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Topological Crossed Semimodules and Schreier Internal Categories in the Category of Topological Monoids

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ABSTRACT

The aim of this paper is to introduce the notion of Schreier internal categories in the category of topological monoids and of topological crossed semimodules and to prove the categorical equivalence between them. This is the generalization of equivalence between the category of internal categories in the category of topological groups and the category of topological crossed modules. Moreover, we obtained a Schreier internal category as a special sort of 2-category with one object in the category of topological monoids.

Keywords: *Schreier Internal Category, Topological Crossed Semimodule.*

1. INTRODUCTION

A group-groupoid, which is also known as $\mathcal{G}$-groupoid [1] or 2-group [2], is an internal category in the category of groups. A crossed module as defined by Whitehead is a pair of groups M, N with boundary map $\partial: M \to N$ and an action $\bullet: N \times M \to M$ such that $\partial(n \bullet m) = n\partial(m)n^{-1}$ and $\partial(m) \bullet m' = mm'm^{-1}$ [3,4]. It is well known that the category of group-groupoids is equivalent to the category of crossed modules (see [1]). Topological aspect of this equivalence was introduced in [5]. One can find in [2] that this equivalence is proved by obtaining a group-groupoid as a 2-category with a single object. The structure of 2-category was first defined by Bénabou in 1967 [6]. A 2-category consists of objects, 1-morphisms between objects as in ordinary category and 2-morphisms between 1-morphisms as follows

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} .$$

Let **Mon** be the category of monoids. The description of Schreier internal categories in **Mon** as crossed semimodules is introduced in [7]. This is the generalization of equivalence between the category of internal categories in the category of groups and the category of crossed modules.

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The main purpose of this paper is to present the topological aspect of the categorical equivalence between Schreier internal categories and crossed semimodules by using topological monoids. In particular, we show in Section 3. that any Schreier internal category may be considered as a 2-category with one object.

2. PRELIMINARIES

An internal category in **Mon** $M=(M_0,M_1,s,t,\varepsilon,\circ)$ consists of a pair of objects M_0 and M_1 together with homomorphisms of monoids $s,t\colon M_1\to M_0$, $\varepsilon\colon M_0\to M_1$ called the source map, the target map and the identity map, respectively, such that $s\varepsilon=t\varepsilon=1_{M_0}$ and the composition map $m\colon M_1\times_{M_0}M_1\to M_1$ as homomorphism of monoids (usually written as $m(f,g)=g\circ f$)

$$x\xrightarrow{f}y\xrightarrow{g}z \quad = \quad x\xrightarrow{g\circ f}z$$

where $M_1\times_{M_0}M_1$ is the pullback of s and t as follows

$$\begin{tikzcd} M_1\times_{M_0}M_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & M_1 \arrow[d, "s"] \\ M_1 \arrow[r, "t"'] & M_0 \end{tikzcd}$$

such that $h\circ(g\circ f)=(h\circ g)\circ f$ and $\varepsilon t(f)\circ f=f=f\circ\varepsilon t(f)$ [7,8].

Let $M=(M_0,M_1)$ and $M'=(M_0',M_1')$ be internal categories in **Mon**. A functor $\gamma=(\gamma_1,\gamma_2)\colon M\to M'$ such that $\gamma_0\colon M_0\to M_0'$ and $\gamma_1\colon M_1\to M_1'$ are both monoid homomorphisms is called morphism of internal categories in **Mon**.

An internal groupoid $M=\ (M_0,M_1,s,t,\varepsilon,\eta,\circ)$ in **Mon** is an internal category with the homomorphism of monoids $\eta\colon M_1\to M_1, \eta(f)=\bar{f}$ called inverse such that $f\circ\bar{f}=1_{s(f)}$ and $f\circ\bar{f}=1_{t(f)}$.

