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Adaptation of the Kantorovich Type Integral to the Dunkl Operator

Gürhan İÇÖZ[1], Esma GÖKMEN[2]

[1]Gazi University, Faculty of Science, Department of Mathematics
[2]Gazi University, Graduate School of Natural and Applied Sciences

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Abstract

The purpose of this article is to show the adaptation of the Kantorovich type integral to the Dunkl operator. This article gives a sequence of operators to get an approximation result. The variant of the operator which is the Kantorovich type integral has been given and examined the approximation ratio by the first and second order modulus of continuity. The approximation order of the operators is shown by the first order modulus of continuity and the Lipschitz class functions.

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Author ID (ORCID Number)

G. İçöz, 0000-0003-1204-9517
E. Gökmen, 0000-0003-4081-8766

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1. INTRODUCTION

The famous people in the approximation theory are defined the well-known operators and given the most important theorems as Bernstein, 1912; Szász, 1950; Korovkin, 1953; Lorentz, 1953; Gadzhiev, 1974; Altomare & Campiti, 1994; Jakimovski & Leviatan, 1969; DeVore & Lorentz, 1993.

ln the recent years, many results about the generalization of the operators including Gamma function have been obtained by mathematicians (İçöz & Çekim, 2016; İçöz et al., 2016; Sucu et al., 2012; Mursaleen et al., 2016; Kanat & Sofyalıoğlu, 2018; 2019; Nasiruzzaman & Aljohani, 2020a, 2020b; Yazici et al., 2022). Sucu (2014) gave the Dunkl-analogue of Szász-Mirakyan operator as follows

$$E_m(\sigma, y) = \frac{1}{e_\mu(my)} \sum_{j=0}^{\infty} \frac{(my)^j}{\gamma_\mu(j)} \sigma\left(\frac{j + 2\mu\theta_j}{m}\right) \tag{1}$$

where $\mu > 0, m \in \mathbb{N}, y \geq 0$ and $\sigma \in C[0, \infty)$ whenever the aforementioned sum converges. Rosenblum (1994) gave the definition of $e_\mu(y)$ by

$$e_\mu(y) = \sum_{j=0}^{\infty} \frac{y^j}{\gamma_\mu(j)}$$

where the coefficients γ_μ are defined as follows for $k \in \mathbb{N}_0$ and $\mu > -\frac{1}{2}$

*Corresponding Author, e-mail: gurhanicoz@gazi.edu.tr

$$\gamma_\mu(2j) = \frac{2^{2j} j!\, \Gamma(j+\mu+1/2)}{\Gamma(\mu+1/2)}$$

and

$$\gamma_\mu(2j+1) = \frac{2^{2j+1} j!\, \Gamma(j+\mu+3/2)}{\Gamma(\mu+1/2)}$$

where

$$\theta_j = \begin{cases} 0; j \in \{0,2,4,\ldots,2n,\ldots\} \\ 1; j \in \{1,3,\ldots,2n+1,\ldots\} \end{cases}$$

Sucu (2014) gave the following identities for E_m operators.

$$E_m\{1; y\} = 1,$$

$$E_m\{t; y\} = y,$$

$$E_m\{t^2; y\} = y^2 + \left(1 + 2\mu \frac{e_\mu(-my)}{e_\mu(my)}\right)\frac{y}{m}. \tag{2}$$

Now, the Kantorovich-type integral generalization of the operators T_m (İçöz & Çekim, 2016) is defined as

$$T_m(f; y) = \frac{m}{e_\mu(my)} \sum_{j=0}^{\infty} \frac{(my)^j}{\gamma_\mu(j)} \int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} f(t)dt. \tag{3}$$

T_m operators satisfy the following equalities

$$T_m(1, y) = 1,$$

$$T_m(t, y) = y + \frac{1}{2m},$$

$$T_m(t^2, y) = y^2 + 2\left(1 + \mu \frac{e_\mu(-my)}{e_\mu(my)}\right)\frac{y}{m} + \frac{1}{3m^2}. \tag{4}$$

The recursion relation is satisfied by

$$\gamma_\mu(j+1) = (j + 1 + 2\mu\theta_{j+1})\gamma_\mu(j), \; j \in \mathbb{N}_0 \tag{5}$$

