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Numerical Solution of High-Order Linear Fredholm Integro-Differential Equations by Lucas Collocation Method

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Abstract. In this paper, a useful matrix approach for high-order linear Fredholm integro-differential equations with initial boundary conditions expressed as Lucas polynomials is proposed. Using a matrix equation which is equivalent to a set of linear algebraic equations the method transforms to integro-differential equation. When compared to other methods that have been proposed in the literature, the numerical results from the suggested technique reveal that it is effective and promising. And also, error estimation of the scheme was derived. These results were compared with the exact solutions and the other numerical methods to the tested problems.

Keywords: Fredholm integro-differential equations · Initial value problem · Lucas matrix method · Lucas polynomials and series

1 Introduction

Integro-differential equations (IDEs), which are a combination of differential and Fredholm-Volterra integral equations are of interest to researchers. This arises frequently in many applied areas, which include biology, astronomy, engineering, chemistry, physics, mechanics, economics, etc. [1–5]. Several numerical methods are used to solve the mentioned integro-differential equations. Such as, Adomian decomposition, Taylor and Euler collocation, Bessel, Legendre polynomial methods, etc. [6–13], were used. A matrix-collocation approach for fractional partial IDEs has been presented by Aslan et al. [14]. The approximate solutions of functional IDEs with variable delay relying on Lucas polynomials have been provided by GÃijmgÃijm et al. [15]. Also, many authors have research for numerical solutions of the partial IDEs [16–18]
The aforementioned techniques are updated and developed for solving the mth order linear FIDE and FIDE with piecewise intervals in this article using the matrix relationships between the Lucas polynomials and their derivatives. The equation that we are going to investigate is

$$\sum_{k=0}^{m} P_k(t)y^{(k)}(t) = g\left(t\right) + \int_{a_f}^{b_f} K_f\left(t,s\right)y\left(s\right)ds \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{m-1}\left(a_{sk}y^{(k)}\left(a\right) + b_{sk}y^{(k)}\left(b\right) + c_{sk}y^{(k)}\left(c\right)\right) = \lambda_s, \quad s = 0,1,...,m-1 \tag{2}$$

where $P_k\left(t\right)$ and $g\left(t\right)$ are functions defined on the interval $a \leq t \leq b$; a_{sk}, b_{sk}, c_{sk} and λ_s are appropriate constants; $y(t)$ is an unknown solution function to be determined.
For our purpose, we assume the approximate solution of the problem Eq.(1) âĂŞ Eq.(2) in the truncated Lucas polynomials form

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n \,\mathrm{L}_n(t), \qquad -1 \leqslant t \leqslant 1 \tag{3}$$

where $a_n, n = 0,1,2,...,N$ are unknown coefficients to be determined and $L_n(t)$ indicates the Lucas polynomials which are originally studied in 1970 by Bicknell. Lucas polynomials are defined recursivelly as follows [19–21].

$$L_{n+1}\left(t\right) = tL_n\left(t\right) + L_{n-1}\left(t\right), \quad n \geqslant 1, \quad L_0\left(t\right) = 2, L_1\left(t\right) = t. \tag{4}$$

Their explicit form for $n \geqslant 1$ is

$$L_n(t) = \sum_{n=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k}t^{n-2k} \tag{5}$$

where x is the largest integer smaller than or equal to x.
By using Eq.(4) and Eq.(5) the first Lucas polynomials respectively are given by

$$L_0(t) = 2, \quad L_1(t) = t, \quad L_2(t) = t^2 + 2, \quad L_3(t) = t^3 + 3t,$$

$$L_4(t) = t^4 + 4t^2 + 2, \quad L_5(t) = t^5 + 5t^3 + 5t, \quad L_6(t) = t^6 + 6t^4 + 9t^2 + 2,$$

2 Materials and Methods

2.1 Matrix Relations

The following process is used in this section to convert the expressions defined in Eq.(1) and Eq.(2) into matrix forms: First, the derivatives of the function $y(t)$ defined by Eq.(3) can be expressed in matrix form.

$$y\left(t\right) \cong y_N\left(t\right) = \mathbf{L}\left(t\right)\mathbf{A} \quad \mathbf{L}\left(t\right) = \mathbf{T}\left(t\right)\mathbf{D}^T \tag{6}$$

where

$$\mathbf{L}\left(t\right) = \left[L_0\left(t\right)\ L_1\left(t\right)\ \cdots\ L_N\left(t\right)\right], \quad \mathbf{A} = \left[a_0\ a_1\ \cdots\ a_N\right]^T$$
$$\mathbf{T}\left(t\right) = \left[1\ t\ t^2\ \cdots\ t^N\right]$$