Let $M=(M_0,M_1,s,t,\varepsilon,\circ)$ be an internal category in **Mon**. If M satisfies the Schreier condition: for any $f\in M_1$ there exists a unique $\tilde{f}\in Kers$ such that

$$f=\tilde{f}\cdot\varepsilon s(f)$$

then M is called a Schreier internal category in **Mon** [7].

Let M and M' be Schreier internal categories in **Mon**. A functor $\gamma=(\gamma_1,\gamma_2)\colon M\to M'$ such that $\gamma_0\colon M_0\to M_0'$ and $\gamma_1\colon M_1\to M_1'$ are both homomorphisms of monoids is called morphism of Schreier internal categories in **Mon**. Hence Schreier internal categories in **Mon** form a subcategory which is denoted by **Sic**.

A Schreier internal groupoid in **Mon** is a Schreier internal category in which each morphism has an inverse.

A crossed semimodule $K=(M,N,\partial,\bullet)$ consists of a pair of monoids M and N with a homomorphism $\partial\colon M\to N$ of monoids and an action $\bullet\colon N\times M\to M$ of monoids such that $\partial(n\bullet m)\cdot n=n\cdot\partial(m)$ and $(\partial(m)\bullet m'))\cdot m=m\cdot m'$.

Let $K=(M,N,\partial,\bullet)$, $K'=(M',N',\partial',\bullet')$ be crossed semimodules. A crossed semimodule morphism is a map $\lambda=(\lambda_2,\lambda_1)\colon K\to K'$ where $\lambda_1\colon N\to N'$ and $\lambda_2\colon M\to M'$ are homomorphisms of monoids such that $\lambda_1\partial=\partial'\lambda_2$ and $\lambda_2(n\bullet m)=\lambda_1(n)\bullet'\lambda_2(m)$. Thus crossed semimodules and their morphisms form a category which is denoted by **Csm**.

Theorem 2.1. *The category of Schreier internal categories in* **Mon** *and of crossed semimodules are equivalent* [7].

Corollary 2.2. *The category of Schreier internal groupoids in* **Mon** *and of crossed semimodules are equivalent where M is a group* [7].

Restricting of this equivalence, the following corollary can be obtained:

Corollary 2.3. *The category of internal categories in the category of groups* **Gp** *and of crossed modules are equivalent* [1].

The following definition is given in [2]:

Definition 2.4. A 2-category $C=(C_0,C_1,C_2)$ consists of the set of objects $C_0=\{x,y,z,\ldots\}$, the set of 1-morphisms between objects $C_1=\{f,g,\ldots\}$ and the set of 2-morphisms between 1-morphisms $C_2=\{\alpha,\beta,\ldots\}$ as follows

$$\begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & y \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd} .$$

with the following maps

1. the source maps

$s: C_1 \to C_0,\ s(f) = x,\ s_h: C_2 \to C_0,\ s_h(\alpha) = x,\ s_v: C_2 \to C_1,\ s_v(\alpha) = f,$

2. the target maps

   $t: C_1 \to C_0,\ t(f) = y,\ t_h: C_2 \to C_0,\ t_h(\alpha) = y,\ t_v: C_2 \to C_1,\ t_v(\alpha) = g,$

3. the composition of 1-morphisms $\circ: C_1 \times_{C_0} C_1 \to C_1$ as in a classical category,

4. the associative horizontal composition of 2-morphisms $\circ_h: C_2 \times_{C_0} C_2 \to C_2$ where $C_2 \times_{C_0} C_2 = \{(\alpha, \delta) \in C_2 \times C_2 :\ s_h(\delta) = t_h(\alpha)\}$ as

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\delta"] & z \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "f_1 \circ f"] \arrow[r, bend right, "g_1 \circ g"'] \arrow[r, phantom, "\Downarrow \delta \circ_h \alpha"] & z \end{tikzcd}$$

