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H-GALOIS EXTENSIONS WITH NORMAL BASIS FOR WEAK HOPF ALGEBRAS

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Abstract. Let H be a weak Hopf algebra and let A be an H-comodule algebra with subalgebra of coinvariants A_H. In this paper we introduce the notion of H-Galois extension with normal basis and we prove that $A_H \hookrightarrow A$ is an H-Galois extension with normal basis if and only if $A_H \hookrightarrow A$ is an H-cleft extension which admits a convolution invertible total integral. As a consequence, if H is cocommutative and A commutative, we obtain a bijective correspondence between the second cohomology group $H^2_{\varphi_{A_H}}(H, A_H)$ and the set of isomorphism classes of H-Galois extensions with normal basis whose left action over A_H is φ_{A_H}.

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1. Introduction

It is a well-known fact in classical Galois theory that if $B \subset A$ is a finite Galois extension of fields with Galois group H, then A/B has a normal basis, i.e., there exists $a \in A$ such that the set $\{x.a \,;\, x \in H\}$ is a basis for A over B. Generalizing finite Galois extension of fields, Kreimer and Takeuchi introduce in [13] the notion of H-Galois extension with normal basis, associated to a Hopf algebra H in a category of modules over a commutative ring, and in [10] Doi and Takeuchi show that there exists an equivalence between the notion of H-Galois extension with normal basis and the one of H-cleft extension for H. This result can be generalized to symmetric closed categories [11] and in [7] we find a more general formulation in the context of entwining structures that was extended to the weak setting in [2] by using the notion of weak C-cleft extensions defined in [1]. On the other hand, being A an algebra, C a coalgebra and $\Gamma_H^A : C \otimes A \to A \otimes C$ a morphism in a strict monoidal category with equalizers and coequalizers, such that (A, C, Γ_H^A) is a weak entwining structure, we have introduced in [2] the notion of weak C-Galois extension with

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normal basis and we proved that, if $A \otimes -$ preserves coequalizers, there exists an equivalence between weak C-Galois extensions and weak C-cleft extensions. Taking into account that every right comodule algebra over a weak Hopf algebra H induces a weak entwining structure, the results obtained in [1] and [2] can be applied for the study of Galois theory for weak Hopf algebras.

In [5] we introduce the notion of H-cleft extension for a weak Hopf algebra H and we prove that this kind of extensions are examples of weak H-cleft extensions like the ones introduced in [1] and satisfying the classical notion of cleftness when particularizing to the Hopf setting. Assuming cocommutativity for H, we give in [5] a bijective correspondence between the equivalence classes of H-cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for H over A_H where A_H denotes the subalgebra of coinvariants of the H-comodule algebra (A, ρ_A) in the weak context. This result permits to generalize the ones proved by Doi [9] about the characterization of equivalence classes of crossed systems as the second Sweedler cohomology group in the cocommutative Hopf algebra setting. To obtain this generalization we need the cohomology theory of algebras over cocommutative weak Hopf algebras we developed in [4] and used in [5] in order to give the weak Hopf version of Doi's result, i.e., a bijection between the isomorphism classes of H-cleft extensions $A_H \hookrightarrow B$, the equivalence classes of crossed systems for H over A_H and the second cohomology group $H^2_{\varphi_{\mathcal{Z}(A_H)}}(H, \mathcal{Z}(A_H))$, where $\mathcal{Z}(A_H)$ is the center of A_H and $\varphi_{\mathcal{Z}(A_H)}$ the corresponding associated action.

As we have pointed above, H-cleft extensions are a kind of weak H-cleft extensions, and these are equivalent to weak H-Galois extensions with normal basis. This leads naturally to the following question: Is there a special class of weak H-Galois extensions with normal basis equivalent to H-cleft extensions? In order to give an affirmative response to this question we introduce the notion of H-Galois extension with normal basis like a special kind of weak H-Galois extension with normal basis and we prove that if $A \otimes -$ preserves coequalizers, the following assertions are equivalent:

(i) $A_H \hookrightarrow A$ is an H-cleft extension that admits a convolution invertible total integral.
(ii) $A_H \hookrightarrow A$ is an H-Galois extension with normal basis.

As a consequence, taking into account that, if H is cocommutative, every H-cleft extension $A_H \hookrightarrow A$ admits a convolution invertible total integral, we obtain that $A_H \hookrightarrow A$ is an H-cleft extension if and only if $A_H \hookrightarrow A$ is an H-Galois extension with normal basis. Therefore, if A is commutative, we obtain a bijective correspondence between the second cohomology group $H^2_{\varphi_{A_H}}(H, A_H)$ and the set of equivalence classes of H-Galois extensions with normal basis with associated left action over the subalgebra of coinvariants φ_{A_H}.

2. Galois extensions with normal basis and Cleft extensions in a weak setting

Throughout this paper $\mathcal{C} = (\mathcal{C}, \otimes, K, c)$ is a symmetric monoidal category with equalizers and coequalizers, where $\otimes$ is the tensor product, K the base object and $c_{M,N} : M \otimes N \to N \otimes M$ the natural isomorphism of symmetry. For any objects A, B and C in $\mathcal{C}$ the natural isomorphism $\mathfrak{a}_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is called the associative constraint, and the natural isomorphisms $\mathfrak{l}_A : K \otimes A \to A$ and $\mathfrak{r}_A : A \otimes K \to A$, are known as the left and right unit constraints, respectively. Moreover, by Theorem XI.5.3 of [12] we know that every monoidal category is monoidally equivalent to a strict one (i.e., a category such that the constraint isomorphisms are identities), and then there is no loss of generality in assuming that $\mathcal{C}$ is strict.