. If N is odd,

$$\mathbf{D} = \left[\begin{array}{cccccccc}
2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & & \\
\frac{N-1}{\left(\frac{N-1}{2}\right)}\binom{\frac{N-1}{2}}{\frac{N-1}{2}} & 0 & \frac{N-1}{\left(\frac{N+1}{2}\right)}\binom{\frac{N+1}{2}}{\frac{N-3}{2}} & 0 & \cdots & \cdots & \frac{N-1}{\left(\frac{2N-2}{2}\right)}\binom{\frac{2N-1}{2}}{0} & 0 \\
0 & \frac{N}{\left(\frac{N+1}{2}\right)}\binom{\frac{N+1}{2}}{\frac{N-1}{2}} & 0 & \frac{N}{\left(\frac{N+3}{2}\right)}\binom{\frac{N+3}{2}}{\frac{N-3}{2}} & \cdots & \cdots & 0 & \frac{N}{N}\binom{N}{0}
\end{array}\right]$$

and if N is even,

$$\mathbf{D} = \left[\begin{array}{cccccccc}
2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & & \\
0 & \frac{N-1}{\left(\frac{N}{2}\right)}\binom{\frac{N}{2}}{\frac{N-2}{2}} & 0 & \frac{N-1}{\left(\frac{N+2}{2}\right)}\binom{\frac{N+2}{2}}{\frac{N-4}{2}} & \cdots & \cdots & 0 & \frac{N-1}{N-1}\binom{N-1}{0} \\
\frac{N}{\left(\frac{N}{2}\right)}\binom{\frac{N}{2}}{\frac{N}{2}} & 0 & \frac{N}{\left(\frac{N+2}{2}\right)}\binom{\frac{N+2}{2}}{\frac{N-2}{2}} & 0 & \cdots & \cdots & \frac{N}{\left(\frac{2N}{2}\right)}\binom{\frac{2N}{2}}{0} & 0
\end{array}\right]$$

From the matrix relations Eq. (6), it follows that

$$y_N\left(t\right) = \mathbf{T}\left(t\right)\mathbf{D}^T\mathbf{A}, \tag{7}$$

Besides, the relation between the matrix $\mathbf{T}(t)$ and its derivatives are

$$\mathbf{T}^{(k)}(t) = \mathbf{T}(t)\,\mathbf{B}^k$$

where

$$\mathbf{B} = \begin{bmatrix} 0\,1\,0 \cdots & 0 \\ 0\,0\,2 \cdots & 0 \\ \vdots\,\vdots\,\vdots \ddots & \vdots \\ 0\,0\,0\;\;0 & N \\ 0\,0\,0\;\;0 & 0 \end{bmatrix} \quad \mathbf{B}^0 = \begin{bmatrix} 1\,0\,0 \cdots 0 \\ 0\,1\,0 \cdots 0 \\ \vdots\,\vdots\,\vdots \ddots \vdots \\ 0\,0\,0 \cdots 0 \\ 0\,0\,0 \cdots 1 \end{bmatrix}$$

By using Eq.(6)-(7), we have the matrix relation

$$y_N^{(k)}(t) = \mathbf{T}(t)\,\mathbf{B}^k\mathbf{D}^T\mathbf{A},\;\; k = 0, 1, 2, \ldots \tag{8}$$

Additionally, the kernel function $K_f(t,s)$ in Eq.(1) is constructed in matrix form as follows

$$K_f(t,s) = \mathbf{T}(t)\mathbf{K}_f\mathbf{T}(s)^T \tag{9}$$

where $\mathbf{K}_f = \mathrm{K} = [k_{mn}]$, $\quad m, n = 0, 1, ..., N$

$$k_{mn} = \frac{1}{m!n!}.\frac{\partial^{m+n}\mathrm{K}(0,0)}{\partial t^m \partial s^n}$$

$$\int_a^b \mathrm{K}_f(t,s)\,y(s)\,ds = \mathbf{T}(t)\mathbf{K}_f\mathbf{Q}_f\mathbf{D}^T\mathbf{A} \tag{10}$$

where

$$\mathbf{Q}_f = \left[q^f{}_{mn}\right] = \int_a^b \mathbf{T}^T(s)\mathbf{T}(s)ds,$$

$$\left. q^f{}_{mn} = \frac{b^{m+n+1} - a^{m+n+1}}{m+n+1} \right\} \quad m, n = 0, 1, \ldots, N$$

To obtain the Lucas polynomial solution of Eq.(1) in the form Eq.(3) we firstly compute the Lucas coefficients by means of the collocation points defined by

$$t_i = a + \frac{b-a}{N}i, \quad i = 0, 1, \ldots, N.$$

The following steps are taken to obtain the matrix equation system:

$$\sum_{k=0}^{m} P_k(t_i)\,y^{(k)}(t_i) = g(t_i) + \int_a^b K_f(t_i, s_i)\,y(s_i)\,ds \tag{11}$$

It is constructed the fundamental matrix equation corresponding to the FIDEs, by substituting the matrix relations Eq.(8)-(10) into Eq.(1):

$$\sum_{k=0}^{m} P_k\left(t_i\right)\mathbf{T}\left(t_i\right)\mathbf{B}^k\mathbf{D}^T\mathbf{A} = g\left(t_i\right) + \mathbf{T}(t_i)\mathbf{K}_f\mathbf{Q}_f\mathbf{D}^T\mathbf{A} \tag{12}$$

or briefly,

$$\sum_{k=0}^{m}\mathbf{P}_k\mathbf{T}\mathbf{B}^k\mathbf{D}^T\mathbf{A} - \mathbf{T}\mathbf{K}_f\mathbf{Q}_f\mathbf{D}^T\mathbf{A} = \mathbf{G} \tag{13}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k\left(t_0\right) & 0 & \cdots & 0 \\ 0 & P_k\left(t_1\right) & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & P_k\left(t_N\right) \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}\left(t_0\right) \\ \mathbf{T}\left(t_1\right) \\ \vdots \\ \mathbf{T}\left(t_N\right) \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} g\left(t_0\right) \\ g\left(t_1\right) \\ \vdots \\ g\left(t_N\right) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix},$$