5. the associative vertical composition of 2-morphisms $\circ_v: C_2 \times_{C_1} C_2 \to C_2$ where $C_2 \times_{C_1} C_2 = \{(\alpha, \beta) \epsilon C_2 \times C_2 :\ s_v(\beta) = t_v(\alpha)\}$

$$\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & y \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \beta \circ_v \alpha"] & y \end{tikzcd}$$

   which satisfies the following interchange rule together with the horizontal composition

$$(\theta \circ_v \delta) \circ_h (\beta \circ_v \alpha) = (\theta \circ_h \beta) \circ_v (\delta \circ_h \alpha)$$

   whenever compositions are defined,

6. identity maps $\varepsilon: C_0 \to C_1$, $\varepsilon(x) = 1_x$, as in ordinary category, $\varepsilon_h: C_0 \to C_2$, $\varepsilon_h(x) = 1_{1_x}$, $\varepsilon_h(y) = 1_{1_y}$ such that $\alpha \circ_h 1_{1_x} = \alpha = 1_{1_y} \circ_h \alpha$ and $\varepsilon_v: C_1 \to C_2$, $\varepsilon_v(f) = 1_f$, $\varepsilon_v(g) = 1_g$ such that $\alpha \circ_v 1_f = \alpha = 1_g \circ_v \alpha$ where the following diagram is the commutative

$$\begin{tikzcd} C_2 \arrow[rr, shift left, "s_h"] \arrow[rr, shift right, "t_h"'] \arrow[dr, shift left, "t_v"] \arrow[dr, shift right, "s_v"'] & & C_0 \arrow[ll, bend right, "\varepsilon_h"'] \arrow[dl, bend left, "\varepsilon"] \\ & C_1 \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] \arrow[ul, bend left, "\varepsilon_v"] & \end{tikzcd}$$

Let C, C' be 2-categories. A 2-functor is a map $F: C \to C'$ sending each object of C to an object of C', each 1-morphism of C to 1-morphism of C' and 2-morphism of C to 2-morphism of C' as follows

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} \mapsto \begin{tikzcd} F(x) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] & F(y) \end{tikzcd}$$

such that $F(f_1 \circ f) = F(f_1) \circ F(f)$, $F(\delta \circ_h \alpha) = F(\delta) \circ_h F(\alpha)$, $F(\beta \circ_v \alpha) = F(\beta) \circ_v F(\alpha)$, $F(1_{1_x}) = 1_{F(1_x)} = 1_{1_{F(x)}}$ and $F(1_f) = 1_{F(f)}$. Thus, 2-categories and 2-functors form a category which is denoted by **2Cat** [10].

A 2-groupoid is a 2-category whose all 1-morphisms and 2-morphisms are invertible [10] as follows

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "\bar{f}"] \arrow[r, bend right, "\bar{g}"'] \arrow[r, phantom, "\Downarrow\bar{\alpha}^h"] & x \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\Downarrow 1_{1_x}"] & x \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\bar{\alpha}^v"] & y \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & y \end{tikzcd}$$

A morphism of 2-groupoids is a 2-functor between them. Thus 2-groupoids form a category which is denoted by **2Gpd** [10].

3. TOPOLOGICAL CROSSED SEMIMODULES AND SCHREIER INTERNAL CATEGORIES IN Mon

Let **TMon** be the category of topological monoids. Then an internal category $M = (M_0, M_1, s, t, \varepsilon, \circ)$ in **TMon** consists of the topological monoid of objects M_0 and the topological monoid of morphisms M_1 with continuous structure maps s, t, ε, m as morphisms of **TMon**. Hence a Schreier internal category in **TMon** can be defined in the following definition.

Definition 3.1. Let $M = (M_0, M_1, s, t, \varepsilon, \circ)$ be an internal category in **TMon**. If for any $f \in M_1$ there exists a unique $\tilde{f} \in Kers$ such that $f = \tilde{f} \cdot \varepsilon s(f)$, then M is called a Schreier internal category in **TMon**.

