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Intuitionistic Fuzzy Metrics Deduced by Combination of Several Distance Criteria

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Abstract

In this paper we give some examples of intuitionistic fuzzy metrics in the sense of Park [17]. All the examples have been classified with respect to their construction. At the same time, most of the well-known intuitionistic fuzzy metrics are given in this paper. Also, some important intuitionistic fuzzy metrics, by the help of classical metrics and some definite special kinds of functions are shown.

1. INTRODUCTION AND PRELIMINARIES

Since the introduction of fuzzy sets by Zadeh [24] 1965, many authors have introduced the concept of fuzzy metric spaces in different ways [4,7,9,10]. Especially George and Veeremani[8] have introduced a notion of fuzzy metric spaces with the help of continuous t-norms. Later Gregori, Romaguera, Sapena and Morillas have made significant contributions to fuzzy metric spaces [11,12,13,14,18,19,20].

Park [17], using the idea of intuitionistic fuzzy sets which was introduced by Atanassov [3], has defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [8]. Later many authors have studied on intuitionistic fuzzy metric spaces [1,2,5,15,22,23].

When it is thought, it can be seen that there are limited examples about intuitionistic fuzzy metrics. In this paper we give some examples of intuitionistic fuzzy metrics in the sense of Park [17]. We show that intuitionistic fuzzy metrics can be attained by using different distance criterias. This situation implies that it is easy to combine different distance criteria that may originally be in quite different ranges, but intuitionistic fuzzy metrics take away a common range which is the interval [0,1]. Therefore, the combination of several distance criteria may be done in an easy way.

We start by recalling some definitions and notions.

Definition 1. ([21]) A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

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Definition 2. ([21])A binary operation $\lozenge: [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\lozenge$ satisfies the following conditions:

(i) $\lozenge$ is commutative and associative;

(ii) $\lozenge$ is continuous;

(iii) $a \lozenge 1 = a$ for all $a \in [0,1]$;

(iv) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

Remark 1. ([17])

(a) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \lozenge r_2$.

(b) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \lozenge r_7 \leq r_5$.

Definition 3. ([17]) A 5-tuple $(X, M, N, *, \lozenge)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$;

(IFM-2) $M(x, y, t) > 0$;

(IFM-3) $M(x, y, t) = 1$ if and only if $x = y$;

(IFM-4) $M(x, y, t) = M(y, x, t)$;

(IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(IFM-6) $M(x, y, .): (0, \infty) \to [0,1]$ is continuous;

(IFM-7) $N(x, y, t) > 0$;

(IFM-8) $N(x, y, t) = 0$ if and only if $x = y$;

(IFM-9) $N(x, y, t) = N(y, x, t)$;

(IFM-10) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$

(IFM-11) $N(x, y, .): (0, \infty) \to [0,1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of nonnearness between x and y with respect to t, respectively.

Remark 2. (i) Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \lozenge)$ such that t-norm $*$ and t-conorm $\lozenge$ are associated ([16]), i.e. $x \lozenge y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in [0,1]$.

(ii) In intuitionistic fuzzy metric space X, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Example 1. ([17]) (Induced Intuitionistic Fuzzy Metric) Let (X, d) be a metric space. Denote $a * b = a.b$ and $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)} \ , N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \lozenge)$ is an intuitionistic fuzzy metric space.

Remark 3. Note the above example holds even with the t-norm $a * b = min\{a, b\}$ and the t-conorm $a \lozenge b = max\{a, b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = k = m = n = 1$, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Definition4. ([17]) Let $(X, M, N,*,\Diamond)$ be an intuitionistic fuzzy metric space, and let $r \in (0,1), t > 0$ and $x \in X$. The set

$$B_{(M,N)}(x,r,t) = \{y \in X: M(x,y,t) > 1 - r \text{ } and \text{ } N(x,y,t) < r\}$$

is called the open ball with center x and radius r with respect to t.

Theorem 1. ([17]) Every open ball $B_{(M,N)}(x,r,t)$ is an open set.

Remark 4. Let $(X, M, N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Define

$$\tau_{(M,N)} = \{A \subset X: \forall\, x \in A, \exists\, t > 0 \text{ } and \text{ } r \in (0,1) \ni \text{ } B_{(M,N)}(x,r,t) \subset A\}.$$

Then $\tau_{(M,N)}$ is a topology on X.

Remark 5. (i) Since $\{B_{(M,N)}(x, \frac{1}{n}, \frac{1}{n}): n = 1,2, \dots\}$ is a local base at x, the topology $\tau_{(M,N)}$ is first countable.

(ii) Let $(X, M, N,*,\Diamond)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n \to x$ if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$.

Theorem 2. ([17])Every intuitionistic fuzzy metric space is Hausdorff.

Definition 5. ([17]) Let $(X, M, N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Then,

(i) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$.

(ii) $(X, M, N,*,\Diamond)$ is called complete if every Cauchy sequence convergent with respect to $\tau_{(M,N)}$.