Besides, the fundamental matrix equation Eq.(13) can be expressed in the form

$$\mathbf{W}\mathbf{A} = \mathbf{G} \quad \Leftrightarrow \quad [\mathbf{W} : \mathbf{G}] \tag{14}$$

where

$$\mathbf{W} = \sum_{k=0}^{m}\mathbf{P}_k\mathbf{T}\mathbf{B}^k\mathbf{D}^T - \mathbf{T}\mathbf{K}_f\mathbf{Q}_f\mathbf{D}^T = \left[w_{mn}\right]; \quad m, n = 0, 1, \ldots N.$$

Now we can obtain the corresponding matrix form for the initial conditions Eq.(2), by means of the relation Eq.(8),

$$\mathbf{U}_s\mathbf{A} = \lambda_s \quad \Leftrightarrow \quad \left[\mathbf{U}_s : \lambda_s\right]; \quad s = 0, 1, \ldots m-1. \tag{15}$$

such that

$$\mathbf{U}_s = \sum_{k=0}^{m-1}\left(a_{sk}\mathbf{T}\left(a\right) + b_{sk}\mathbf{T}\left(b\right) + c_{sk}\mathbf{T}\left(c\right)\right)\mathbf{B}^k\mathbf{D}^T\mathbf{A} = \begin{bmatrix} u_{s0} & u_{s1} & \cdots & u_{sN} \end{bmatrix} \tag{16}$$

After substituting any m rows of the augmented matrix (14) with the m row matrices (16), we finally get the new matrix as the answer to the problems (1)-(2).

$$\widetilde{\mathbf{W}}\,\mathbf{A} = \widetilde{\mathbf{G}} \Rightarrow \left[\widetilde{\mathbf{W}} : \widetilde{\mathbf{G}}\right] \tag{17}$$

In Eq.(17), if $rank\widetilde{\mathbf{W}} = rank\left[\widetilde{\mathbf{W}} : \widetilde{\mathbf{G}}\right] = N+1$, then the coefficient matrix $\mathbf{A}$ is uniquely determined and the solution of the problem Eq.(1)-(2) is obtained as

$$y_N(t) = \mathbf{L}(t)\mathbf{A} = \mathbf{T}(t)\mathbf{D}^T\mathbf{A}$$

3 Residual Error Analysis

By employing the residual correction method, we build an error estimation strategy for the Lucas polynomial approximations of the problem Eq.(1)-(2), and we then use this technique to improve the approximation.
To begin with, the residual function of the method is

$$R_N(t) = L[y_N(t)] - g(t) \tag{18}$$

where $L[y_N(t)] \cong g(t)$ and $y_N(t)$ is the Lucas polynomial solution Eq.(3) of the problems Eq.(1)-(2). For $t = t_l \in [-1, 1], \quad l = 0, 1, 2, \ldots; \quad R_N(t_l) \leq 10^{-k_l}$ (k_l is any positive integer). When the difference $R_N(t_l)$ at each point is lower than the recommended 10^{-k_l}, the truncation limit N is increased.
Further, the error function $e_N(t)$ can be determined as

$$e_N(t) = y(t) - y_N(t) \tag{19}$$

where $y(t)$ is the exact solution of the problem Eq.(1)-(2). From Eqs.(1), (2), (18) and (19), we obtain the system of the error differential equations

$$L[e_N(t)] = L[y(t)] - L[y_N(t)] = -R_N(t) \tag{20}$$

and the error problem

$$\begin{aligned} &\sum_{k=0}^{m} P_k(t) e_N^{(k)}(t) - \int_{a_f}^{b_f} K_f(t, s)\, e_N(s)\, ds = -R_N(t) \\ &e_{jN}^{(k)}(a) = 0, \quad j = 1, 2, \ldots J, \quad k = 0, 1, \ldots m-1 \end{aligned} \tag{21}$$

The error problem Eq.(21) can be settled by using the presented method in Section 2. So, we obtain the approximation $e_{N,M}(t)$ to $e_N(t)$ as follows:

$$e_{N,M}(t) = \sum_{n=0}^{M} a_N^* L_N(t), \quad M > N, \quad j = 1, 2, \ldots J. \tag{22}$$

As a result, using the polynomials $y_N(t)$ and $e_{N,M}(t)$, the corrected Lucas polynomial solution $y_{N,M}(t) = y_N(t) + e_{N,M}(t)$ is achieved. Additionally, the error function $e_N(t) = y(t) - y_N(t)$, the estimated error function $e_{N,M}(t)$ and the corrected error function $E_{N,M}(t) = e_N(t) - e_{N,M}(t) = y(t) - y_{N,M}(t)$ constructed [21–24].

