\right] = 0,$$

where $X = (x, y, z)$ is the coordinate system. □

Now we can give the geometric meaning of the $n$-th derivative of a polynomial space curve of degree $n$.

**Theorem 2.8.** *Let $\alpha(t) : I \to R^3$ be a polynomial space curve of degree n. A new circle $\beta(\psi)$ with radius $\|\alpha^n\|$ can be parametrized by*

$$\beta(\psi) = p_0 - \left\|\alpha^{(n)}\right\| (\sin(\psi)\mathbf{t}(t_0) + \cos(\psi)\mathbf{D}_2(t_0)) \tag{2.19}$$

*in the osculating-like plane of the curve at the point $t_0 \in I$ and the center of the circle $p_0$ is given by*

$$p_0 = \alpha(t_0) + \left\|\alpha^{(n)}\right\| \mathbf{D}_2(t_0)$$

*In Figure 2, the circle has the same tangent and normal-like vectors with the curve at the point $t_0$.*

*Proof.* If $\psi = 0$, then we have $\beta(0) = p_0 - \|\alpha^n\| \mathbf{D}_2(t_0) = \alpha(t_0)$ which implies

$$p_0 = \alpha(t_0) + \left\|\alpha^{(n)}\right\| \mathbf{D}_2(t_0).$$

Differentiating (2.19) with respect to $\psi$ gives

$$\beta'(\psi) = -\left\|\alpha^{(n)}\right\| (d\psi \cos(\psi)\mathbf{t}(t_0) + d\psi \sin(\psi)\mathbf{D}_2(t_0)),$$

which implies that the tangent vector $\mathbf{t}_\beta$ of the circle coincides with the tangent vector of the curve as follows

$$\mathbf{t}_\beta(0) = \frac{\beta'(0)}{\|\beta'(0)\|} = \mathbf{t}(t_0),$$

and this concludes that the curve and circle have the common normal-like vector $\mathbf{D}_2(t_0)$. □

Note that, the new circle is the unique circle which passes through $t_0$, has the same tangent in $t_0$ as $\alpha$ as well as the same curvature $d_1$, and whose center lies in the direction of the unit normal-like vector.

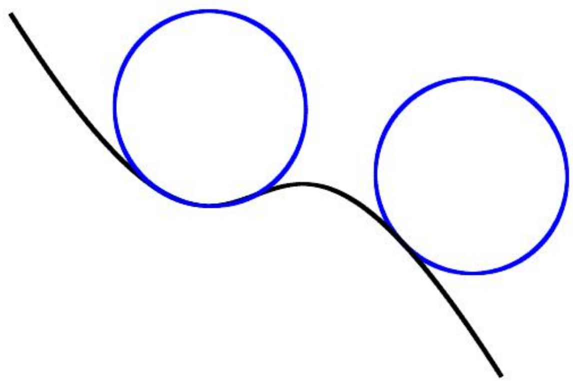
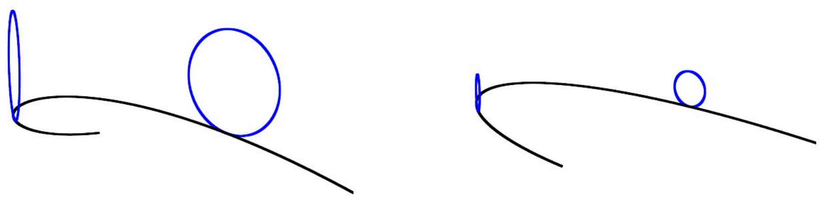


Figure 3. The curve $\alpha(t) = (t, t, t^3/10)$ with two circles at the points $t_0 = -1.3$ and $t_0 = 0.9$



(a) The curve $a(t) = (t, t^2, t^3/10)$ with circles. (b) The curve $a(t) = (t, t^2, t^3/30)$ with circles.

Figure 4. The circles at the points t=-1.3 and t=0.9. The effect of of leading coefficients.