We assume that the reader is familiar with the notions of (co)algebra and (co)module and morphisms between them in this monoidal setting (see [1], [2]). Note that if $\mathcal{C}$ admits equalizers then every idempotent morphism in $\mathcal{C}$ splits, i.e., for every morphism $q : Y \to Y$ such that $q = q \circ q$ there exists an object Z (image of q) and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

For each object M in $\mathcal{C}$, we denote the identity morphism by $id_M : M \to M$ and for simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

Let $D = (D, \varepsilon_D, \delta_D)$ be a coalgebra, with counit $\varepsilon_D : D \to K$ and coproduct $\delta_D : D \to D \otimes D$, and let $A = (A, \eta_A, \mu_A)$ be an algebra with unit $\eta_A : K \to A$ and product $\mu_A : A \otimes A \to A$. If $f, g : D \to A$ in $\mathcal{C}$ are morphisms in $\mathcal{C}$, $f * g$ denotes the usual convolution product in the category, that is, $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$.

For an algebra A, the category of right (resp. left) A-modules will be denoted by $\mathcal{M}_A$ (resp. ${}_A\mathcal{M}$). Similarly, if D is a coalgebra we denote by $\mathcal{M}^D$ (resp. ${}^D\mathcal{M}$) the category of right (resp. left) D-comodules.

Definition 2.1. *A weak bialgebra H in $\mathcal{C}$ is an algebra (H, η_H, μ_H) and a coalgebra $(H, \varepsilon_H, \delta_H)$ satisfying:*

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$,

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

If moreover,

(a4) *there exists a morphism* $\lambda_H : H \to H$ *in* $\mathcal{C}$ *(called antipode of* H*) satisfying:*

(a4-1) $id_H * \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(a4-2) $\lambda_H * id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(a4-3) $\lambda_H * id_H * \lambda_H = \lambda_H$.

we say that the weak bialgebra H *is a weak Hopf algebra in the category* $\mathcal{C}$*. Note that in a strict monoidal category the associativity of the convolution product follows by the associativity of the product* μ_H *and the coassociativity of the coproduct* δ_H*.*

In a similar way to the Hopf algebra case, the antipode λ_H of a weak Hopf algebra H is unique, antimultiplicative ($\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}$), anti-comultiplicative ($\delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H$) and leaves the unit η_H and the counit ε_H invariable ($\lambda_H \circ \eta_H = \eta_H$, $\varepsilon_H \circ \lambda_H = \varepsilon_H$).

Moreover, we can define the idempotent morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\begin{aligned}
\Pi_H^L &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H);\\
\Pi_H^R &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H));\\
\overline{\Pi}_H^L &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H);\\
\overline{\Pi}_H^R &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H));
\end{aligned}$$

which satisfy the equalities $\Pi_H^L = id_H * \lambda_H$, $\Pi_H^R = \lambda_H * id_H$ and then $\Pi_H^L * \Pi_H^L = \Pi_H^L$, $\Pi_H^R * \Pi_H^R = \Pi_H^R$. In what follows we denote by H_L the image of the target morphism and by p_L and i_L the morphisms such that $i_L \circ p_L = \Pi_H^L$ and $p_L \circ i_L = id_{H_L}$.

Finally, we have that (see [6]),

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L; \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R; \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L; \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R; \tag{1}$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L; \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R; \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L; \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R. \tag{2}$$

Definition 2.2. *Let* H *be a weak bialgebra and let* A *be an algebra with coaction* $\rho_A : A \to A \otimes H$ *such that* (A, ρ_A) *is a right* H*-comodule satisfying the equality* $\mu_{A\otimes H} \circ (\rho_A \otimes \rho_A) = \rho_A \circ \mu_A$*. The object* (A, ρ_A) *is called a right* H*-comodule algebra if one of the following equivalent conditions holds (see* [8]*, Proposition 4.10):*

(b1) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes (\delta_H \circ \eta_H))$,

(b2) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ ((\rho_A \circ \eta_A) \otimes (\delta_H \circ \eta_H))$,

(b3) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$,

(b4) $(A \otimes \Pi_H^L) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A)$,

(b5) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$,

(b6) $(A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

For a right H-comodule algebra A we define $\Gamma_A^H : H\otimes A\to A\otimes H$ as $\Gamma_A^H = (A\otimes\mu_H)\circ(c_{H,A}\otimes H)\circ(H\otimes\rho_A)$. Then the triple (A,H,Γ_A^H) is a right-right weak entwining structure (see [8], Theorem 4.14), i.e., it satisfies

$$\Gamma_A^H\circ(H\otimes\mu_A) = (\mu_A\otimes H)\circ(A\otimes\Gamma_A^H)\circ(\Gamma_A^H\otimes A), \tag{3}$$

$$(A\otimes\delta_H)\circ\Gamma_A^H = (\Gamma_A^H\otimes H)\circ(H\otimes\Gamma_A^H)\circ(\delta_H\otimes A), \tag{4}$$

$$\Gamma_A^H\circ(H\otimes\eta_A) = (e_A\otimes H)\circ\delta_H, \tag{5}$$

$$(A\otimes\varepsilon_H)\circ\Gamma_A^H = \mu_A\circ(e_A\otimes A), \tag{6}$$

where

$$e_A = (A\otimes\varepsilon_H)\circ\Gamma_A^H\circ(H\otimes\eta_A). \tag{7}$$

Let A and H be fixed. We denote by $\mathcal{M}_A^H(\Gamma_A^H)$ the category of right-right weak entwined modules, i.e., the objects M in $\mathcal{C}$ together with two morphisms $\phi_M : M\otimes A\to A$ and $\rho_M : M\to M\otimes H$ such that (M,ϕ_M) is a right A-module, (M,ρ_M) is a right H-comodule and the following equality

$$\rho_M\circ\phi_M = (\phi_M\otimes H)\circ(M\otimes\Gamma_A^H)\circ(\rho_M\otimes A) \tag{8}$$

holds. Obviously, if (A,ρ_A) is a right H-comodule algebra, (A,μ_A,ρ_A) is an object of $\mathcal{M}_A^H(\Gamma_A^H)$.