4 Numerical Illustrations

In order to demonstrate the correctness and efficiency of the procedure, some numerical examples of the problem Eq. (1) are provided in this section.

Example 4.1. [25] Let us first consider the third-order linear FIDE

$$y'''(t) - y(t) = -3t^2 - 5t + 2 + 5\int_{-1}^{1}(ts + t^2s^2)y(s)\,ds + 12\int_{-1}^{0}(t+s)y(s)\,ds$$
$$+4\int_{0}^{1}(ts)y(s)\,ds + 12\int_{\frac{-1}{2}}^{\frac{1}{2}}y(s)\,ds$$

$-1 \le t, s \le 1$ with the initial conditions $y(0) = y'(0) = 0, \quad y''(0) = 2.$

We approximate the solution $y(t)$ by the polinomial

$$y(t) = y_N(t) = \sum_{n=0}^{3} a_n L_n(t), \quad -1 \le t \le 1$$

$P_3(t) = 1, \quad P_2(t) = P_1(t) = 0, \quad P_0(t) = -1, \quad g(t) = -3t^2 - 5t + 2$
$K_1(t,s) = 5(ts + t^2s^2), \quad K_2(t,s) = 12(t+s), \quad K_3(t,s) = 4ts, \quad K_4(t,s) = 12$
and the collocation points for $a = -1, \quad b = 1$ and $N = 3$ are computed as

$$\left\{t_0 = -1, \quad t_1 = -\frac{1}{3}, \quad t_2 = \frac{1}{3} \quad t_3 = 1\right\}.$$

Following the procedure in Section2, the fundamental matrix equation of the given equation becomes

$$\sum_{k=0}^{3}\mathbf{P}_k\mathbf{T}\mathbf{B}^k\mathbf{D}^T\mathbf{A} - \mathbf{T}\mathbf{K}_f\mathbf{Q}_f\mathbf{D}^T\mathbf{A} = \mathbf{G} \quad 1 \le f \le 4.$$

where

$$\mathbf{P}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{3} & \frac{1}{9} & -\frac{1}{27} \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}^T = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{K}_1=\begin{bmatrix}0\,0\,0\,0\\0\,5\,0\,0\\0\,0\,5\,0\\0\,0\,0\,0\end{bmatrix},\quad \mathbf{Q}_1=\begin{bmatrix}2&0&\frac{2}{3}&0\\0&\frac{2}{3}&0&\frac{2}{5}\\\frac{2}{3}&0&\frac{2}{5}&0\\0&\frac{2}{5}&0&\frac{2}{7}\end{bmatrix},\quad \mathbf{K}_2=\begin{bmatrix}0&12&0&0\\12&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \mathbf{Q}_2=\begin{bmatrix}1&\frac{-1}{2}&\frac{1}{3}&\frac{-1}{4}\\\frac{-1}{2}&\frac{1}{3}&\frac{-1}{4}&\frac{1}{5}\\\frac{1}{3}&\frac{-1}{4}&\frac{1}{5}&\frac{-1}{6}\\\frac{-1}{4}&\frac{1}{5}&\frac{-1}{6}&\frac{1}{7}\end{bmatrix},$$

$$\mathbf{K}_3=\begin{bmatrix}0\,0\,0\,0\\0\,4\,0\,0\\0\,0\,0\,0\\0\,0\,0\,0\end{bmatrix},\quad \mathbf{Q}_3=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}\\\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}\end{bmatrix},\quad \mathbf{K}_4=\begin{bmatrix}0\,0\,0\,0\\0\,4\,0\,0\\0\,0\,0\,0\\0\,0\,0\,0\end{bmatrix},\quad \mathbf{Q}_4=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}\\\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}\end{bmatrix},$$

$$\mathbf{W}=\begin{bmatrix}\frac{22}{3}&-\frac{13}{3}&\frac{34}{3}&-\frac{43}{5}\\-\frac{146}{27}&-\frac{37}{9}&-\frac{56}{27}&-\frac{1183}{135}\\-\frac{650}{27}&-\frac{35}{9}&-\frac{650}{27}&-\frac{217}{27}\\-\frac{146}{3}&-\frac{11}{3}&-\frac{164}{3}&-\frac{41}{5}\end{bmatrix},\mathbf{G}=\begin{bmatrix}4\\\frac{10}{3}\\2.22e-16\\-6\end{bmatrix}$$

The augmented matrix for this fundamental matrix equation is

$$\left[\widetilde{\mathbf{W}}\quad;\quad\widetilde{\mathbf{G}}\right]=\begin{bmatrix}\frac{22}{3}&-\frac{13}{3}&\frac{34}{3}&-\frac{43}{5}&;4\\2&0&2&0&;0\\0&1&0&3&;0\\0&0&2&0&;2\end{bmatrix}$$

Solving this system, $\mathbf{A}$ is obtained as $\mathbf{A}=[-1\quad 0\quad 1\quad 0]$
Thus, the solution of the problem becomes

$$y_3(t)=t^2$$

which is the exact solution. Thus, it can be seen that the current approach is precise, effective, and useful.