Definition 6. ([6]) Let $(X, M, N,*,\Diamond)$ be an intuitionistic fuzzy metric space. The intuitionistic fuzzy metric $(M, N,*,\Diamond)$ is said to be stationary if M and N don't depend on t, in other words the functions $M_{x,y}$ and $N_{x,y}$ are constant for each $x, y \in X$.

If $(X, M, N,*,\Diamond)$ is a stationary intuitionistic fuzzy metric space, we will denote $M(x,y)$, $N(x,y)$ and $B_{(M,N)}(x,r)$ instead of $M(x,y,t)$, $N(x,y,t)$ and $B_{(M,N)}(x,r,t)$, respectively.

2. MAIN RESULTS

Throughout this section $*$ and $\Diamond$ will be continuous t-norm and continuous t-conorm, respectively. At the same time X will be a nonempty set, $\mathbb{N}$ the positive integers, $\mathbb{R}^+$ the set of positive real numbers and M, N functions defined on $X \times X \times \mathbb{R}^+$ with values in $(0,1]$.

2.1 Generalizing well-known intuitionistic fuzzy metrics

Proposition 1. ([6]) Let $f: X \to \mathbb{R}^+$ be a one-to-one function and let $\varphi: \mathbb{R}^+ \to [0, +\infty)$ be an increasing continuous function. Fixed $\alpha, \beta > 0$, denote $a * b = ab$ and $a \Diamond b = a + b - ab$ for all $a, b \in [0,1]$, define M and N by

$$M(x,y,t) = \left(\frac{(min\{f(x), f(y)\})^{\alpha} + \varphi(t)}{(max\{f(x), f(y)\})^{\alpha} + \varphi(t)}\right)^{\beta}, N(x,y,t) = 1 - \left(\frac{(min\{f(x), f(y)\})^{\alpha} + \varphi(t)}{(max\{f(x), f(y)\})^{\alpha} + \varphi(t)}\right)^{\beta} \qquad (1)$$

Then, $(M, N,*,\Diamond)$ is an intuitionistic fuzzy metric on X.

Remark 6. Note that for above proposition, it is easy to say open ball $B_{(M,N)}(x,r,t)$ is

$$B_{(M,N)}(x,r,t) = \left\{y \in X: \left[(1-r)^{\frac{1}{\beta}}\left(f(x)^{\alpha} + \varphi(t)\right) - \varphi(t)\right]^{\frac{1}{\alpha}} \leq f(y) \leq \left[\frac{f(x)^{\alpha} + \varphi(t)}{(1-r)^{\frac{1}{\beta}}} - \varphi(t)\right]^{\frac{1}{\alpha}}\right\}$$

for all $r \in (0,1)$ and $t > 0$.

Now, if we take f as the corresponding identity function and $\alpha = \beta = 1$ then we obtain the next three examples as particular cases of this proposition.

Example 2. Let $X = \mathbb{R}^+$ and φ be the identity function. Then (1) becomes

$$M(x,y,t) = \frac{min\{x,y\} + t}{max\{x,y\} + t} \ , N(x,y,t) = \frac{max\{x,y\} - min\{x,y\}}{max\{x,y\} + t}$$

and so, $(M, N, *, \lozenge)$ is an intuitionistic fuzzy metric on $\mathbb{R}^+$. At the same time it is easy to see $\tau_{(M,N)}$ coincides with the usual topology of $\mathbb{R}$ relative to $\mathbb{R}^+$.

Example 3. Let $X = \mathbb{N}$ and $\varphi(t) = 0$ (zero function), for all $t > 0$. Then (1) becomes

$$M(x,y,t) = \frac{min\{x,y\}}{max\{x,y\}} \ , N(x,y,t) = \frac{max\{x,y\} - min\{x,y\}}{max\{x,y\}}$$

and so, $(M, N, *, \lozenge)$ is a stationary intuitionistic fuzzy metric on $\mathbb{N}$. Also, if we choose $0 < r < 1 - \frac{n}{n+1} (n \in \mathbb{N})$ the open ball $B_{(M,N)}(x, r, t)$ becomes $B_{(M,N)}(x, r, t) = \{n\}$. Then the topology $\tau_{(M,N)}$ is a discrete topology.

Example 4. Let $X = (-k, +\infty), k > 0$ and $\varphi(t) = k$ (constant function), for all $t > 0$. Then (1) becomes

$$M(x,y) = \frac{min\{x,y\} + k}{max\{x,y\} + k} \ , N(x,y) = \frac{max\{x,y\} - min\{x,y\}}{max\{x,y\} + k}$$

and so, $(M, N, *, \lozenge)$ is a stationary intuitionistic fuzzy metric on the interval $(-k, +\infty)$. Since the other conditions of intuitionistic fuzzy metric are clear, we will show only the condition (IFM-10).