Let (A,ρ_A) be a right H-comodule algebra. We define the subalgebra of coinvariants of A by the equalizer:

$$\begin{tikzcd} A_H \arrow[r, "i_A"] & A \arrow[r, shift left, "\rho_A"] \arrow[r, shift right, "\zeta_A"'] & A\otimes H \end{tikzcd}$$

where $\zeta_A = (\mu_A\otimes H)\circ(A\otimes(\rho_A\circ\eta_A))$. Note that, by (b3), $\zeta_A = (A\otimes\overline{\Pi}_H^R)\circ\rho_A$ and also, by (1) and (2), (A_H, i_A) is the equalizer of ρ_A and $(A\otimes\Pi_H^L)\circ\rho_A$.

It is not difficult to see that $(A_H,\eta_{A_H},\mu_{A_H})$ is an algebra, being η_{A_H} and μ_{A_H} the factorizations through the equalizer i_A of the morphisms η_A and $\mu_A\circ(i_A\otimes i_A)$, respectively. As a consequence, $\varphi_A = \mu_A\circ(i_A\otimes A)$ (respectively $\phi_A = \mu_A\circ(A\otimes i_A)$) defines a left (right) A_H-module structure for A.

Note that the weak Hopf algebra H is a right H-comodule algebra with comodule structure giving by $\rho_H = \delta_H$ and subalgebra of coinvariants $H_H = H_L$. In this case $i_H = i_L$.

The morphism $\Delta_{A\otimes H} = (\mu_A\otimes H)\circ(A\otimes\Gamma_A^H)\circ(A\otimes H\otimes\eta_A) : A\otimes H\to A\otimes H$ is an idempotent and, as a consequence, there exist an object $A\square H$ and morphisms $i_{A\otimes H} : A\square H\to A\otimes H$, $p_{A\otimes H} : A\otimes H\to A\square H$ such that $\Delta_{A\otimes H} = i_{A\otimes H}\circ p_{A\otimes H}$ and $id_{A\square H} = p_{A\otimes H}\circ i_{A\otimes H}$. Moreover $A\square H$ is a right A-module, where the action is defined by $\phi_{A\square H} = p_{A\otimes H}\circ(\mu_A\otimes H)\circ(A\otimes\Gamma_A^H)\circ(i_{A\otimes H}\otimes A)$, and a right H-comodule, with coaction $\rho_{A\square H} = (p_{A\otimes H}\otimes H)\circ(A\otimes\delta_H)\circ i_{A\otimes H}$, and $(A\square H, \phi_{A\square H}, \rho_{A\square H})$ is a weak entwined module and a left A-module with action $\varphi_{A\square H} = p_{A\otimes H}\circ(\mu_A\otimes H)\circ(A\otimes i_{A\otimes H})$. On the other hand, the equality

$$\Delta_{A\otimes H} = (\mu_A\otimes H)\circ(A\otimes((e_A\otimes H)\circ\delta_H)), \tag{9}$$

comes directly from (5).

The morphism (lifted canonical morphism) $r_A = p_{A\otimes H}\circ(\mu_A\otimes H)\circ(A\otimes\rho_A) : A\otimes A\to A\square H$ factorizes through the coequalizer morphism $q_{A,A} : A\otimes A\to A\otimes_{A_H}A$ of the morphisms $\theta^1_{A,A} = A\otimes\varphi_A$ and $\theta^2_{A,A} = \phi_A\otimes A$. As a consequence, there exists a unique morphism, called the canonical morphism, $\gamma_A : A\otimes_{A_H} A\to A\square H$ such that $\gamma_A\circ q_{A,A} = r_A$. Further, r_A and γ_A are morphisms of right H-comodules being $\rho_{A\otimes A} = A\otimes\rho_A$ and $\rho_{A\otimes_{A_H}A}$ the factorization of $(q_{A,A}\otimes H)\circ(A\otimes\rho_A)$ through the coequalizer $q_{A,A}$. If the functor $A\otimes -$ preserves coequalizers, γ_A is a morphism of left A-modules where $\varphi_{A\otimes_B A}$ is the factorization of $q_{A,A}\circ(\mu_A\otimes A)$ through the coequalizer $A\otimes q_{A,A}$. Finally, γ_A is a morphism of right A-modules where $\phi_{A\otimes_B A}$ is the factorization of $q_{A,A}\circ(A\otimes\mu_A)$ through the coequalizer $q_{A,A}\otimes A$.

Definition 2.3. *If the functor $A\otimes -$ preserves coequalizers, we say that $A_H\hookrightarrow A$ is a weak H-Galois extension if the canonical morphism γ_A is an isomorphism.*

Note that, if $\mathcal{C}$ is a closed category, the functor $A\otimes -$ preserves coequalizers. Also, if A is a finite object, i.e., there exists an object A^* and an adjunction $A\otimes - \dashv A^*\otimes -$, we have that $A\otimes -$ preserves coequalizers.

Let H be a weak Hopf algebra and let (A,ρ_A) be a right H-comodule algebra. In Definition 1.8 of [1] we introduce the set $Reg^{WR}(H,A)$ as the one whose elements are the morphisms $h : H\to A$ such that there exists $h^{-1} : H\to A$, called the left weak inverse of h, satisfying $h^{-1}*h = e_A$ where e_A is the morphism defined in (7) for the right-right weak entwining structure Γ_A^H associated to (A,ρ_A).

Definition 2.4. *We say that $A_H\hookrightarrow A$ is a weak H-cleft extension if there exists a morphism $h : H\to A$ in $Reg^{WR}(H,A)$ (called the cleaving morphism) of right H-comodules such that*

$$\Gamma_A^H\circ(H\otimes h^{-1})\circ\delta_H = \zeta_A\circ(e_A * h^{-1}). \tag{10}$$

Also, by (2.9) of [5], we can assume without loss of generality that $e_A*h^{-1} = h^{-1}$ and as a consequence (10) can be expressed as

$$\Gamma_A^H\circ(H\otimes h^{-1})\circ\delta_H = \zeta_A\circ h^{-1}. \tag{11}$$

Then, if the extension $A_H\hookrightarrow A$ is weak H-cleft, by Proposition 1.12 of [1], we get that $q_A = \mu_A\circ(A\otimes h^{-1})\circ\rho_A : A\to A$ factorizes through i_A. Therefore, there exists a unique morphism $p_A : A \to A_H$ such that $q_A = i_A \circ p_A$. Then, $h * h^{-1} = q_A \circ h$ and, as a consequence, $h * h^{-1}$ admits a factorization through i_A.