The Dunkl derivation τ_μ has the form

$$(\tau_\mu v)(y) = \tau_\mu v(y) = \frac{dv}{dy} + \mu \frac{v(y) - v(-y)}{y}, \tag{6}$$

where μ is real number satisfying $\mu > -1/2$ and v is an entire function. Repeating this process we can have

$$(\tau^2{}_\mu v)(y) = \tau^2{}_\mu v(y) = \frac{d^2v}{dy} + \frac{2\mu}{y}\frac{dv}{dy} + \mu \frac{v(y) - v(-y)}{y} \tag{7}$$

Ben Cheikh and Gaied (2007) defined Dunkl-Appell polinomial set. (p_j) is called Dunkl-Appell polynomial set if and only if for $j \in \mathbb{N}_0$

$$\tau_\mu\, p_{j+1}(y) = \frac{\gamma_\mu(j+1)}{\gamma_\mu(j)} p_j(y). \tag{8}$$

The polynomials p_j can be written as

$$p_j(y) = \sum_{j=0}^{k} \binom{k}{j}_\mu b_{k-j} y^j, (b_0 \neq 0), \tag{9}$$

(p_j) is has the following generation function by

$$P(t)e_\mu(yt) = \sum_{j=0}^{\infty} \frac{p_j(y)}{\gamma_\mu(j)} t^j,$$

where

$$P(t) = \sum_{j=0}^{\infty} \frac{a_j}{\gamma_\mu(j)} t^j, (a_0 \neq 0).$$

In 2020, Sucu (2020) introduced a new generalization of the operator S_m with the Dunkl-Appell polynomials by

$$S_m(\varphi, y) = \frac{1}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \varphi\left(\frac{j + 2\mu\theta_j}{m}\right). \tag{10}$$

Let $\{S_m\}_{m\geq 1}$ be the sequence of operators defined by (10). Then, in (İçöz & Çekim, 2016) there are following assertions:

$$S_m\{1; y\} = 1,$$

$$S_m\{\psi; y\} = y + \frac{\left(e_\mu(my) - e_\mu(-my)\right) P'(1) + e_\mu(-my)\left(\tau_\mu P\right)(1)}{P(1)me_\mu(my)},$$

$$\begin{aligned} S_m(\psi^2; y) = y^2 &+ \frac{e_\mu(my)(2P'(1)P(1) + 2\mu P(-1)e_\mu(-my)}{P(1)me_\mu(my)} y \\ &+ \frac{\left(\tau_\mu P\right)(1)e_\mu(-my)}{P(1)m^2 e_\mu(my)} \\ &+ \frac{\left[2P''(1) - \left(\tau_\mu P\right)'(1) - \left(\tau_\mu P'\right)(1) + P(1) - 2\mu P'(-1)\right]\left(e_\mu(my) - e_\mu(-my)\right)}{P(1)m^2 e_\mu(my)} \\ &+ \frac{\left(\tau_\mu^2 P\right)(1) + 2\mu\left(\tau_\mu P\right)(-1)}{P(1)m^2}. \end{aligned} \tag{11}$$

Now, we construct the Kantorovich type generalization of the operators S_m given by (10). For this purpose, we get $f \in C_{\mathfrak{z}}[0,\infty) = \{f \in C[0,\infty): f(\mathfrak{z}) = O(\lambda^{\mathfrak{z}})\}$ as $\mathfrak{z} \to \infty$. Then, for all $y \in [0,\infty)$, $\mathfrak{z} > m$, $m \in \mathbb{N}, P(1) \neq 0$ and $\mu \geq 0$. Our operators are given by

$$E_m(\varphi; y) = \frac{m}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} \varphi(t)dt. \tag{12}$$

In here, we set up the operators E_m to the other operators in Nasiruzzaman & Aljohani (2020b), similary. But in the paper, we examine the operator E_m with the help of Dunkl derivation. In Nasiruzzaman & Aljohani (2020b), the authors use the classical derivation. Therefore, we have done a wider and more accurate study.