Example 4.2. [25] Let us consider the another third-order linear FIDE

$$y'''(t) = e^t - \frac{t}{2} - 4\int_0^{\frac{1}{4}} e^{(t+s)} y(s)\,ds + \int_0^{\frac{1}{2}} te^s y(s)\,ds - \int_0^1 e^{(s-t)} y(s)\,ds$$

$0 \le t, s \le 1$ with the initial conditions $y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1.$

Following the procedure, for different values of N the polynomial solution is obtained as follows:

$$y_4(t) = 0.03686t^4 - 0.16666t^3 + 0.5t^2 - t + 1$$

$$y_8(t) = 1.82*10^{-5}t^8 - 0.00019t^7 + 0.00138t^6 - 0.00833t^5 + 0.04167t^4 - 0.16667t^3 + 0.5t^2 - t + 1$$

$$\begin{aligned} y_{10}(t) = {} & 1.99 * 10^{-7}t^{10} - 2.64 * 10^{-6}t^9 + 0.00002t^8 - 0.00020t^7 + 0.00139t^6 \\ & -0.00833t^5 + 0.04167t^4 + 0.16667t^3 + 0.5t^2 + t + 1 \end{aligned}$$

which are the approximate solution expanded for $N = 4, 8, 10$ as $y(t) = e^{-t}$
From Eq. (18)

$$R_7(t) = \sum_{k=0}^{m} P_k(t) y_7^{(k)}(t) - \int_{a_f}^{b_f} K_f(t,s)\, y_7(s)\,ds - g(t)$$

we construct the error problem

$$\sum_{k=0}^{m} P_k(t) e_7^{(k)}(t) - \int_{a_f}^{b_f} K_f(t,s)\, e_7(s)\,ds = -R_7(t)$$

$$e_{j7}^{(k)}(a) = 0, \quad j = 1, 2, \ldots J, \quad k = 0, 1, \ldots m-1$$

The error problem is solved for the truncated limited $M = 8$ and we obtain the approximation

$$e_{7,8}(t) = 1.82*10^{-5}t^8 - 4.19*10^{-5}t^7 + 3.69*10^{-5}t^6 - 1.51*10^{-5}t^5 + 2.66*10^{-6}t^4 - 2.13*10^{-7}t^3$$

and the corrected solution

$$y_{7,8}(t) = 1.82*10^{-5}t^8 - 0.00019t^7 + 0.00138t^6 - 0.00833t^5 + 0.04167t^4 - 0.16667t^3 + 0.5t^2 - t + 1$$

Some results from the solutions and the CPU running time results of the example are tabulated for $N = 5, 8, 10$ in Table 1. Furthermore, the maximum absolute errors for some values of N, M are tabulated in Table 2. The tables show that, the result obtained by the current approach is almost the same as the results of the exact solution. The current approach is practical and efficient as well.

Table 1. Comparisons of numerical results for N= 5, 8, 10 in Example 4.2.

t	**Exact solution**	**Present method (N=5)**	**Present method (N=8)**	**Present method (N=10)**
0	1	1	1	1
0.25	0.7788007830714	0.7788003812038	0.7788007832448	0.7788007830732
0.5	0.6065306597126	0.6065295650246	0.6065306613067	0.6065306597265
0.75	0.4723665527410	0.4723534494236	0.4723665582548	0.4723665527955
1.0	0.3678794411714	0.3676949212457	0.3678794827292	0.36787944165404
CPU time		0.920 s	0.952 s	0.961 s

Table 2. Absolute errors for some values of N, M in Example 4.2.

	Absolute errors (Actual, Estimated, Improved)					
t	e_7	e_{10}	$e_{7,8}$	$e_{10,11}$	$E_{7,8}$	$E_{10,11}$
-1.0	1.11e-16	0	0	0	0	0
-0.5	8.97e-10	1.77e-12	1.00e-09	1.77e-12	1.05e-10	4.33e-15
0.0	1.31e-08	1.38e-11	1.36e-08	1.39e-11	5.02e-10	1.75e-14
0.5	4.77e-08	5.45e-11	4.89e-08	5.46e-11	1.18e-09	3.77e-14
1.0	4.57e-07	4.83e-10	4.83e-07	4.84e-10	2.64e-08	1.60e-12

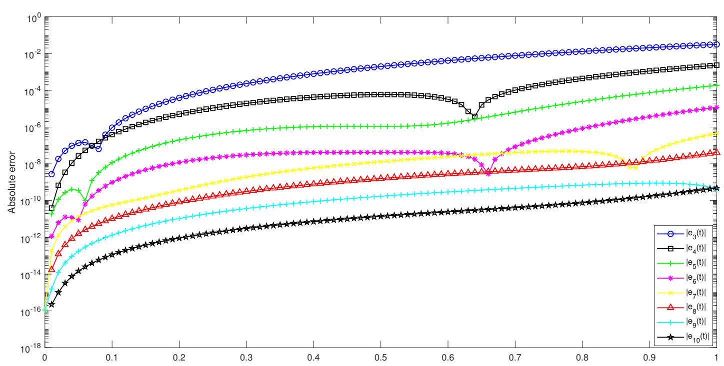

Fig. 1. The absolute errors of Example 4.2 for $3 \leq N \leq 10$.