Now we recall the definition of H-Galois extension with normal basis we formulate in [2].

Definition 2.5. *A weak H-Galois extension $A_H \hookrightarrow A$ has a normal basis if there exists an idempotent morphism of left A_H-modules and right H-comodules $\Omega_{A_H\otimes H} : A_H \otimes H \to A_H \otimes H$ ($\varphi_{A_H\otimes H} = \mu_{A_H} \otimes H$, $\rho_{A_H\otimes H} = A_H \otimes \delta_H$) and an isomorphism of left A_H-modules and right H-comodules $b_A : A \to A_H \times H$, where $A_H \times H$ is the image of $\Omega_{A_H\otimes H}$ and $\varphi_{A_H\times H} = p_{A_H,H} \circ \varphi_{A_H\otimes H} \circ (A_H \otimes i_{A_H,H})$, $\rho_{A_H\times H} = (p_{A_H,H} \otimes H) \circ \rho_{A_H\otimes H} \circ i_{A_H,H}$, being $i_{A_H,H} : A_H \times H \to A_H \otimes H$ and $p_{A_H,H} : A_H \otimes H \to A_H \times H$ the morphisms such that $i_{A_H,H} \circ p_{A_H,H} = \Omega_{A_H\otimes H}$ and $p_{A_H,H} \circ i_{A_H,H} = id_{A_H\times H}$.*

For a weak H-Galois extension with normal basis, if we define $\omega_A = b_A^{-1} \circ p_{A_H,H} : A_H \otimes H \to A$ and $\omega'_A = i_{A_H,H} \circ b_A : A \to A_H \otimes H$, the morphism $\omega'_A \circ \omega_A = \Omega_{A_H\otimes H}$, $\omega_A \circ \omega'_A = id_A$ and $m'_A = \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ \omega'_A)) : A \otimes A \to A$ factorizes through the coequalizer $q_{A,A}$. Then there exists a unique morphism of left A-modules $m_A : A \otimes_{A_H} A \to A$ such that

$$m_A \circ q_{A,A} = \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ \omega'_A)) \tag{12}$$

(see Lemma 1.9 of [2]). Note that, in these conditions, we have that $p_{A_H,H}$, $i_{A_H,H}$, ω_A and ω'_A are also morphisms of left A_H-modules and right H-comodules.

As we have showed in [1], there is a close connection between weak H-cleft extensions and weak H-Galois extensions with normal basis. More precisely, the main result in [1] establishes that, if $A \otimes -$ preserves coequalizers, $A_H \hookrightarrow A$ is a weak H-cleft extension if and only if $A_H \hookrightarrow A$ is a weak H-Galois extension with normal basis. For clarity we briefly review the proof:

Let $A_H \hookrightarrow A$ be a weak H-Galois extension with normal basis. We define the cleaving morphism $h_A = \omega_A \circ (\eta_{A_H} \otimes H) : H \to A$ and its left weak inverse is $h_A^{-1} = m_A \circ \gamma_A^{-1} \circ p_{A\otimes H} \circ (\eta_A \otimes H) : H \to A$. Note that in this part of the proof we obtain that $m_A \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ \rho_A = ((i_C^A \otimes \varepsilon_C) \circ \omega'_A$ and

$$h_A * h_A^{-1} = (i_A \otimes \varepsilon_H) \circ \Omega_{A_H\otimes H} \circ (\eta_{A_H} \otimes H). \tag{13}$$

Conversely, if $A_H \hookrightarrow A$ is a weak H-cleft extension with cleaving morphism h, the morphisms of left A_H-modules and right H-comodules defined by $\omega_A = \mu_A \circ (i_A \otimes h)$ and $\omega'_A = (p_A \otimes H) \circ \rho_A$ satisfy the equality $\omega_A \circ \omega'_A = id_A$. As a consequence, the morphism $\Omega_{A_H\otimes H} = \omega'_A \circ \omega_A$ is idempotent and we have a commutative diagram

$$\begin{tikzcd} & A \arrow[dr, "\omega'_A"] & \\ A_H \otimes H \arrow[ur, "\omega_A"] \arrow[rr, "\Omega_{A_H\otimes H}"] \arrow[dr, "p_{A_H,H}"'] & & A_H \otimes H \\ & A_H \times H \arrow[ur, "i_{A_H,H}"'] & \end{tikzcd}$$

where $p_{A_H,H} \circ i_{A_H,H} = id_{A_H\times H}$. Therefore, the morphism $b_A = p_{A_H,H} \circ \omega'_A$ is an isomorphism of right H-comodules and left A_H-modules with inverse $b_A^{-1} = \omega_A \circ i_{A_H,H}$. Moreover, the inverse of the canonical morphism γ_A is $\gamma_A^{-1} = q_{A,A} \circ (\mu_A \otimes A) \circ (A \otimes h_A^{-1} \otimes h_A) \circ (A \otimes \delta_H) \circ i_{A\otimes H}$.

In the second section of [5], we introduce the notion of H-cleft extension for a weak Hopf algebra H and we prove that this kind of extensions are examples of weak H-cleft extensions. To define H-cleft extensions we need convolution invertible integrals. As in the Hopf setting, for a weak Hopf algebra H and a right H-comodule algebra (A, ρ_A), an integral is a morphism of right H-comodules $f : H \to A$. If moreover $f \circ \eta_H = \eta_A$ we will say that the integral is total.

An integral $f : H \to A$ is convolution invertible if there exists a morphism $f^{-1} : H \to A$ (called the convolution inverse of f) such that

(c1) $f^{-1} * f = e_A$.
(c2) $f * f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$.
(c3) $f^{-1} * f * f^{-1} = f^{-1}$.