2. APPROXIMATION PROPERTIES OF E_m OPERATORS

Lemma1. E_m operators are linear and positive operators. E_m operators satisfy the following equalties:

$$E_m\{1; y\} = 1, \tag{13}$$

$$E_m\{t; y\} = y + \frac{1}{P(1)me_\mu(my)}\{e_\mu(-my)(\tau_\mu P)(1) \quad + P'(1)[e_\mu(my) - e_\mu(-my)]\} + \frac{1}{2m}, \tag{14}$$

$$\begin{aligned} E_m\{t^2; y\} = y^2 &+ \frac{e_\mu(my)\big(2P'(1)+P(1)\big)+2\mu P(-1)e_\mu(-my)}{mP(1)e_\mu(my)} y + \frac{(\tau_\mu P)(1)e_\mu(-my)}{m^2P(1)e_\mu(my)} \\ &+ \frac{\left[2P''(1) - (\tau_\mu P)'(1) - (\tau_\mu P')(1) + P(1) - 2\mu P'(-1)\right]\left(e_\mu(my) - e_\mu(-my)\right)}{m^2P(1)e_\mu(my)} \\ &+ \frac{(\tau_\mu^2 P)(1) + 2\mu(\tau_\mu P)(-1)}{P(1)m^2} \\ &+ \frac{y}{m} + \frac{1}{m^2P(1)e_\mu(my)}\{e_\mu(-my)(\tau_\mu P)(1) \quad + P'(1)[e_\mu(my) - e_\mu(-my)]\} + \frac{1}{3m^2}. \end{aligned} \tag{15}$$

Proof . For $\varphi(t) = 1$, from the generation function (9), we have

$$\begin{aligned} E_m(1; y) &= \frac{m}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} dt \\ &= \frac{m}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \frac{1}{m} \\ &= \frac{1}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \end{aligned}$$

$$=\frac{e_\mu(my)}{e_\mu(my)}=1.$$

For $\varphi(t)=t$, we obtain

$$E_m(t;y)=\frac{m}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} tdt$$

$$=\frac{m}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left[\frac{t^2}{2}\right]_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}$$

$$=\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\frac{1}{2m}\left[2\left(j+2\mu\theta_j\right)+1\right]$$

$$=\frac{1}{m}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)$$

$$+\frac{1}{2m}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}. \tag{16}$$

If we use the equation

$$\gamma_\mu(j+1)=\left(j+1+2\mu\theta_{j+1}\right)\gamma_\mu(j),$$

we can obtain

$$\gamma_\mu(j)=\left(j+2\mu\theta_j\right)\gamma_\mu(j-1).$$

Let's use the above equation in the operator, then we get

$$E_m(t;y)=\frac{1}{m}\frac{1}{P(1)e_\mu(my)}\sum_{j=1}^{\infty}\frac{p_j(my)}{\left(j+2\mu\theta_j\right)\gamma_\mu(j-1)}\left(j+2\mu\theta_j\right)+\frac{1}{2m}$$

$$=\frac{1}{m}\frac{1}{P(1)e_\mu(my)}\sum_{j=1}^{\infty}\frac{p_j(my)}{\gamma_\mu(j-1)}+\frac{1}{2m}$$

$$=\frac{1}{m}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_{j+1}(my)}{\gamma_\mu(j)}+\frac{1}{2m}.$$

Taking the Dunkl derivative with respect to t of both sides of the equation

$$P(t)e_\mu(yt)=\sum_{j=0}^{\infty}\frac{p_j(y)}{\gamma_\mu(j)}t^j.$$

Then we obtain

$$\sum_{j=0}^{\infty}\frac{p_j(y)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)t^{j-1}$$

$$= yP(t)e_\mu(yt)+e_\mu(-yt)\left(\tau_\mu P\right)(t)+P'(t)\left[e_\mu(yt)-e_\mu(-yt)\right]. \quad (17)$$

In the above expression, if ***y*** is replaced by ***my*** and ***t*** is replaced by ***1*** then we can find

$$myP(1)e_\mu(my)+e_\mu(-my)\left(\tau_\mu P\right)(1)+P'(1)\left[e_\mu(my)-e_\mu(-my)\right]$$

$$=\sum_{j=1}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)$$

$$=\sum_{j=0}^{\infty}\frac{p_{j+1}(my)}{\gamma_\mu(j+1)\,\gamma_\mu(j)}\left(j+1+2\mu\theta_{j+1}\right)$$

$$=\sum_{j=0}^{\infty}\frac{p_{j+1}(my)}{\gamma_\mu(j)}.$$

Using the above equation in $E_m(t;y)$, we see that

$$E_m(t;y)=\frac{1}{m}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_{j+1}(my)}{\gamma_\mu(j)}+\frac{1}{2m}$$