Fig.1 depicts the numerical solution of the absolute errors in Example 4.2. As the integer N is increased, the error goes down.

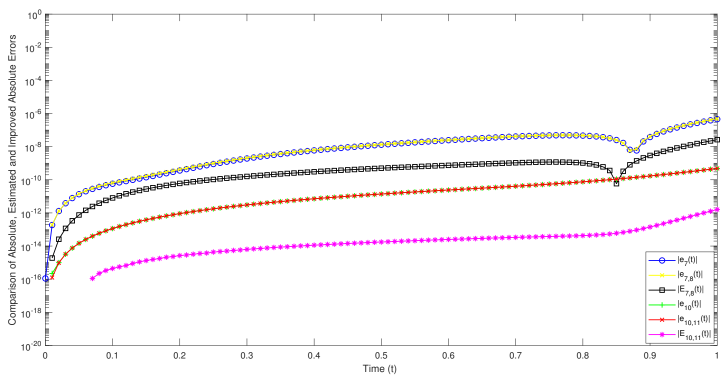

Fig. 2. Comparison of Absolute, Estimated and Improved Absolute Errors of Example 4.2.

Additionally, the residual error analysis provides the improved numerical results as seen in Fig 2.

Example 4.3. [25, 26] Consider the problem

$$y'''(t) - y'(t) = 2t(\cos(1) - \sin(1)) - 2\cos(t) + \int_{-1}^{1} tsy(s)ds$$

with the initial conditions y(0) = 0, $\quad y'(0) = 1, \quad y''(0) - 2y'(0) = -2.$

The solution of the problem for different values of N becomes as follows:

$$y_3(t) = -0.04263t^3 + 6.89 * 10^{-18}t^2 + t$$

$$\begin{aligned} y_{10}(t) = {} & 8.05 * 10^{-8}t^{10} + 2.83 * 10^{-6}t^9 - 6.98 * 10^{-10}t^8 - 0.00020t^7 - 1.58 * 10^{-8}t^6 \\ & + 0.00833t^5 + 3.47 * 10^{-9}t^4 - 0.16667t^3 + 1.58 * 10^{-16}t^2 + t + 4.24 * 10^{-22} \end{aligned}$$

which are the approximate solution expanded for $N = 3, 10$ as $y(t) = \sin(t)$ In Table 3, we compare our obtained results with other methods (Taylor collocation method given in [25] and the Legendre polynomial method given in [26]). From these comparison, it is seen that the proposed method gives better results than other methods.

The numerical solution of the absolute errors in Example 4.3 are depicted in Fig. 3. As the integer N is increased, the error goes down.

Absolute errors of the approximate solutions, the estimated solutions and the improved approximate solutions will be given in Fig.4

Table 3. Comparison of the results of the present method with Taylor[25] and Legendre polynomials [26] in Example 4.3.

t	Exact solution	Present method (N=9)	Taylor method (N=9)	Legendre method (N=9)
-1	-0.8414709848	-0.8414709858	-0.8414709878	-0.8414709892
-0.8	-0.7173560909	-0.7173560912	-0.7173560891	-0.7173560862
-0.6	-0.5646424734	-0.5646424734	-0.5646424729	-0.5646424722
-0.4	-0.3894183423	-0.3894183423	-0.3894183414	-0.3894183400
-0.2	-0.1986693308	-0.1986693308	-0.1986693308	-0.1986693307
0.0	0.0	-9.5291207e-22	-1.9919833e-18	-1.9921875e-17
0.2	0.1986693308	0.1986693308	0.1986693308	0.1986693309
0.4	0.3894183423	0.3894183423	0.3894183438	0.3894183530
0.6	0.5646424734	0.5646424714	0.5646424473	0.5646424233
0.8	0.7173560909	0.7173560449	0.7173560214	0.7173547372
1.0	0.8414709848	0.8414706249	0.8414709801	0.8414709848

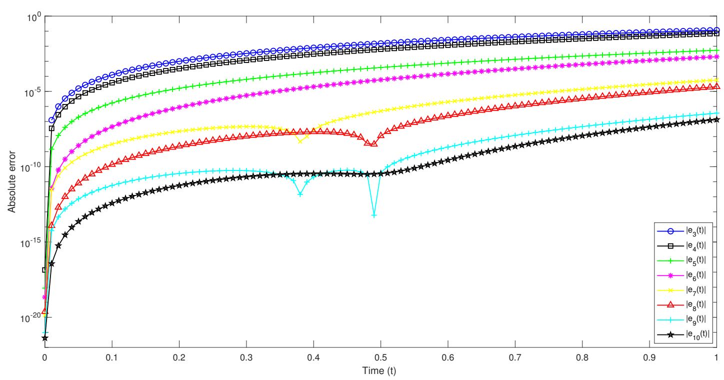

Fig. 3. The absolute errors of Example 4.3 for $3 \leq N \leq 10$.

The CPU running time results of the example for $N = 5, 8, 10$ are 0.875 s, 0.900 s, 0.938 s respectively.