In a Schreier internal category, that continuous monoid product $\cdot$ is an internal functor in **TMon** gives the usual interchange rule

$$(g \circ f) \cdot (g' \circ f') = (g \cdot g') \circ (f \cdot f')$$

whenever $g \circ f$ and $g' \circ f'$ are defined. Using this rule, composition of morphisms can be written in terms of the monoid operation as follows

$$g \circ f = \tilde{g} \cdot \tilde{f} \cdot \varepsilon s(f)$$

Example 3.2. Consider any abelian topological monoid $(M, \cdot)$. If we assume that pair (x, y) is a morphism from x to $x \cdot y$ as follows

$$x \xrightarrow{(x,y)} x \cdot y$$

Then $M = (M, M \times M, s, t, \varepsilon, \circ)$ is a Schreier internal category with the direct product. The continuous composition of morphisms is defined as

$$(x \cdot y, z) \circ (x, y) = (x, y \cdot z)$$

where continuous structure maps are given by

$$s(x, y) = x, t(x, y) = x \cdot y, \varepsilon(x) = (x, e).$$

Since $(x, y) = (e, y) \cdot (x, e)$, all morphisms satisfy the Schreier condition.

Definition 3.3. Let $M = (M_0, M_1)$ and $M' = (M_0', M_1')$ be Schreier internal categories in **TMon**. A morphism of Schreier internal categories in **TMon** is an internal functor $\gamma = (\gamma_0, \gamma_1): M \to M$ where $\gamma_0: M_0 \to M_0'$ and $\gamma_1: M_1 \to M_1'$ are both homomorphisms of topological monoids. Therefore Schreier internal categories in **TMon** form a subcategory which we denoted by **TSic**.

Definition 3.4. A topological crossed semimodule $K = (M, N, \partial, \bullet)$ consists of topological monoids M and N together with a continuous homomorphism of topological monoids $\partial: M \to N$ and a continuous action $\bullet: N \times M \to M$ of topological monoids satisfying the following identities

1. $\partial(n \bullet m) \cdot n = n \cdot \partial(m)$
2. $(\partial(m) \bullet m') \cdot m = m \cdot m'$.

Definition 3.5. Let $K = (M, N, \partial, \bullet)$ and $K' = (M', N', \partial', \bullet')$ be topological crossed semimodules. A homomorphism of topological crossed semimodules is a mapping $\lambda = (\lambda_2, \lambda_1): K \to K'$ where $\lambda_1: N \to N'$ and $\lambda_2: M \to M'$ are homomorphisms of topological monoids satisfying $\lambda_1 \partial = \partial' \lambda_2$ and $\lambda_2(n \bullet m) = \lambda_1(n) \bullet' \lambda_2(m)$. Thus topological crossed semimodules and their morphisms form a category which we denoted by **TCsm**.

Theorem 3.6. *The category of Schreier internal categories in* **TMon** *is equivalent to the category of topological crossed semimodules.*

Proof. A functor δ:**TSic**→**TCsm** is defined as follows. For any Schreier internal category $M = (M_0, M_1, s, t, \varepsilon, \circ)$ in **TMon**, $\delta(M) = (Kers, M_0, \partial, \bullet)$ is a topological crossed semimodule where $\partial = t|_{Kers}$ and $\bullet: M_0 \times Kers \to Kers$,

$(x \bullet f) \cdot \varepsilon(x) = \varepsilon(x) \cdot f$, for any $x \in M_0$ and $f \in Kers$. Since the following diagram is commutative, $\bullet$ is a continuous action of topological monoids.

$$\begin{tikzcd} M_0 \times Kers \arrow[r, "(\bullet,\varepsilon)"] \arrow[d, "(\varepsilon\times 1)"'] & Kers \times M_1 \arrow[d, "\cdot"] \\ M_1 \times Kers \arrow[r, "\cdot"] & M_1 \end{tikzcd}$$

It is routine to check that $\partial(x \bullet f) \cdot x = x \cdot \partial(f)$ and $(\partial(f) \bullet f') \cdot f = f \cdot f'$.