$$=\frac{1}{m}\frac{1}{P(1)e_\mu(my)}\left\{myP(1)e_\mu(my)+e_\mu(-my)\left(\tau_\mu P\right)(1)+P'(1)\left[e_\mu(my)-e_\mu(-my)\right]\right\}$$

$$+\frac{1}{2m}$$

$$=y+\frac{1}{P(1)e_\mu(my)}\left\{e_\mu(-my)\left(\tau_\mu P\right)(1)+P'(1)\left[e_\mu(my)-e_\mu(-my)\right]\right\}+\frac{1}{2m}. \quad (18)$$

For φ(t)=t^2, we obtain

$$E_m(t^2;y)=\frac{m}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} t^2dt.$$

$$=\frac{m}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left[\frac{t^3}{3}\right]_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}$$

$$=\frac{m}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\frac{1}{3m^2}\left[3\left(j+2\mu\theta_j\right)^2+3\left(j+2\mu\theta_j\right)+1\right]$$

$$=\frac{1}{m^2}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)^2$$

$$+\frac{1}{m^2}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)$$

$$+\frac{1}{3m^2}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}. \qquad (19)$$

If we use the equation

$$\gamma_\mu(j+1)=\left(j+1+2\mu\theta_{j+1}\right)\gamma_\mu(j),$$

we obtain

$$\gamma_\mu(j-1)=\left(j-1+2\mu\theta_{j-1}\right)\gamma_\mu(j-2).$$

Taking the Dunkl derivative with respect to t of both sides of the equation

$$P(t)e_\mu(yt)=\sum_{j=0}^{\infty}\frac{p_j(y)}{\gamma_\mu(j)}t^j,$$

we get

$$\sum_{j=1}^{\infty}\frac{p_j(y)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)t^{j-1}$$

$$=\ yP(t)e_\mu(yt)+e_\mu(-yt)\left(\tau_\mu P\right)(t)+P'(t)\left[e_\mu(yt)-e_\mu(-yt)\right]. \qquad (20)$$

Again taking the derivative of both sides of the above equation with respect to $\boldsymbol{t}$, we obtain

$$\sum_{j=2}^{\infty}\frac{p_j(y)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)\left(j-1+2\mu\theta_j\right)t^{j-2}$$

$$=y^2P(t)e_\mu(yt)+ye_\mu(-yt)\left(\tau_\mu P\right)(t)+yP'(t)\left[e_\mu(yt)-e_\mu(-yt)\right]$$

$$-ye_\mu(-yt)\left(\tau_\mu P\right)(t)+e_\mu(yt)\left(\tau_\mu^2 P\right)(t)+\left[e_\mu(-yt)-e_\mu(yt)\right]\left(\tau_\mu P\right)'(t)$$

$$+\left[ye_\mu(yt)+ye_\mu(-yt)\right]P'(t)+\left[e_\mu(-yt)-e_\mu(yt)\right]\left(\tau_\mu P'\right)$$

$$+2\left[e_\mu(yt)-e_\mu(-yt)\right]P''(t).$$

In the above expression, if $\boldsymbol{y}$ is replaced by $\boldsymbol{my}$ and $\boldsymbol{t}$ is replaced by $\mathbf{1}$ we can find

$$\frac{1}{m^2}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)^2$$

$$=y^2+\frac{e_\mu(my)\big(2P'(1)+P(1)\big)+2\mu P(-1)e_\mu(-my)}{mP(1)e_\mu(my)}y+\frac{\left(\tau_\mu P\right)(1)e_\mu(-my)}{m^2P(1)e_\mu(my)}$$

$$+\frac{\left[2P''(1)-\left(\tau_\mu P\right)'(1)-\left(\tau_\mu P'\right)(1)+P(1)-2\mu P'(-1)\right]\left(e_\mu(my)-e_\mu(-my)\right)}{m^2P(1)e_\mu(my)}$$

$$+\frac{\left(\tau_\mu^2 P\right)(1)+2\mu\left(\tau_\mu P\right)(-1)}{P(1)m^2}$$