**Corollary 2.9.** *Let $\alpha(t)$ be a polynomial space curve of degree n parametrized by*

$$\alpha(t) = (\sum_{i=0}^{n} a_i t^i, \sum_{i=0}^{n} b_i t^i, \sum_{i=0}^{n} c_i t^i),$$

*where $a_i, b_i$ and $c_i$ are coefficients.*

*The circle $\beta(\psi)$ can be characterized by the following properties:*

- *Since the radius of the circle is $\left\|\alpha^{(n)}\right\| = n!c$ where $c = \sqrt{a_n^2 + b_n^2 + c_n^2}$, the radius of the circle is constant. Figure 3 demonstrated that just the position of the circle changes when the circle moves along the curve.*
- *The radius of the circle is just depend on leading coefficients $a_n, b_n$ or $c_n$ and degree of the curve.*
- *This circle is a global property of polynomial space curves. For the curves have the same degree, if c decreases then the radius of the circle decreases. The Figure 4 demonstrates that if c decrease then the curve approximates the osculating-like plane.*

**Theorem 2.10.** *The Darboux vector $\mathbf{d}_{Flc}$ of the Flc-frame can be obtained as in the following form*

$$\mathbf{d}_{Flc} = v(d_3\mathbf{t} - d_2\mathbf{D}_2 + d_1\mathbf{D}_1).$$

*Proof.* The variation of the Flc-frame in terms of its Darboux vector $\mathbf{d}_{Flc}$ can be written as

$$\mathbf{t}' = \mathbf{d}_{Flc} \wedge \mathbf{t}, \mathbf{D}_2' = \mathbf{d}_F \wedge \mathbf{D}_2, \mathbf{D}_1' = \mathbf{d}_F \wedge \mathbf{D}_1. \tag{2.20}$$

Since $\{\mathbf{t}, \mathbf{D}_2, \mathbf{D}_1\}$ are mutually orthogonal, they form a basis for the vector fields along. Hence, there exist functions $a$,$b$, $c$ such that

$$\mathbf{d}_{Flc} = a\mathbf{t} + b\mathbf{D}_2 + c\mathbf{D}_1. \tag{2.21}$$

Thus, from (1.2), (2.20) and (2.21) we have

$$\mathbf{d}_{Flc} = v(d_3\mathbf{t} - d_2\mathbf{D}_2 + d_1\mathbf{D}_1).$$

□

It is easy to see that, the Darboux vector $\mathbf{d}_{Flc}$ of the Flc-frame does not satisfy $\langle \mathbf{d}_{Flc}, \mathbf{t} \rangle = 0$, therefore it is not a RMF.

Thus, the instantaneous angular speed of the Flc-frame is calculated as follows

$$\|\mathbf{d}_{Flc}\| = v\sqrt{d_1^2 + d_2^2 + d_3^2}.$$

In this section we will compare the angular speed of the frames: Frenet $\|\mathbf{d}_F\| = v\sqrt{\tau^2 + \kappa^2}$, Bishop $\|\mathbf{d}_F\| = v\sqrt{\kappa^2}$ and Flc-frame $\|\mathbf{d}_{Flc}\| = v\sqrt{d_1^2 + d_2^2 + d_3^2}$.

**Example 2.11.** Let us consider a curve given by

$$\alpha(t) = (2t, t^2, t^3).$$

In Figure 5, we are able to compare the instantaneous angular speed of the Flc-frame against two standard methods of curve framing: the RMF and the Frenet frame.