Given a homomorphism $\gamma = (\gamma_0, \gamma_1)$ of Schreier internal categories in **TMon**, $\delta(\gamma_0, \gamma_1) = (\gamma_1|_{Kers}, \gamma_0)$ is a homomor-phism of topological crossed semimodules.

Now, let us define a functor θ:**TCsm**$\rightarrow$ **TSic** as an equivalence of categories. Given any topological crossed semimodule $K = (M, N, \partial, \bullet)$, then $\theta(K) = (N, N \ltimes M, s, t, \varepsilon, \circ)$ is a Schreier internal category where $N \ltimes M$ is the semi-direct product of topological monoids with the multiplication

$$(N \ltimes M)_s\times_t(N \ltimes M) \to N \ltimes M,$$
$$((n, m), (n', m')) \mapsto (n \cdot n', m \cdot (n \bullet m')).$$

Since the monoid product and the action are continuous, the multiplication of the semi-direct product is continuous. The source and the target maps are defined by $s(n, m) = n$, $t(n, m) = \partial(m) \cdot n$, respec-tively, and the identity map is defined by $\varepsilon(n) = (n, e_M)$ where the composition is given by

$$(\partial(m) \cdot n, m_1) \circ (n, m) = (n, m_1 \cdot m)$$

Clearly, the above maps are continuous. Since $(n, m) = (e_N, m) \cdot (n, e_M)$, morphisms satisfy the Schreier condition.

Let $\lambda = (\lambda_2, \lambda_1): K \to K'$ be a homomor-phism of topological crossed semimodules. Then $\theta(\lambda_2, \lambda_1) = (\lambda_1, \lambda_1 \times \lambda_2)$ is a homomorphism of Schreier internal categories in **TMon**.

It is easy to check that $\delta\theta \cong 1$. To prove $1 \cong \theta\delta$, we must define a natural equivalence $S: \theta\delta \to 1_{TSic}$. Let M be a Schreier internal category in **TMon**. Then the map $S_M: \theta\delta(M) \to M$ is defined to be the identity on objects and is defined by $f \mapsto (s(f), \tilde{f})$ on morphisms. Clearly S_M is a homeomorphism and preserves the monoid operation and composition due to Schreier condition as follows: for $f, g, f' \in M_1$ such that $s(g) = t(f)$,

$$\begin{aligned} S_M(f \cdot f') &= S_M\left(\tilde{f} \cdot \varepsilon s(f) \cdot \tilde{f}' \cdot \varepsilon s(f')\right) = S_M\left(\tilde{f} \cdot (s(f) \bullet \tilde{f}') \cdot \varepsilon s(f) \cdot \varepsilon s(f')\right) \\ &= S_M\left(\tilde{f} \cdot (s(f) \bullet \tilde{f}') \cdot \varepsilon s(f \cdot f')\right) = \left(s(f \cdot f'), \tilde{f} \cdot (s(f) \bullet \tilde{f}')\right) \\ &= \left(s(f) \cdot s(f'), \tilde{f} \cdot (s(f) \bullet \tilde{f}')\right) = S_M(f) \cdot S_M(f') \end{aligned}$$

$$S_M(g \circ f) = \left(s(f), \tilde{g} \cdot \tilde{f}\right) = (s(g), \tilde{g}) \circ (s(f), \tilde{f}) = S_M(g) \circ S_M(f)$$

■

Example 3.7. Given monoids $(\mathbb{Z}, \cdot)$ and $(\mathbb{Z}_4, \cdot)$ with discrete topologies, a topological crossed semimodule $(\mathbb{Z}, \mathbb{Z}_4, \partial, \bullet)$ can be constructed with continuous homomorphism $\partial(z) = \bar{z}$ and continuous action $\bar{\omega} \bullet z = z$. Then $(\mathbb{Z}_4, \mathbb{Z}_4 \ltimes \mathbb{Z}, s, t, \varepsilon)$ is a Schreier internal category in **TMon** together with continuous structure maps

$$s(\bar{\omega}, z) = \bar{\omega}, t(\bar{\omega}, z) = \bar{z} \cdot \bar{\omega}, \varepsilon(\bar{\omega}) = (\bar{\omega}, 1), (\bar{z} \cdot \bar{\omega}, z_1) \circ (\bar{\omega}, z) = (\bar{\omega}, z_1 \cdot z)$$

Definition 3.8. A Schreier internal groupoid in **TMon** is a Schreier internal category in which each morphism is invertible.