For

$$\frac{1}{m^2}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left(j+2\mu\theta_j\right)$$

if we use the equation (20), we have

$$\frac{y}{m}+\frac{1}{m^2P(1)e_\mu(my)}\left\{e_\mu(-my)\left(\tau_\mu P\right)(1)+P'(1)\left[e_\mu(my)-e_\mu(-my)\right]\right\}.$$

Lastly, we get that

$$\frac{1}{3m^2}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}=\frac{1}{3m^2}\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}P(1)e_\mu(my)$$

$$=\frac{1}{3m^2}.$$

Here is the following result:

$$E_m(t^2;y)=y^2+\frac{e_\mu(my)\big(2P'(1)+P(1)\big)+2\mu P(-1)e_\mu(-my)}{mP(1)e_\mu(my)}y+\frac{\left(\tau_\mu P\right)(1)e_\mu(-my)}{m^2P(1)e_\mu(my)}$$

$$+\frac{\left[2P''(1)-\left(\tau_\mu P\right)'(1)-\left(\tau_\mu P'\right)(1)+P(1)-2\mu P'(-1)\right]\left(e_\mu(my)-e_\mu(-my)\right)}{m^2P(1)e_\mu(my)}$$

$$+\frac{\left(\tau_\mu^2 P\right)(1)+2\mu\left(\tau_\mu P\right)(-1)}{P(1)m^2}$$

$$+\frac{y}{m}+\frac{1}{m^2P(1)e_\mu(my)}\{e_\mu(-my)\big(\tau_\mu P\big)(1)+P'(1)\big[e_\mu(my)-e_\mu(-my)\big]\}$$

$$+\frac{1}{3m^2}.$$

So, the proof of the lemma is completed.

Let us denote, the first and second moments of operator E_m as follows $\nabla_1 = E_m(\Im - y; y)$ and

$\nabla_2 = E_m((\Im - y)^2; y)$.

Lemma2: For E_m operators, the following relations are hold:

$$\nabla_1 = \frac{1}{mP(1)e_\mu(my)}\{e_\mu(-my)\big(\tau_\mu P\big)(1)+P'(1)\big[e_\mu(my)-e_\mu(-my)\big]\}+\frac{1}{2m} \tag{21}$$

$$\nabla_2 = \frac{e_\mu(my)\big(2P'(1)+P(1)\big)+2\mu P(-1)e_\mu(-my)}{mP(1)e_\mu(my)}y+\frac{\big(\tau_\mu P\big)(1)e_\mu(-my)}{m^2P(1)e_\mu(my)}$$

$$+\frac{\left[2P''(1)-\big(\tau_\mu P\big)'(1)-\big(\tau_\mu P'\big)(1)+P(1)-2\mu P'(-1)\right]\Big(e_\mu(my)-e_\mu(-my)\Big)}{m^2P(1)e_\mu(my)}$$

$$+\frac{\big(\tau_\mu^2 P\big)(1)+2\mu\big(\tau_\mu P\big)(-1)}{P(1)m^2}-\frac{2y}{mP(1)e_\mu(my)}\{e_\mu(-my)\big(\tau_\mu P\big)(1)+P'(1)\big[e_\mu(my)-e_\mu(-my)\big]\}$$

$$+\frac{1}{m^2P(1)e_\mu(my)}\{e_\mu(-my)\big(\tau_\mu P\big)(1)+P'(1)\big[e_\mu(my)-e_\mu(-my)\big]\}+\frac{1}{3m^2}. \tag{22}$$

3. MAIN RESULTS

In this section, the rate of convergence is given with the help of the classical modulus continuity and Lipschitz class functions.

Lipschitz class of order α, $\mathrm{Lip}_m(\alpha)$ $(0<\alpha\le 1, M>0)$ is defined as follows:

$$\mathrm{Lip}_M(\alpha)=\{f: |f(\varrho)-f(\iota)|\le M|\varrho-\iota|^\alpha, \varrho,\iota\in[0,\infty)\}.$$

Theorem 1. Let $\varpi\in\mathrm{Lip}_M(\beta)$. Then we have

$$|E_m(\varpi;y)-\varpi(y)|\le M.\big(\nabla_2(y)\big)^{\beta/2},$$

Where ∇_2 is given as (22).