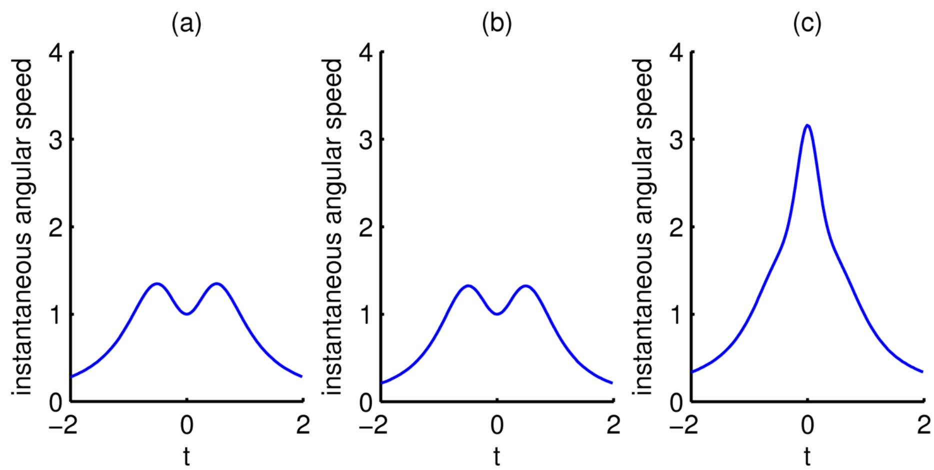



Figure 5. Comparison of the instantaneous angular speed of the Flc-frame (left), RMF (center) and the Frenet frame (right).

Observe that although the Flc-frame is not rotation-minimizing frame with respect to **t**, there is almost no difference between the instantaneous angular speeds of the frames: the RMF and the Flc-frame.

## 3. Conclusion

In this paper we propose a new method to examine the geometric meaning of the $n$-th derivative of a polynomial space curve of degree $n$. Summarizing, the norm of $n$-th derivative of a polynomial space curve of degree $n$ is a radius of a circle in the osculating-like plane of the curve.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

## References

[1] Ayvacı, K.H., Senyurt, S., Canlı, D., *Some characterizations of spherical indicatrix curves generated by Flc frame*, Turk. J. Math. Comput. Sci., **13**(2021), 379–387.
[2] Bishop, R.L., *There is more than one way to frame a curve*, Amer. Math. Monthly, **82**(1975), 246–251.
[3] Dede, M., *A new representation of tubular surfaces*, Houston Journal of Mathematics, **45**(2018), 707–720.
[4] Dede, M., Ekici, C., Görgülü, A., *Directional q-frame along a space curve*, IJARCSSE, **5**(2015), 775–780.
[5] Farouki, R. T., Pythagorean-Hodograph Curves: Algebra and Geometry, Springer, 2008.
[6] Gray, A., Modern Differential Geometry of Curves and Surfaces with Mathematica, Second Edition, CRC Press, Boca Raton, 1998.
[7] Guggenheimer, H., *Computing frames along a trajectory*, Comput. Aided Geom. Des., **6**(1989), 77–78.
[8] Jüttler, B., Mäurer, C., *Cubic Pythagorean Hodograph spline curves and applications to sweep surface modeling*, Comput. Aided Design, **31**(1999), 73–83.
[9] Li, Y., Eren, K., Ayvacı, K.H., Ersoy, S., *Simultaneous characterizations of partner ruled surfaces using Flc frame*, AIMS Math., **7**(2022), 20213–20229.
[10] Ravani, R., Meghdari A., Ravani, B., *Rational Frenet-Serret curves and rotation minimizing frames in spatial motion design*, IEEE international conference on Intelligent engineering systems, (2004), 186–192.
[11] Schot, S.H., *Geometry of the third derivative*, Mathematics Magazine, **51**(1978), 259–275.
[12] Schot, S.H., *Geometrical properties of the penosculating tonics of a plane curve*, Amer. Math. Monthly, **82**(1979), 449–457.
[13] Senyurt, S., Ayvacı, K.H., *On geometry of focal surfaces due to Flc frame in Euclidean 3-space*, Authorea, (2022).
[14] Wang, W., Juttler, B., Zheng, D., Liu, Y., *Computation of rotation minimizing frame*, ACM Trans. Graph., **27**(2008), article no. 2.