Let **TCsm*** be the category of topological crossed semimodules $K = (M, N, \partial, \bullet)$ such that M is a topological group. Then, restricting of Theorem 3.6. gives the following Corollary:

Corollary 3.9. *The category of Schreier internal groupoids in* **TMon** *is equivalent to* **TCsm***.

Proof. Let M be a topological group and $K = (M, N, \partial, \bullet)$ be a topological crossed semimodule. Then $(N, N \ltimes M, s, t, \varepsilon, \circ)$ is a Schreier internal groupoid in **TMon** where $\eta(n, m) = (\partial(m) \cdot n, m^{-1})$.

■

Definition 3.10. Given the group $(\mathbb{Z}_5,\cdot)$ and the monoid $(\mathbb{Z}_4,\cdot)$ with discrete topologies, a topological crossed semimodule $(\mathbb{Z}_5,\mathbb{Z}_4,\partial,\bullet)$ is constructed with continuous homomorphism $\partial(\bar{z}^5)=\bar{z}^4$ and continuous action $\bar{\omega}^4\bullet\bar{z}^5=\bar{z}^5$. Then $(\mathbb{Z}_4,\mathbb{Z}_4\ltimes\mathbb{Z}_5,s,t,\varepsilon,\circ)$ is a Schreier internal groupoid in **TMon** as in Example 3.7. with continuous map $\eta(\bar{\omega}^4,\bar{z}^5)=(\bar{z}^4\cdot\bar{\omega}^4,(\bar{z}^5)^{-1})$.

Restricting of this Corollary, the following result is obtained:

Corollary 3.11. *The category of internal categories in the category of topological groups is equivalent to the category of topological crossed modules* [4].

4. SCHREIER INTERNAL CATEGORIES AS 2-CATEGORIES

In this section, a Schreier internal category in **TMon** is obtained as a special sort of 2-category similar way as in [2].
A topological monoid M can be thought of as a category with a single object [8]. Here elements of M are considered as morphisms when continuous composition is defined by multiplication of M as follows:

$$\star\xrightarrow{x}\star\xrightarrow{y}\star\quad=\quad\star\xrightarrow{y\cdot x}\star$$

Then a Schreier internal category in **TMon** can be thought of as 2-category with one object. In this 2-category, 2-morphisms are determined by morphisms of the Schreier internal category in **TMon** as follows:

$$\begin{tikzcd}\star \arrow[r, bend left, "x"{name=U}] \arrow[r, bend right, "y"'{name=D}] & \star \arrow[from=U, to=D, phantom, "\Downarrow f"]\end{tikzcd}$$

The Schreier condition can be expressed by $f=\tilde{f}\cdot\varepsilon_v s_v(f)$ where $\tilde{f}\in Kers_v$. We can now define a Schreier internal category in **TMon**:

Definition 4.1. A Schreier internal category $M=(M_0,M_1)$ in **TMon** is a 2-category with one object such that for any $f\in M_1$ as a 2-morphism, there exists a unique 2-morphism $\tilde{f}\in Kers_v$ which satisfies the Schreier condition $f=\tilde{f}\cdot\varepsilon_v s_{v(f)}$.