Trivially, the inverse is unique and we get that $f * f^{-1} * f = f$ (see Definition 2.4 of [5]). Note that, when f is a total integral, we can rewrite (c1) as $f^{-1} * f = f \circ \Pi_H^R$ and (c2) as $f * f^{-1} = f \circ \overline{\Pi}_H^L$.

Definition 2.6. *We say that $A_H \hookrightarrow A$ is an H-cleft extension if there exists a convolution invertible integral $h : H \to A$ such that the morphism $h * h^{-1}$ factorizes through the equalizer i_A.*

Obviously, $H_L \hookrightarrow H$ is a weak H-cleft extension with $h = id_H$ and $h^{-1} = \lambda_H$. By Proposition 2.2 of [5] we know that if H is a cocommutative weak Hopf algebra and there exists a convolution invertible integral $f : H \to A$ then $A_H \hookrightarrow A$ is an H-cleft extension. Also, by Corollary 2.1 of [5], we have that an H-cleft extension is also a weak H-cleft extension. Finally, Proposition 2.3 of [5] asserts that, if the antipode of H is an isomorphism and $A_H \hookrightarrow A$ is an H-cleft extension with convolution invertible integral f, then $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H)))$ is a total integral. Moreover, if H is cocommutative h is convolution invertible.

In the following definition we introduce the notion of H-Galois extension with normal basis.

Definition 2.7. *Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra such that $A \otimes -$ preserves coequalizers. We say that a weak H-Galois extension with normal basis $A_H \hookrightarrow A$ is an H-Galois extension with normal basis if the following identities hold:*

(d1) $b_A \circ \eta_A = p_{A_H,H} \circ (\eta_{A_H} \otimes \eta_H)$.
(d2) $((b_A \circ i_A) \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes H) = p_{A_H,H} \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)$.

3. Galois and Cleft extensions and cohomology

In this section we give the main results of the paper and a cohomological interpretation of Cleft extensions.

Lemma 3.1. *Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra such that $A\otimes -$ preserves coequalizers. If $A_H \hookrightarrow A$ is an H-Galois extension with normal basis the following equality holds:*

$$(A_H \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (A_H \otimes \eta_H) = id_{A_H}. \tag{14}$$

Proof. First of all, note that by the definition of the morphism $\overline{\Pi}_H^L$ and the properties of the (co)unit η_H (ε_H), it is easy to see that $\overline{\Pi}_H^L \circ \eta_H = \eta_H$. Now, by composing (d2) of Definition 2.7 with η_H and using (d1) we have that

$$((b_A \circ i_A) \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes \eta_H) = p_{A_H,H} \circ (\eta_{A_H} \otimes \eta_H) = b_A \circ \eta_A.$$

Therefore,

$$(i_A \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes \eta_H) = \eta_A \tag{15}$$

holds and as a consequence we have:

$$\begin{aligned}
& i_A \\
&= \mu_A \circ (i_A \otimes \eta_A) \\
&= ((\mu_A \circ (i_A \otimes i_A) \otimes \varepsilon_H) \circ (A_H \otimes (\Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes \eta_H))) \\
&= (i_A \otimes \varepsilon_H) \circ (\mu_{A_H} \otimes H) \circ (A_H \otimes (\Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes \eta_H))) \\
&= (i_A \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (A_H \otimes \eta_H).
\end{aligned}$$

where the first equality follows by the properties of the unit η_A, the second one by (15), the third one by the properties of μ_{A_H}, the fourth one because $\Omega_{A_H \otimes H}$ is a morphism of left A_H-modules.

Then, using that i_A is a monomorphism we conclude the proof. □

Theorem 3.2. *Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra such that $A \otimes -$ preserves coequalizers. The following are equivalent.*

(i) $A_H \hookrightarrow A$ *is an* H*-cleft extension that admits a convolution invertible total integral.*

(ii) $A_H \hookrightarrow A$ *is an* H*-Galois extension with normal basis.*

Proof. (i) $\Rightarrow$ (ii) By Corollary 2.1 of [5] we know that if $A_H \hookrightarrow A$ is an H-cleft extension with convolution invertible integral h then it is a weak H-cleft extension with cleaving morphism h. Therefore, by Theorem 2.11 of [2], we obtain that $A_H \hookrightarrow A$ is a weak H-Galois extension with normal basis where $\omega_A = \mu_A \circ (i_A \otimes h)$, $\omega'_A = (p_A \otimes H) \circ \rho_A$, $\Omega_{A_H \otimes H} = \omega'_A \circ \omega_A$, $b_A = p_{A_H,H} \circ \omega'_A$ and $b_A^{-1} = \omega_A \circ i_{A_H,H}$. Moreover, for $\Omega_{A_H \otimes H}$ we have the following identity:

$$\Omega_{A_H \otimes H} = ((p_A \circ \mu_A \circ (i_A \otimes h)) \otimes H) \circ (A_H \otimes \delta_H). \tag{16}$$

Indeed:

$$\begin{aligned}
&\Omega_{A_H \otimes H} \\
&= (p_A \otimes H) \circ \mu_{A \otimes H} \circ ((\rho_A \circ i_A) \otimes (\rho_A \circ h)) \\
&= (p_A \otimes H) \circ \mu_{A \otimes H} \circ ((\zeta_A \circ i_A) \otimes (\rho_A \circ h)) \\
&= ((p_A \circ \mu_A) \otimes H) \circ (i_A \otimes (\mu_{A \otimes H} \circ ((\rho_A \circ \eta_A) \otimes (\rho_A \circ h)))) \\
&= ((p_A \circ \mu_A) \otimes H) \circ (i_A \otimes (\rho_A \circ h)) \\
&= ((p_A \circ \mu_A) \otimes H) \circ (i_A \otimes ((h \otimes H) \circ \delta_H)) \\
&= ((\mu_{A_H} \circ (A_H \otimes (p_A \circ h))) \otimes H) \circ (A_H \otimes \delta_H),
\end{aligned}$$

where the first equality follows by the structure of right H-comodule algebra of A, the second one by the definition of A_H, the third one by the associativity of μ_A, the fourth one by the structure of right H-comodule algebra of A and the properties of the unity η_A. In the fifth equality we used that h is a right H-comodule morphism and finally, the last one follows by (e1) of Lemma 3.11 of [3].