Example 4.4. [27] To compare the results of the proposed method, the example is taken from ÃĞimen et al. Consider the first-order linear FIDE

$$y'(t) + 2y(t) = \frac{1}{4}(e^{-(1+t)} - e^{-t}) + \frac{1}{4}\int_0^1 e^{s-t} y(s)ds \qquad 0 < t \leq 1$$

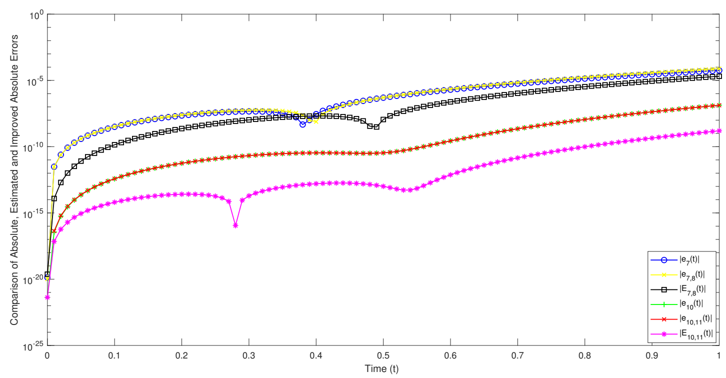

Fig. 4. Comparison of Absolute, Estimated and Improved Absolute Errors of Example 4.3.

with the initial condition y(0) = 1.
The solution of the problem for different values of N becomes as follows:

$$y_6(t) = 0.03991t^6 - 0.21948t^5 + 0.64288t^4 - 1.32730t^3 + 1.99938t^2 - 2t + 1$$

$$\begin{aligned} y_{10}(t) = {} & 0.00012t^{10} - 0.00111t^9 + 0.00601t^8 - 0.02515t^7 + 0.08877t^6 \\ & -0.26663t^5 + 0.66666t^4 - 1.33333t^3 + 2t^2 - 2t + 1 \end{aligned}$$

which are the approximate solution expanded for $N = 6, 10$ as $y(t) = e^{-2t}$
Fig. 5 depicts the absolute errors to solution of Example 4.4. As the number N is increased, the error decreases.

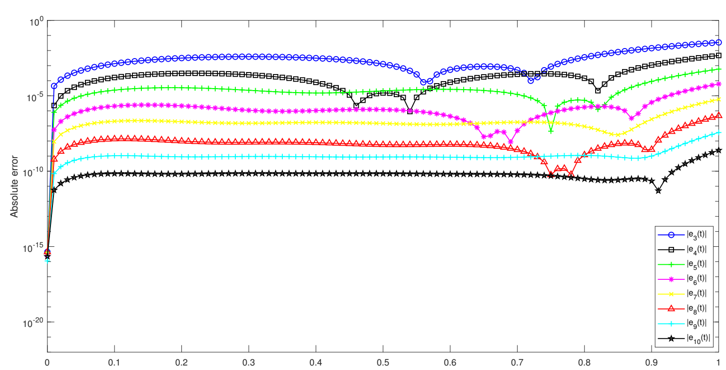

Fig. 5. The absolute errors of Example 4.4 for $3 \leq N \leq 13$.

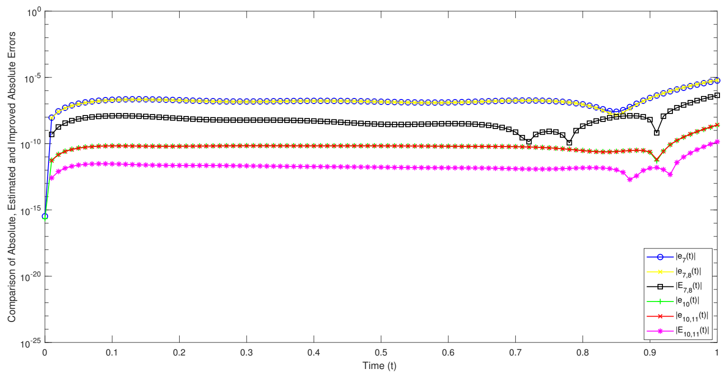

Fig. 6. Comparison of Absolute, Estimated and Improved Absolute Errors of Example 4.4.

In Table 4, we compare our obtained results with other methods (ÃĞimen et.al.'s proposed method and Euler method [27]). From these comparison, it is seen that the proposed method gives better results than other methods.

The CPU running time results of the example for $N = 6, 10$ are 0.878 s and 0.981 s respectively.