$$\begin{tikzcd}\star \arrow[r, bend left, "x"{name=U}] \arrow[r, bend right, "y"'{name=D}] & \star \arrow[from=U, to=D, phantom, "\Downarrow f"]\end{tikzcd} = \begin{tikzcd}\star \arrow[r, bend left, "x"{name=U1}] \arrow[r, bend right, "x"'{name=D1}] & \star \arrow[r, bend left, "e"{name=U2}] \arrow[r, bend right, "\tilde{y}"'{name=D2}] & \star \arrow[from=U1, to=D1, phantom, "\Downarrow 1_x"] \arrow[from=U2, to=D2, phantom, "\Downarrow \tilde{f}"]\end{tikzcd}$$

Theorem 4.2. *There is a categorical equivalence between topological crossed semimodules and Schreier internal categories which are 2-categories with one object.*

Proof. Given a Schreier internal category $M=(M_0,M_1)$ in **TMon** as 2-category, a topological crossed semimodule $(M,N,\partial,\bullet)$ can be extracted where $N=M_0$ consisting of all morphisms of M, the topological monoid $M=Kers_v$ consisting of all 2-morphisms whose source is the identity morphism e with the restriction of the target map $\partial=t_v|_{Kers_v}$ and the continuous action $\bullet\colon N\times M\to M$ is given by

$$\begin{tikzcd}\star \arrow[r, bend left, "x"{name=U1}] \arrow[r, bend right, "x"'{name=D1}] & \star \arrow[r, bend left, "e"{name=U2}] \arrow[r, bend right, "\partial(x\bullet f)"'{name=D2}] & \star \arrow[from=U1, to=D1, phantom, "\Downarrow 1_x"] \arrow[from=U2, to=D2, phantom, "\Downarrow x\bullet f"]\end{tikzcd} := \begin{tikzcd}\star \arrow[r, bend left, "e"{name=U1}] \arrow[r, bend right, "\partial(f)"'{name=D1}] & \star \arrow[r, bend left, "x"{name=U2}] \arrow[r, bend right, "x"'{name=D2}] & \star \arrow[from=U1, to=D1, phantom, "\Downarrow f"] \arrow[from=U2, to=D2, phantom, "\Downarrow 1_x"]\end{tikzcd}.$$

Conversely, given a topological crossed semimodule $K=(M,N,\partial,\bullet)$, a Schreier internal category M in **TMon** can be constructed as 2-category with one object where topological monoid of 1-morphisms is N and topological monoid of 2-morphisms is the semi-direct product $N\ltimes M$ of topological monoids with

$$(n,m)\odot(n',m')=\big(n\cdot n',m\cdot(n\bullet m')\big).$$

The source and the target maps are defined by $s_v(n,m)=n$, $t_v(n,m)=\partial(m)\cdot n$, respectively and the identity map is defined by $\varepsilon_v(n)=(n,e_M)$ where the horizontal composition of 2-morphisms is given by

$$(n_1,m_1)\circ_h(n,m)=\big(n_1 n,m_1\cdot(n_1\bullet m)\big)$$

and the vertical composition of 2-morphisms is defined by

$$(\partial(m)\cdot n,m_2)\circ_v(n,m)=(n,m_2\cdot m)$$

It is easy to check that the horizontal composition and the vertical composition satisfy the usual interchange rule. Obviously all of maps and above the compositions are continuous.

Since $(n, m) = (e_N, m) \cdot (n, e_M)$ all 2-morphisms satisfy the Schreier condition.

$$\begin{tikzcd} \star \arrow[r, bend left, "n"] \arrow[r, bend right, "\partial(m)\cdot n"'] \arrow[r, phantom, "\Downarrow(n,m)"] & \star \end{tikzcd} = \begin{tikzcd} \star \arrow[r, bend left, "n"] \arrow[r, bend right, "n"'] \arrow[r, phantom, "\Downarrow(n,e_M)"] & \star \arrow[r, bend left, "e"] \arrow[r, bend right, "\partial(m)"'] \arrow[r, phantom, "\Downarrow(e_N,m)"] & \star \end{tikzcd}$$

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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