Then,

$$\begin{aligned}
&i_{A_H,H} \circ b_A \circ \eta_A \\
&= \Omega_{A_H \otimes H} \circ (p_A \otimes H) \circ \rho_A \circ \eta_A \\
&= ((p_A \circ \mu_A \circ (i_A \otimes h)) \otimes H) \circ (p_A \otimes \delta_H) \circ \rho_A \circ \eta_A \\
&= ((p_A \circ \mu_A \circ (q_A \otimes h) \circ \rho_A) \otimes H) \circ \rho_A \circ \eta_A \\
&= (p_A \otimes H) \circ \rho_A \circ \eta_A,
\end{aligned}$$

where the first equality follows by the definition of b_A, the second one by (16), the third one by the properties of ρ_A, the fourth one by the equality (d2) of Lemma 3.9 of [3], i.e. $\mu_A \circ (q_A \otimes h) \circ \rho_A = id_A$. On the other hand, using (16), the properties of η_{A_H} and the condition of total integral for h we have

$$\begin{aligned}
&i_{A_H,H} \circ p_{A_H,H} \circ (\eta_{A_H} \otimes \eta_H) \\
&= \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes \eta_H) \\
&= (p_A \otimes H) \circ \rho_A \circ h \circ \eta_H
\end{aligned}$$

$$= (p_A \otimes H) \circ \rho_A \circ \eta_A.$$

Therefore, we get (d1), because $i_{A_H,H}$ is a monomorphism. Moreover,

$$\begin{aligned}
&((b_A \circ i_A) \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes H)\\
&= b_A \circ i_A \circ p_A \circ h\\
&= b_A \circ q_A \circ h\\
&= b_A \circ (h * h^{-1})\\
&= b_A \circ h \circ \overline{\Pi}_H^L\\
&= b_A \circ \omega_A \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)\\
&= p_{A_H,H} \circ \omega_A' \circ \omega_A \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)\\
&= p_{A_H,H} \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)\\
&= p_{A_H,H} \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L),
\end{aligned}$$

and (d2) holds. Note that in the first equality we used (16), in the fourth one we applied that h is total and in the fifth one we use that $\omega_A \circ (\eta_{A_H} \otimes H) = h$.

Therefore, $A_H \hookrightarrow A$ is an H-Galois extension with normal basis.

(ii) $\Rightarrow$ (i) Conversely, assume that $A_H \hookrightarrow A$ is an H-Galois extension with normal basis. Then, $A_H \hookrightarrow A$ is a weak H-Galois extension with normal basis and by Theorem 2.11 of [2] is a weak H-cleft extension with cleaving morphism $h = \omega_A \circ (\eta_A \otimes H)$ and $h^{-1} = m_A \circ \gamma_A^{-1} \circ p_{A\otimes H} \circ (\eta_A \otimes H)$. Therefore $h^{-1} * h = e_A$. Also $h * h^{-1}$ factorizes through the equalizer i_A.

By (13) and (d2) of Definition 2.7,

$$\begin{aligned}
&h * h^{-1}\\
&= (i_A \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes H)\\
&= b_A^{-1} \circ p_{A_H,H} \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)\\
&= \omega_A \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)\\
&= h \circ \overline{\Pi}_H^L.
\end{aligned}$$

On the other hand, using that b_A and $\Omega_{A_H \otimes H}$ are morphisms of right H-comodules and (d1) of Definition 2.7 we have:

$$\begin{aligned}
&(A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= ((b_A^{-1} \circ b_A) \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= (b_A^{-1} \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_{A_H \times H} \circ b_A \circ \eta_A) \otimes H)\\
&= ((b_A^{-1} \circ p_{A_H,H}) \otimes (\varepsilon_H \circ \mu_H)) \circ (((A_H \otimes \delta_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes \eta_H)) \otimes H)\\
&= b_A^{-1} \circ p_{A_H,H} \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)\\
&= \omega_A \circ (\eta_{A_H} \otimes \overline{\Pi}_H^L)\\
&= h \circ \overline{\Pi}_H^L,
\end{aligned}$$

where the third equality uses the definitions of $\Omega_{A_H \otimes H}$ and $\rho_{A_H \times H}$.

Therefore, $h * h^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$ and $h * h^{-1} = h \circ \overline{\Pi}_H^L$. As a consequence,

$$h \circ \eta_H = h \circ \overline{\Pi}_H^L \circ \eta_H = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes \eta_H) = \eta_A,$$

and h is a total integral.

To finish the proof it only remains to show that $h^{-1} * h * h^{-1} = h^{-1}$. First we proceed by showing the identity

$$m_A = \mu_A \circ (m_A \otimes (h \circ \overline{\Pi}_H^L)) \circ \rho_{A \otimes_{A_H} A}. \tag{17}$$