(IFM-10) We show that $N(x,z) \le N(x,y) + N(y,z) - [N(x,y).N(y,z)]$, for all $x, y \in X = (-k, +\infty)$. Firsty we find that the equivalent of $N(x,y) \lozenge N(y,z) = N(x,y) + N(y,z) - [N(x,y).N(y,z)]$ and then we examine cases.

$$N(x,y) + N(y,z) - [N(x,y).N(y,z)] =$$

$$= \frac{kmax\{x,y\} - kmin\{x,y\} + max\{x,y\}max\{y,z\}}{[max\{x,y\} + k][max\{y,z\} + k]} + \frac{kmax\{y,z\} - kmin\{y,z\} - min\{x,y\}min\{y,z\}}{[max\{x,y\} + k][max\{y,z\} + k]}$$

$$= \frac{[max\{y,z\} + k][max\{x,y\} + k]}{[max\{x,y\} + k][max\{y,z\} + k]} - \frac{[min\{y,z\} + k][min\{x,y\} + k]}{[max\{x,y\} + k][max\{y,z\} + k]} \tag{2}$$

Now, supposed that $x \le z$. In such a case there are three cases:

Case 1. If $x \le y \le z$, by using (2)

$$N(x,y) + N(y,z) - [N(x,y).N(y,z)] = \frac{(y+k)(z+k) - (x+k)(y+k)}{(y+k)(z+k)} = 1 - \frac{(x+k)}{(y+k)}\frac{(y+k)}{(z+k)}$$

$$= \frac{z-x}{z+k} = \frac{max\{x,z\} - min\{x,z\}}{max\{x,z\} + k} = N(x,z)$$

Case 2. If $y \le x \le z$, by using (2)

$$N(x,y) + N(y,z) - [N(x,y).N(y,z)] = \frac{(x+k)(z+k) - (y+k)(y+k)}{(x+k)(z+k)} \ge 1 - \frac{(x+k)}{(x+k)}\frac{(x+k)}{(z+k)}$$

$$= \frac{z-x}{z+k} = \frac{max\{x,z\} - min\{x,z\}}{max\{x,z\} + k} = N(x,z)$$

Case 3. If $x \le z \le y$, by using (2)

$$N(x,y) + N(y,z) - [N(x,y).N(y,z)] = \frac{(y+k)(y+k) - (x+k)(z+k)}{(y+k)(y+k)}$$

$$= 1 - \frac{(x+k)}{(y+k)}\frac{(z+k)}{(y+k)} \geq 1 - \frac{(x+k)}{(z+k)}\frac{(z+k)}{(z+k)}$$

$$= \frac{z-x}{z+k} = \frac{max\{x,z\} - min\{x,z\}}{max\{x,z\}+k} = N(x,z).$$

Similar operations are performed if $z < x$.

With respect to the topology $\tau_{(M,N)}$ which induced by the (M,N), the open ball $B_{(M,N)}(x,r)$ is as follows.

$$B_{(M,N)_{x \leq y}}(x,r) = \left\{y \in X: \frac{x+k}{y+k} > 1-r, \frac{y-x}{y+k} < r\right\} = \left\{y \in X: y < \frac{x+kr}{1-r}\right\}$$

and

$$B_{(M,N)_{y \leq x}}(x,r) = \left\{y \in X: \frac{y+k}{x+k} > 1-r, \frac{x-y}{x+k} < r\right\} = \{y \in X: y > x - r(x+k)\}$$

and so

$$B_{(M,N)}(x,r) = \left(x - r(x+k), \frac{x+kr}{1-r}\right).$$

Then we can find $r \in (0,1)$, $t > 0$ such that

$$(x - r(x+k), \frac{x+kr}{1-r}) \subset (x-\delta, x+\delta)$$

for all $x \in X$, $\delta > 0$ and we can find $\delta > 0$ such that

$$(x-\delta, x+\delta) \subset (x - r(x+k), \frac{x+kr}{1-r})$$

for all $x \in X$, $r \in (0,1)$, $t > 0$. Consequently $\tau_{(M,N)}$ coincides with the usual topology of $\mathbb{R}$ relative to $(-k, +\infty)$.

It is easy to see that $(M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric with the continuous t-norm $a * b = min\{a,b\}$ and continuous t-conorm $a \lozenge b = max\{a,b\}$ for all $a, b \in [0,1]$ for the last three examples.

2.2 Intuitionistic fuzzy metrics defined by means of a metric

Proposition 2. Let $g: \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function, d is a metric on X, $m \in \mathbb{R}^+$ and $t, s > 0$. Define the functions M, N by

$$M(x,y,t) = \frac{g(t)}{g(t) + md(x,y)}, N(x,y,t) = \frac{md(x,y)}{g(t) + md(x,y)} \tag{3}$$

and $a * b = ab$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$. Then $(M, N, *, \lozenge)$ is an intuitionistic fuzzy metric on X.

Proof. We only show the condition (IFM-10).