Table 4. Comparison of the results of the present method with ÃĞimen et.al.'s proposed method and Euler method [27] in Example 4.4.

t	Exact solution	Present method (N=10)	Method in [27] (N=64)	Euler method [27] (N=64)
0.125	0.778800783071405	0.778800783002729	0.7785212	0.7785212
0.250	0.606530659712633	0.606530659644057	0.6060662	0.6060662
0.375	0.472366552741015	0.472366552671275	0.4717871	0.4717871
0.500	0.367879441171442	0.367879441102222	0.3672360	0.3672360
0.625	0.286504796860190	0.286504796796931	0.2858341	0.2858341
0.750	0.223130160148430	0.223130160099518	0.2224582	0.2224582
0.875	0.173773943450445	0.173773943419333	0.1731185	0.1731185
1.000	0.135335283236613	0.135335285687288	0.1347083	0.1347083

Example 4.5. [28] To compare the results of the proposed method, the example is taken from ÂăFarshadmoghadam et al. Consider the eight-order linear FIDE

$$y^{(8)}(t) - y(t) = -8e^{t} + t^{2} + \int_{0}^{1} y(s)ds \qquad 0 \le t \le 1$$

with the initial conditions $y(0) = 1, y'(0) = 0, y''(0) = -1, y'''(0) = -2,$
$y^{(4)}(0) = -3, y^{(5)}(0) = -4, y^{(6)}(0) = -5, y^{(7)}(0) = -6.$
The solution of the problem for different values of N becomes as follows:

$$y_8(t) = -0.00017t^8 - 0.00119t^7 - 0.00694t^6 - 0.03333t^5 - 0.125t^4 - 0.33333t^3 - 0.5t^2 + 1$$

$$\begin{aligned} y_{12}(t) = &-2.76 * 10^{-8}t^{12} - 2.46 * 10^{-7}t^{11} - 2.34 * 10^{-6}t^{10} - 0.00002t^9 - 0.00017t^8 \\ &-0.00119t^7 - 0.00694t^6 - 0.03333t^5 - 0.125t^4 - 0.33333t^3 - 0.5t^2 + 1 \end{aligned}$$

which are the approximate solution expanded for $N = 5, 10$ as $y(t) = \sin(t)$.

Table 5. Comparison maximum absolute errors of the present method with radial basis functions method [28] in Example 4.5.

Method	**N=10**	**N=12**	**N=13**
Proposed Method	2.2693e-7	1.4580e-7	1.4564e-7
Radial Basis Functions Method	1.6777e-04	8.4842e-06	1.1661e-06

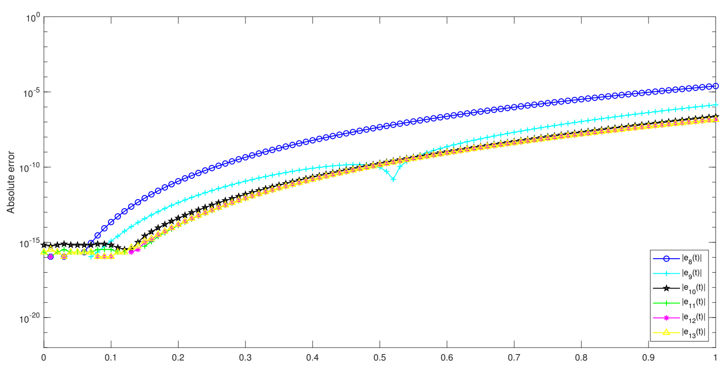

Fig. 7. The absolute errors of Example 4.5 for $8 \le N \le 13$.

Table 6. Comparisons of absolute errors for N= 8, 10, 13 in Example 4.5.

t	**Exact solution**	$\lvert e_8 \rvert$	$\lvert e_{10} \rvert$	$\lvert e_{13} \rvert$
0.1	0.994653826268	2.22E-14	9.99E-16	0
0.2	0.977122206528	1.15E-11	4.01E-14	1.49E-14
0.3	0.944901165303	4.49E-10	1.54E-12	8.60E-13
0.4	0.895094818585	6.05E-09	2.15E-11	1.53E-11
0.5	0.824360635350	4.56E-08	1.78E-10	1.42E-10
0.6	0.728847520156	2.38E-07	1.07E-09	8.81E-10
0.7	0.604125812241	9.65E-07	5.13E-09	4.11E-09
0.8	0.445108185698	3.25E-06	2.07E-08	1.56E-08
0.9	0.245960311116	9.49E-06	7.25E-08	5.08E-08
1	0	2.48E-05	2.27E-07	1.46E-07

In Table 5, we compare our obtained results with other methods (radial basis functions method [28]). From these comparison, it is seen that the proposed method gives better results than other methods.
Fig. 6 depicts the absolute errors to solution of Example 4.5. As the number N is increased, the error decreases. The CPU running time results of the example for $N = 8, 11, 13$ are 0.850 s, 0.934 s and 0.943 s respectively.

5 Conclusion

A collocation method based on the Lucas polynomial is proposed in this study to solve the linear FIDE and FIDE with piecewise intervals. The residual error function also provides an estimation of the error. We presented five numerical examples to demonstrate the method. In Example 4.1, we find the analytical solution for $N = 3$. In the second example, we computed the approximate solutions for $N = 4, 8, 10$. The problem in Example 4.1 has an exact solution but it has not an exact solution in Example 4.2. Therefore, we measured the reliability of the solutions by means of the estimated absolute error functions. We compared the actual and estimated absolute error functions and their values in Fig. 1-2 and Table 1-2. The values of the estimated errors closely match those of the actual errors. Additionally, comparisons between our method, the exact solution and other method in Example 4.3, Example 4.4 and Example 4.5. When results from the tables and figures are compared, it is clear that the proposed method is very efficient and practical. Besides all these, it is seen from the CPU times that the problems solved efficiently and rapidly without the need for detailed procedures.

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