Indeed, composing with the coequalizer $q_{A,A}$ we have that

$$\begin{aligned}
&\mu_A \circ (m_A \otimes (h \circ \overline{\Pi}_H^L)) \circ \rho_{A \otimes_{A_H} A} \circ q_{A,A} \\
&= \mu_A \circ ((m_A \circ q_{A,A}) \otimes (h \circ \overline{\Pi}_H^L)) \circ (A \otimes \rho_A) \\
&= \mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((i_A \otimes \varepsilon_H \otimes (h \circ \overline{\Pi}_H^L)) \circ (\omega'_A \otimes H) \circ \rho_A)) \\
&= \mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((i_A \otimes (\varepsilon_H \otimes (h \circ \overline{\Pi}_H^L) \circ \delta_H)) \circ \omega'_A)) \\
&= \mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((i_A \otimes (h \circ \overline{\Pi}_H^L)) \circ \omega'_A)) \\
&= \mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((i_A \otimes ((i_A \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes H))) \circ \omega'_A)) \\
&= \mu_A \circ (A \otimes i_A) \circ (A \otimes ((\mu_{A_H} \otimes \varepsilon_H) \circ (A_H \otimes (\Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes H))) \circ \omega'_A)) \\
&= \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ \omega'_A)) \\
&= \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ \omega'_A)) \\
&= m_A \circ q_{A,A},
\end{aligned}$$

and then (17) holds. In the last equalities, the first one follows by the definition of $\rho_{A \otimes_{A_H} A}$, the second one by (12) and in the third one we used that ω'_A is a morphism of right H-comodules. The fourth equality follows by the properties of the counit and the fifth one by the identity

$$h \circ \overline{\Pi}_H^L = (i_A \otimes \varepsilon_H) \circ \Omega_{A_H \otimes H} \circ (\eta_{A_H} \otimes H), \tag{18}$$

obtained in the proof that $h * h^{-1} = h \circ \overline{\Pi}_H^L$. The sixth one relies on the definition of μ_{A_H}, in the seventh one we applied that $\Omega_{A_H \otimes H}$ is a morphism of left A_H-modules and in the eighth one we used that $\Omega_{A_H \otimes H} = \omega'_A \circ \omega_A$ and $\omega_A \circ \omega'_A = id_A$. Finally, the last one follows by (12).

Therefore,

$$\begin{aligned}
&h^{-1} * h * h^{-1} \\
&= \mu_A \circ (h^{-1} \otimes (h \circ \overline{\Pi}_H^L)) \circ \delta_H \\
&= \mu_A \circ ((m_A \circ \gamma_A^{-1} \circ p_{A \otimes H}) \otimes (h \circ \overline{\Pi}_H^L)) \circ (A \otimes \delta_H) \circ \Delta_{A \otimes H} \circ (\eta_A \otimes H) \\
&= \mu_A \circ ((m_A \circ \gamma_A^{-1}) \otimes (h \circ \overline{\Pi}_H^L)) \circ \rho_{A \square H} \circ p_{A \otimes H} \circ (\eta_A \otimes H) \\
&= \mu_A \circ (m_A \otimes (h \circ \overline{\Pi}_H^L)) \circ \rho_{A \otimes_{A_H} A} \circ \gamma_A^{-1} \circ p_{A \otimes H} \circ (\eta_A \otimes H) \\
&= m_A \circ \gamma_A^{-1} \circ p_{A \otimes H} \circ (\eta_A \otimes H) \\
&= h^{-1},
\end{aligned}$$

where the first equality follows using that $h*h^{-1}=h\circ\overline{\Pi}_H^L$, the second one because $\Delta_{A\otimes H}$ is a morphism of right H-comodules, the third one by the definition of $\rho_{A\square H}$ and in the fourth one we applied that γ_A^{-1} is a morphism of right H-comodules. Finally, the fifth equality follows by (17) and the last one by the definition of h^{-1}. $\square$

As a consequence, by Propositions 2.2 and 2.3 of [5], we have the following corollary.

Corollary 3.3. *Let H be a cocommutative weak Hopf algebra and let (A,ρ_A) be a right H-comodule algebra such that $A\otimes -$ preserves coequalizers. The following are equivalent.*

(i) *$A_H\hookrightarrow A$ is an H-cleft extension.*
(ii) *$A_H\hookrightarrow A$ is an H-Galois extension with normal basis.*

To finish this paper, we will give a cohomological interpretation of Cleft extensions. For clarity, we briefly describe the construction of the cohomology groups in the weak setting. The interested reader can find the details in [4]. Assume that H is a cocommutative weak Hopf algebra and let (A,φ_A) be a commutative weak left H-module algebra. Let H^0 be the unit object of $\mathcal{C}$ and for $n\geq 1$ denote by H^n the n-fold tensor power $H\otimes\cdots\otimes H$. If $n\geq 2$, m_H^n denotes the morphism $m_H^n: H^n\to H$ defined by $m_H^2=\mu_H$ and by $m_H^3=m_H^2\circ(H\otimes\mu_H),\cdots,m_H^n=m_H^{n-1}\circ(H^{n-2}\otimes\mu_H)$ for $k>2$. Analogously, with δ_{H^n} we denote the coproduct defined for the coalgebra H^n. Finally, φ_A^n will be the morphism $\varphi_A^n: H^n\otimes A\to A$ defined as $\varphi_A^1=\varphi_A$ and $\varphi_A^n=\varphi_A\circ(H\otimes\varphi_A^{n-1})$. For brevity, we denote the morphisms $\varphi_A\circ(m_H^n\otimes\eta_A)$ and $\varphi_A\circ(H\otimes\eta_A)$ by u_n and u_1, respectively.

For $n\geq 1$, let $Reg_{\varphi_A}(H^n,A)$ be the set of morphisms $\sigma: H^n\to A$ such that there exists a morphism $\sigma^{-1}: H^n\to A$ (the convolution inverse of σ) satisfying the following equalities:

(c1) $\sigma\wedge\sigma^{-1}=\sigma^{-1}\wedge\sigma=u_n$.
(c2) $\sigma\wedge\sigma^{-1}\wedge\sigma=\sigma$.
(c3) $\sigma^{-1}\wedge\sigma\wedge\sigma^{-1}=\sigma^{-1}$,

and $Reg_{\varphi_A}(H_L,A)$ will be the set of morphisms $g: H_L\to A$ such that there exists a morphism $g^{-1}: H_L\to A$ satisfying

$$g\wedge g^{-1}=g^{-1}\wedge g=u_0,\;\; g\wedge g^{-1}\wedge g=g,\;\; g^{-1}\wedge g\wedge g^{-1}=g^{-1}$$

where $u_0=u_1\circ i_L$.