(IFM-10) We show that $N(x,z,t+s) \leq N(x,y,t) + N(y,z,s) - N(x,y,t).N(y,z,s)$. N is a decreasing function. So indeed, since g is an increasing function, $g(t+s) \geq g(t)$ for all $t, s > 0$.

$$\Rightarrow g(t+s).m.d(x,y) \geq g(t).m.d(x,y)$$

$$\Rightarrow md(x,y)[md(x,y) + g(t+s)] \geq md(x,y)[md(x,y) + g(t)]$$

$$\Rightarrow \frac{md(x,y)}{[md(x,y) + g(t)]} \geq \frac{md(x,y)}{[md(x,y) + g(t+s)]}$$

$$\Rightarrow N(x,y,t) \geq N(x,y,t+s)$$

$$\Rightarrow N \text{ is decreasing} \tag{4}$$

At the same time $N(x,z,t) \le N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t)$ is satisfied for all $x,y,z \in X$ and $t > 0$. Indeed,

$$N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t) \ \ge 1 - \frac{g(t)^2}{g(t)^2 + mg(t)[d(x,y) + d(y,z)]}$$

$$\ge 1 - \frac{g(t)^2}{g(t)^2 + mg(t)d(x,z)}$$

$$= \frac{md(x,z)}{g(t) + md(x,z)} = N(x,z,t). \qquad (5)$$

From (4) and (5) $N(x,z,t+s) \le N(x,y,t) + N(y,z,s) - N(x,y,t).N(y,z,s)$.

Now we will give the next two examples as a particular cases of this proposition.

We will need the following lemma.

Lemma 1. ([18]) Let (X,d) be a metric space and $t,s > 0$. The following inequality holds, for all $n \ge 1$;

$$\frac{d(x,z)}{(t+s)^n} \le max\left\{\frac{d(x,y)}{t^n}, \frac{d(y,z)}{s^n}\right\}.$$

Example 5. As a particular case if we take $m = 1$ and $g(t) = t^n$ where $n \in \mathbb{N}$, in the above proposition (3) becomes

$$M(x,y,t) = \frac{t^n}{t^n + d(x,y)}, N(x,y,t) = \frac{d(x,y)}{t^n + d(x,y)}$$

so, $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = ab$, $a \lozenge b = a + b - ab$ for all $a,b \in [0,1]$.

Note that for this example $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = min\{a,b\}$, $a \lozenge b = max\{a,b\}$ for all $a,b \in [0,1]$. Now we will proof $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X for minimum t-norm and maximum t-conorm. We will show only the condition (IFM-10).

(IFM-10) We show that $N(x,z,t+s) \le max\{N(x,z,t), N(x,z,s)\}$ for all $x,y,z \in X$ and $t,s > 0$. By the previous lemma

$$1 + \frac{d(x,z)}{(t+s)^n} \le max\left\{1 + \frac{d(x,y)}{t^n}, 1 + \frac{d(y,z)}{s^n}\right\}$$

$$\Rightarrow \frac{(t+s)^n + d(x,z)}{(t+s)^n} \ \le max\left\{\frac{t^n + d(x,y)}{t^n}, \frac{s^n + d(y,z)}{s^n}\right\}$$

$$\Rightarrow \frac{(t+s)^n}{(t+s)^n + d(x,z)} \ge min\left\{\frac{t^n}{t^n + d(x,y)}, \frac{s^n}{s^n + d(y,z)}\right\}$$

$$\Rightarrow 1 - \frac{(t+s)^n}{(t+s)^n + d(x,z)} \le 1 - min\left\{\frac{t^n}{t^n + d(x,y)}, \frac{s^n}{s^n + d(y,z)}\right\}$$

$$\Rightarrow \frac{d(x,z)}{(t+s)^n + d(x,z)} \le max\left\{\frac{d(x,y)}{t^n + d(x,y)}, \frac{d(y,z)}{s^n + d(y,z)}\right\}$$

$$\Rightarrow N(x,z,t+s) \le max\{N(x,y,t), N(y,z,s)\}.$$

In particular, for $n = 1$ the well-known standard intuitionistic fuzzy metric, defined by in ([17]) Remark 2.9 is attained.

Example 6. As a particular case if we take $m = 1$ and $g(t) = k > 0$ (constant function) in above proposition, (3) becomes

$$M(x,y,t)=\frac{k}{k+d(x,y)},N(x,y,t)=\frac{d(x,y)}{k+d(x,y)}$$

Then $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a*b=ab$, $a\lozenge b=a+b-ab$ for all $a,b\in[0,1]$. But in general $(M,N,*,\lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a*b=min\{a,b\}$, $a\lozenge b=max\{a,b\}$ for all $a,b\in[0,1]$. So indeed, if we take $X=\mathbb{R}$, d is the usual metric on $\mathbb{R}$ and choose $x=1,y=10$ and $z=100$

$$N(x,z,t+s)=\frac{d(x,z)}{k+d(x,z)}=\frac{99}{k+99} \quad ,$$

$$N(x,y,t)\lozenge N(y,z,s)=max\{N(x,y,t)\lozenge N(y,z,s)\} \quad =max\left\{\frac{d(x,y)}{k+d(x,y)},\frac{d(y,z)}{k+d(y,z)}\right\}$$

$$=max\left\{\frac{9}{k+9},\frac{90}{k+90}\right\}=\frac{90}{k+90}$$

and $N(x,z,t+s)-N(x,y,t)\lozenge N(y,z,s)>0$.