Proof. From the monotonity property of E_m and Lemma 1, we obtain

$$|E_m(\varpi;y)-\varpi(y)| = |E_m(\varpi(\Im))-\varpi(y);y|$$

$$\le E_m(|(\varpi(\Im))-\varpi(y)|;y)$$

$$\le M.E_m\big(|\Im-y|^\beta;y\big).$$

From the Hölder inequality the following expression is deduced

$$|E_m(\varpi; y) - \varpi(y)| \leq \frac{m}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} |\varpi(t) - \varpi(y)| dt$$

$$\leq M.\frac{1}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} m^{\frac{\beta}{2}} |t-y|^\beta m^{\frac{2-\beta}{2}} dt$$

$$\leq M.\frac{1}{P(1)e_\mu(my)} \sum_{j=0}^{\infty} \frac{p_j(my)}{\gamma_\mu(j)} \left(\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} m(t-y)^2 dt \right)^{\frac{\beta}{2}}$$

$$\times \left(\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} m dt \right)^{\frac{2-\beta}{2}}$$

$$\leq M \sum_{j=0}^{\infty} \left(\frac{p_j(my)}{P(1)e_\mu(my)\gamma_\mu(j)} \int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} m \left(\frac{j+2\mu\theta_j}{m} - y \right)^2 dt \right)^{\frac{\beta}{2}}$$

$$\times \left(\frac{p_j(my)}{P(1)e_\mu(my)\gamma_\mu(j)} \right)^{\frac{2-\beta}{2}}$$

$$\leq M. \left(\sum_{j=0}^{\infty} \frac{p_j(my)}{P(1)e_\mu(my)\gamma_\mu(j)} \int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}} m \left(\frac{j+2\mu\theta_j}{m} - y \right)^2 dt \right)^{\frac{\beta}{2}}$$

$$\times \left(\sum_{j=0}^{\infty} \frac{p_j(my)}{P(1)e_\mu(my)\gamma_\mu(j)} \right)^{\frac{2-\beta}{2}}$$

$$= M.(\nabla_2)^{\frac{\beta}{2}}.$$

If ∇_2 is selected as (23), the desired result is obtained.

The modulus of continuity of function $\varphi \in C[0,\infty)$ is defined (DeVore & Lorentz, 1993) as follows

$$\omega(\varphi;\delta)=\sup_{y,h\in[0,\infty)}\{|\varphi(y)-\varphi(h)|,|y-h|\leq\delta\}.$$

Theorem 2. Let f be a uniformly continuos function on $[0,\infty)$. Then the following holds;

$$|E_m(\varphi;y)-\varphi(y)|\leq\left(1+\partial_m(y)\right)\omega\left(\varphi;\frac{1}{\sqrt{m}}\right),$$

where $\partial_m(y)=\sqrt{(\mathrm{m}\nabla_2)}$.

Proof. According to Lemma 1, it follows that

$$|E_m(\varphi;y)-\varphi(y)|\leq\frac{m}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}|\varphi(t)-\varphi(y)|dt$$

$$\leq\left\{1+\frac{1}{\delta}\frac{m}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}|(t)-y|dt\right\}\omega(\varphi;\delta).$$

In the right hand side of above inequality, if we use Schwarz inequality we get

$$\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}\sqrt{m}|t-y|\sqrt{m}dt$$

$$\leq\quad\frac{1}{P(1)e_\mu(my)}\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)}\left(\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}m\,(t-y)^2dt\right)^{\frac{1}{2}}$$

$$\times\left(\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}mdt\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{j=0}^{\infty}\frac{p_j(my)}{P(1)e_\mu(my)\gamma_\mu(j)}\int_{\frac{j+2\mu\theta_j}{m}}^{\frac{j+1+2\mu\theta_j}{m}}m\,(t-y)^2dt\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{j=0}^{\infty}\frac{p_j(my)}{\gamma_\mu(j)P(1)e_\mu(my)}\right)^{\frac{1}{2}}$$

$$= \sqrt{(\nabla_2)}.$$

From the above inequality and $\delta = \frac{1}{\sqrt{m}}$, we get

$$|E_m(\varphi; y) - \varphi(y)| \leq \left\{1 + \frac{1}{\delta}\sqrt{(\nabla_2)}\right\}\omega(f;\delta)$$

$$\leq (1 + \partial_m(y))\omega\left(\varphi; \frac{1}{\sqrt{m}}\right).$$

So, the proof is completed.

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