The sets $Reg_{\varphi_A}(H_L,A)$, $Reg_{\varphi_A}(H^n,A)$ are abelian groups with neutral elements u_0 and u_n respectively. Moreover, we can define a cosimplicial complex of abelian groups with coface operators defined by

$$\partial_{0,i} : Reg_{\varphi_A}(H_L, A) \to Reg_{\varphi_A}(H, A), \ \ i \in \{0,1\}$$

$$\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_H^R)) \circ \delta_H, \ \ \partial_{0,1}(g) = g \circ p_L,$$

$$\partial_{k-1,i} : Reg_{\varphi_A}(H^{k-1}, A) \to Reg_{\varphi_A}(H^k, A), \ \ k \geq 2, \ \ i \in \{0, 1, \cdots, k\}$$

$$\partial_{k-1,i}(\sigma) = \begin{cases} \varphi_A \circ (H \otimes \sigma), \ \ i = 0 \\ \\ \sigma \circ (H^{i-1} \otimes \mu_H \otimes H^{k-i-1}), \ \ i \in \{1, \cdots, k-1\} \\ \\ \sigma \circ (H^{k-2} \otimes (\mu_H \circ (H \otimes \Pi_H^L))), \ \ i = k, \end{cases}$$

and codegeneracy operators defined by $s_{1,0} : Reg_{\varphi_A}(H, A) \to Reg_{\varphi_A}(H_L, A)$,

$$s_{1,0}(h) = h \circ i_L,$$

and $s_{k+1,i} : Reg_{\varphi_A}(H^{k+1}, A) \to Reg_{\varphi_A}(H^k, A), \ \ k \geq 1, \ \ i \in \{0, 1, \cdots, k\}$

$$s_{k+1,i}(\sigma) = \sigma \circ (H^i \otimes \eta_H \otimes H^{k-i}).$$

Let $D^k_{\varphi_A} = \partial_{k,0} \wedge \partial^{-1}_{k,1} \wedge \cdots \wedge \partial^{(-1)^{k+1}}_{k,k+1}$ be the coboundary morphisms of the cochain complex

$$Reg_{\varphi_A}(H_L, A) \stackrel{D^0_{\varphi_A}}{\longrightarrow} Reg_{\varphi_A}(H, A) \stackrel{D^1_{\varphi_A}}{\longrightarrow} Reg_{\varphi_A}(H^2, A) \stackrel{D^2_{\varphi_A}}{\longrightarrow} \cdots$$
$$\cdots \stackrel{D^{k-1}_{\varphi_A}}{\longrightarrow} Reg_{\varphi_A}(H^k, A) \stackrel{D^k_{\varphi_A}}{\longrightarrow} Reg_{\varphi_A}(H^{k+1}, A) \stackrel{D^{k+1}_{\varphi_A}}{\longrightarrow} \cdots$$

associated to the cosimplicial complex $Reg_{\varphi_A}(H^\bullet, A)$.

Then, $(Reg_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})$ gives the Sweedler cohomology of H in (A, φ_A). Therefore, the kth group will be defined by

$$H^k_{\varphi_A}(H, A) = \frac{Ker(D^k_{\varphi_A})}{Im(D^{k-1}_{\varphi_A})}$$

for $k \geq 1$ and $Ker(D^0_{\varphi_A})$ for $k = 0$.

Remark 3.4. *Let H be a weak Hopf algebra. Two H-cleft extensions $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent if $A_H = B_H$ and there exists a morphism of right H-comodule algebras T such that $T \circ i_A = i_B$. Under these conditions T is an isomorphism (see Definition 2.13 of* [5]*). If H is cocommutative and h is a convolution invertible total integral for an H-cleft extension $A_H \hookrightarrow A$, by Proposition 2.18 of* [5]*, we know that $(A_H, \varphi_{A_H} = p_A \circ \mu_A \circ (h \otimes i_A))$ is a left H-module algebra and, if $\sigma_{A_H} = p_A \circ \mu_A \circ (h \otimes h)$, the pair $(\varphi_{A_H}, \sigma_{A_H})$ is a crossed system for H over A_H (see Definition 3.2 and Theorem 3.2 of* [5]*). If we denote by $Cleft(A_H, \varphi_{A_H})$ the set of equivalence classes of H-cleft extensions with a fixed structure φ_{A_H}, by Corollary 3.3 of* [5]*, we obtain that there exists a bijective correspondence between*

$Cleft(A_H, \varphi_{A_H})$ and the set of equivalence classes of crossed systems for H over A_H with a common action φ_{A_H}. Therefore, if A is commutative, there exists a bijective correspondence between the second cohomology group $H^2_{\varphi_{A_H}}(H, A_H)$, defined in [4], *and $Cleft(A_H, \varphi_{A_H})$ (see Theorem 4.1 of* [5]*). As a consequence, by Corollary 3.3, there exists a bijection between $H^2_{\varphi_{A_H}}(H, A_H)$ and the set of equivalence classes of H-Galois extensions with normal basis $A_H \hookrightarrow A$ such that for their associated convolution invertible total integrals h we have $\varphi_{A_H} = p_A \circ \mu_A \circ (h \otimes i_A)$ (the equivalence relation between H-Galois extensions with normal basis is the one induced by the equivalence relation between H-cleft extensions).*

For example, if H is cocommutative, H_L is commutative and $(H_L, \varphi_{H_L} = p_L \circ \mu_H \circ (H \otimes i_L))$ is a left H-module algebra. Therefore, there exists a bijection between $H^2_{\varphi_{H_L}}(H, H_L)$ and the set of equivalence classes of H-Galois extensions with normal basis $H_L \hookrightarrow A$ such that $\varphi_{H_L} = p_A \circ \mu_A \circ (h \otimes i_A)$. In the Hopf algebra setting, if H is a finite Hopf algebra, the last bijection is the isomorphism between the second cohomology group $H^2(H, K)$, introduced by Sweedler [14]*, and the group of isomorphism classes of Galois H-objects with normal basis.*

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