It means that the condition (IFM-10) $(N(x,z,t+s)\le N(x,y,t)\lozenge N(y,z,s))$ is not satisfies for all $x,y,z\in\mathbb{R}$.

Proposition 3. Let $g:\mathbb{R}^+\to\mathbb{R}^+$ is an increasing continuous function, d is a metric on X, $m\in\mathbb{R}^+$ and $t,s>0$. Define the functions M,N by

$$M(x,y,t)=e^{\frac{-d(x,y)}{g(t)}},N(x,y,t)=\frac{e^{\frac{d(x,y)}{g(t)}}-1}{e^{\frac{d(x,y)}{g(t)}}} \tag{6}$$

for all $x,y\in X$ and let $a*b=ab$, $a\lozenge b=a+b-ab$ for all $a,b\in[0,1]$. Then $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X. At the same time the topology $\tau_{(M,N)}$ coincides with the topology τ_d.

Proof. We only show the condition (IFM-10).

(IFM-10) We will show $N(x,z,t+s)\le N(x,y,t)+N(y,z,s)-N(x,y,t).N(y,z,s)$ for all $x,y,z\in X$.

$$N(x,y,t)+N(y,z,s)-N(x,y,t).N(y,z,s)=\frac{e^{\frac{d(x,y)}{g(t)}}-1}{e^{\frac{d(x,y)}{g(t)}}}+\frac{e^{\frac{d(y,z)}{g(s)}}-1}{e^{\frac{d(y,z)}{g(s)}}}-\left[\frac{e^{\frac{d(x,y)}{g(t)}}-1}{e^{\frac{d(x,y)}{g(t)}}}.\frac{e^{\frac{d(y,z)}{g(s)}}-1}{e^{\frac{d(y,z)}{g(s)}}}\right]$$

$$=\frac{e^{\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}}-1}{e^{\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}}}$$

and at the same time since g is an increasing fuction and d is a metric on X,

$$\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}\ge\frac{d(x,y)}{g(t+s)}+\frac{d(y,z)}{g(t+s)}\ge\frac{d(x,z)}{g(t+s)}$$

$$\Rightarrow-\frac{d(x,z)}{g(t+s)}\ge-\left[\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}\right]$$

$$\Rightarrow e^{\frac{-d(x,z)}{g(t+s)}}\ge e^{-\left[\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}\right]}$$

$$\Rightarrow 1-e^{\frac{-d(x,z)}{g(t+s)}}\le 1-e^{-\left[\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}\right]}$$

$$\Rightarrow \frac{e^{\frac{d(x,z)}{g(t+s)}}-1}{e^{\frac{d(x,z)}{g(t+s)}}} \leq \frac{e^{\left[\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}\right]}-1}{e^{\left[\frac{d(x,y)}{g(t)}+\frac{d(y,z)}{g(s)}\right]}}.$$

It means that $N(x,z,t+s) \leq N(x,y,t) + N(y,z,s) - N(x,y,t).N(y,z,s)$.

Now we show $\tau_{(M,N)}$ coincides with the τ_d. For this it is sufficient to show $B_{(M,N)}(x,r,t) = B_d(x,R)$ for all $x \in X, r \in (0,1), t > 0, R > 0$.

$$\begin{aligned} B_{(M,N)}(x,r,t) &= \{y \in X: M(x,y,t) > 1-r, N(x,y,t) < r\} \\ &= \left\{y \in X: e^{\frac{-d(x,y)}{g(t)}} > 1-r, 1-e^{\frac{-d(x,y)}{g(t)}} < r\right\} \\ &= \left\{y \in X: ln(e^{\frac{-d(x,y)}{g(t)}}) > ln(1-r)\right\} \\ &= \left\{y \in X: -\frac{d(x,y)}{g(t)} > \ln(1-r)\right\} \\ &= \left\{y \in X: 0 < d(x,y) < g(t).ln\left(\frac{1}{1-r}\right)\right\} \end{aligned}$$

then if we take $R = g(t).ln\left(\frac{1}{1-r}\right)$, $B_{(M,N)}(x,r,t) = B_d(x,R)$. At the same time for all $x \in X$ and $R > 0$;

$$\begin{aligned} B_d(x,R) &= \{y \in X: d(x,y) < R, R > 0\} \\ &= \{y \in X: -d(x,y) > -R\} \\ &= \left\{y \in X: -\frac{d(x,y)}{g(t)} > -\frac{R}{g(t)}\right\} \\ &= \left\{y \in X: 1-e^{\frac{-d(x,y)}{g(t)}} < 1-e^{\frac{-R}{g(t)}}\right\} \end{aligned}$$

Then if we take $r = 1-e^{\frac{-R}{g(t)}}$, $B_d(x,R) = B_{(M,N)}(x,r,t)$.

Now we will give the next two examples as a particular cases of this proposition.

Example 7. As a particular case if we take $g(t) = k(k > 0)$ as a constant function in previous proposition, (6) becomes

$$M(x,y) = e^{\frac{-d(x,y)}{k}}, N(x,y) = \frac{e^{\frac{d(x,y)}{k}}-1}{e^{\frac{d(x,y)}{k}}}$$

so, $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = ab$, $a \lozenge b = a + b - ab$ for all $a,b \in [0,1]$. But in general $(M,N,*,\lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = min\{a,b\}$, $a \lozenge b = max\{a,b\}$ for all $a,b \in [0,1]$. So indeed, if we take $X = \mathbb{R}$, $k = 1$ and d is the usual metric on $\mathbb{R}$ and choose $x = 0, y = \frac{1}{2}$ and $z = 1$,

$$N(x,z) = \frac{e^{\frac{d(x,z)}{k}}-1}{e^{\frac{d(x,z)}{k}}} = \frac{e-1}{e} \cong 0,63$$

and

$$N(x,y) \lozenge \mathrm{N}(\mathrm{y},\mathrm{z}) = max\{N(x,y),N(y,z)\} = \frac{e^{\frac{1}{2}}-1}{e^{\frac{1}{2}}} \cong 0.39$$

so we find that $N(x,z) > max\{N(x,y),N(y,z)\}$ contradicts with (IFM-10).

Example 8. As a particular case if we take $g(t) = t$ in previous proposition, (6) becomes

$$M(x,y) = e^{\frac{-d(x,y)}{t}}, N(x,y) = \frac{e^{\frac{d(x,y)}{t}}-1}{e^{\frac{d(x,y)}{t}}}$$

so, $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = min\{a,b\}$, $a \lozenge b = max\{a,b\}$ for all $a,b \in [0,1]$. At the same time $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = ab$, $a \lozenge b = a + b - ab$ for all $a,b \in [0,1]$ because of the previous preposition.

Proposition 4. Let (X,d) be a bounded metric space, $d(x,y) < k$ for all $x,y \in X$ and $g: \mathbb{R}^+ \to (k,+\infty)$ be an increasing continuous function. Define the functions M,N by

$$M(x,y,t) = 1 - \frac{d(x,y)}{g(t)}, N(x,y,t) = \frac{d(x,y)}{g(t)} \tag{7}$$

and denote $a * b = max\{0,a+b-1\}$, $a \lozenge b = min\{1,a+b\}$ for all $a,b \in [0,1]$. Then $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X. At the same time the topology $\tau_{(M,N)}$ coincides with the topology τ_d.

Proof. We only show the (IFM-10).

(IFM-10) Since g is increasing function $\frac{g(t+s)}{g(t)} \geq 1$ and $\frac{g(t+s)}{g(t)} \geq 1$. And from the triangle inequlity

$$d(x,z) \leq d(x,y) + d(y,z) \leq \frac{g(t+s)}{g(t)} d(x,y) + \frac{g(t+s)}{g(s)} d(y,z)$$

$$\Rightarrow \frac{d(x,z)}{g(t+s)} \leq \frac{d(x,y)}{g(t)} + \frac{d(y,z)}{g(s)}$$

$$\Rightarrow N(x,z,t+s) \leq N(x,y,t) + N(y,z,s)$$

This means that $N(x,z,t+s) = min\{1, N(x,y,t) + N(y,z,s)\}$.

Now we show $\tau_{(M,N)}$ coincides with the τ_d. For this, it is sufficient to show $B_{(M,N)}(x,r,t) = B_d(x,R)$ for all $x \in X, r \in (0,1), t > 0$.

$$B_{(M,N)}(x,r,t) = \left\{y \in X: 1 - \frac{d(x,y)}{g(t)} > 1 - r, \frac{d(x,y)}{g(t)} < r\right\} = \{y \in X: d(x,y) < g(t).r\}$$

then if we take $R = g(t).r$, we find $R > 0$ for all $x,y \in X$ so $B_{(M,N)}(x,r,t) = B_d(x,R)$. At the same time for all $x \in X$and $R > 0$;

$$\begin{aligned} B_d(x,R) &= \{y \in X: d(x,y) < R, R > 0\} \\ &= \left\{y \in X: -\frac{d(x,y)}{g(t)} > -\frac{R}{g(t)}\right\} \\ &= \left\{y \in X: 1 - \frac{d(x,y)}{g(t)} > 1 - \frac{R}{g(t)}\right\} \\ &= \left\{y \in X: M(x,y,t) > 1 - \frac{R}{g(t)}, N(x,y,t) < \frac{R}{g(t)}\right\}, \end{aligned}$$

we have taken $r = \frac{R}{g(t)}$ so, $B_d(x,R) = B_{(M,N)}(x,r,t)$ for all $x \in X$ and $R > 0$.

Now we will give the next example as a particular case of this proposition.

Example 9. If we take $g = K > k$ as a constant function then (7) becomes

$$M(x,y,t) = 1 - \frac{d(x,y)}{K}, N(x,y,t) = \frac{d(x,y)}{K}$$

so, $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = max\{0, a+b-1\}$, $a \lozenge b = min\{1, a+b\}$ for all $a, b \in [0,1]$. But in general $(M,N,*,\lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$. So indeed, if we take $X = [0,10]$, d is the usual metric on X and choose $x = 0, y = 1$ and $z = 2$;

$$N(x,z) = \frac{d(x,z)}{K} = \frac{2}{K}$$

and

$$N(x,y) + \mathrm{N(y,z)} - N(x,y).\mathrm{N(y,z)} = \frac{d(x,y)}{K} + \frac{d(y,z)}{K} - \left[\frac{d(x,y)}{K}.\frac{d(y,z)}{K}\right] = \frac{2}{K} - \frac{1}{K^2}$$

so we find that $N(x,z) > N(x,y) + N(y,z) - N(x,y).N(y,z)$ contradicts with (IFM-10).

2.3 The discrete intuitionistic fuzzy metric

Proposition 5. Let $\varphi: \mathbb{R}^+ \to (0,1)$ be an increasing continuous function. Define the functions M, N by

$$M(x,y,t) = \begin{cases} 1 & x = y \\ \varphi(\mathrm{t}) & x \neq y \end{cases}, N(x,y,t) = \begin{cases} 0 & x = y \\ 1 - \varphi(\mathrm{t}) & x \neq y \end{cases} \tag{8}$$

and denote $a * b = min\{a, b\}$, $a \lozenge b = max\{a, b\}$ for all $a, b \in [0,1]$. Then $(M,N,*,\lozenge)$ is an intuitionistic fuzzy metric on X.

Proof. We only show the condition (IFM-10).

(IFM-10) We show that $N(x,z,t+s) \leq N(x,y,t) \lozenge N(y,z,s) = max\{N(x,y,t), N(y,z,s)\}$ for all $x, y, z \in X$ and $t, s > 0$. For this, we examine the following five cases.

Case 1.If $x = y = z$; $N(x,z,t+s) = 0$ and

$$N(x,y,t) \lozenge N(y,z,s) = max\{N(x,y,t), N(y,z,s)\} = 0$$

then $N(x,z,t+s) = max\{N(x,y,t), N(y,z,s)\}$.

Case 2.If $x = y \neq z$; $N(x,z,t+s) = 1 - \varphi(t+s)$ and

$$\begin{aligned} N(x,y,t) \lozenge N(y,z,s) &= max\{N(x,y,t), N(y,z,s)\} \\ &= max\{0, 1-\varphi(s)\} = 1 - \varphi(s). \end{aligned}$$

Since φ is increasing $\varphi(t+s) \geq \varphi(s)$, so $N(x,z,t+s) \leq max\{N(x,y,t), N(y,z,s)\}$.

Case 3.If $x \neq y = z$; $N(x,z,t+s) = 1 - \varphi(t+s)$ and

$$\begin{aligned} N(x,y,t) \lozenge N(y,z,s) &= max\{N(x,y,t), N(y,z,s)\} \\ &= max\{1-\varphi(t), 0\} = 1 - \varphi(t). \end{aligned}$$

Since φ is increasing $\varphi(t+s) \geq \varphi(t)$, $N(x,z,t+s) \leq max\{N(x,y,t), N(y,z,s)\}$.

Case 4. If $x = z \neq y$; $N(x,z,t+s) = 0$ and

$$\begin{aligned} N(x,y,t) \lozenge N(y,z,s) &= max\{N(x,y,t), N(y,z,s)\} \\ &= max\{1-\varphi(t), 1-\varphi(s)\} \end{aligned}$$

then $N(x,z,t+s) \leq max\{N(x,y,t), N(y,z,s)\}$.

Case 5.If $x \neq y \neq z \neq x$; $N(x,z,t+s) = 1 - \varphi(t+s)$ and

$$N(x,y,t) \lozenge N(y,z,s) = max\{1-\varphi(t), 1-\varphi(s)\}.$$

Since φ is increasing $\varphi(t+s) \geq \varphi(t)$ and $\varphi(t+s) \geq \varphi(s)$ so $1-\varphi(t+s) \leq 1-\varphi(t)$ and $1-\varphi(t+s) \leq 1-\varphi(s)$, $N(x,z,t+s) \leq max\{N(x,y,t), N(y,z,s)\}$.

Now we will give the next example as a particular case of this proposition.

Example 10. As a particular case if we take $\varphi(t) = k$ $(k \in (0,1))$ as a constant function then (8) becomes

$$M(x,y,t) = \begin{cases} 1 & x = y \\ \mathrm{k} & x \neq y \end{cases}, N(x,y,t) = \begin{cases} 0 & x = y \\ 1-\mathrm{k} & x \neq y \end{cases}$$

then $(M,N,*,\lozenge)$ is an intuitionistic fuzyy metric on X with $a * b = min\{a,b\}$, $a \lozenge b = max\{a,b\}$ for all $a,b \in [0,1]$. We will call the $(M,N,*,\lozenge)$ discrete intuitionistic fuzzy metric due to its analogy with the classical discrete metric. For this it is sufficient to see that $\{x\}$ is open respect to $\tau_{(M,N)}$ for all $x \in X$. Indeed, $B_{(M,N)}(x, 1-\varphi(t), t) = \{x\}$.

2.4 Intuitionistic fuzzy metrics deduced by symmetric functions

Proposition 6. Let $F: X \times X \to (0,\frac{1}{2})$ be a symmetric function (i.e. $F(x,y) = F(y,x)$ for all $x,y \in X$). Define the functions M,N by

$$M(x,y) = \begin{cases} 1 & x = y \\ F(x,y) & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1-F(x,y) & x \neq y \end{cases} \quad (9)$$

And denote $a * b = max\{0, a+b-1\}$, $a \lozenge b = min\{1, a+b\}$ for all $a,b \in [0,1]$. Then $(M,N,*,\lozenge)$ is a stationary intuitionistic fuzzy metric on X.

Proof. We only show the condition (IFM-10).

(IFM-10) We show that $N(x,z) \leq N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\}$ for all $x,y \in X$. For this we examine the following five cases. Take $x,y,z \in X$.

Case 1. If $x = y = z$; $N(x,z) = 0$ and

$$N(x,y) \lozenge N(y,z) = \min\{1, N(x,y) + N(y,z)\} = min\{1, 0+0\} = 0,$$

so $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 2. If $x = y \neq z$; $N(x,z) = 1 - F(x,z) \in (\frac{1}{2}, 1)$ and

$$\begin{aligned} N(x,y) \lozenge N(y,z) &= min\{1, N(x,y) + N(y,z)\} \\ &= \min\{1, 0+1-F(y,z)\} = 1-F(y,z). \end{aligned}$$

Since $x = y$, $F(x,z) = F(y,z)$ and so $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 3.If $x \neq y = z$; $N(x,z) = 1 - F(x,z) \in (\frac{1}{2}, 1)$ and

$$\begin{aligned} N(x,y) \lozenge N(y,z) &= min\{1, N(x,y) + N(y,z)\} \\ &= min\{1, 1-F(x,y)+0\} = 1-F(x,y) \end{aligned}$$

Since $y = z$, $1 - F(x,z) = 1 - F(x,y)$ and so $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 4.If $x = z \neq y$; $N(x,z) = 0$ and

$$\begin{aligned} N(x,y) \lozenge N(y,z) &= min\{1, N(x,y) + N(y,z)\} \\ &= \min\{1, 1-F(x,y)+1-F(y,z)\} = 1 \end{aligned}$$

(Remember that $F(x,y), F(y,z) \in (0,\frac{1}{2})$ for all $x,y,z \in X$ and so $1 < 2 - F(x,y) - F(y,z) < 2$)

Case 5.If $x \neq y \neq z \neq x$; $N(x,z) = 1 - F(x,z) \in (\frac{1}{2}, 1)$ and

$$N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\}$$

$$= min\{1,1 - F(x,y) + 1 - F(y,z)\} = 1.$$

Example 11. As a particular case, if we take $X = (0,\frac{\pi}{4})$ and $F(x,y) = cos^2x - sin^2y$ then (9) becomes

$$M(x,y) = \begin{cases} 1 & x = y \\ cos^2x - sin^2y & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1 - (cos^2x - sin^2y) & x \neq y \end{cases}$$

so, $(M, N,*,\lozenge)$ is a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. But in general $(M, N,*,\lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$

It is clear that $F(x,y) = cos^2x - sin^2y$ is a symmetric function for all $x, y \in X = (0,\frac{\pi}{4})$ and the function F provides the above proposition and if we take $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$ then $(M, N,*,\lozenge)$ be an intuitionistic fuzzy metric on X. But, if we take $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$ then $(M, N,*,\lozenge)$ is not a stationary fuzzy metric on X. So indeed, if we choose $x = \frac{11\pi}{45}, y = \frac{\pi}{180}, z = \frac{43\pi}{180}$;

$$N(x,z) \cong 0{,}0130$$

and

$$N(x,y) + N(y,z) - N(x,y).N(y,z) \cong 0{,}0037$$

so $N(x,z) > N(x,y) + N(y,z) - N(x,y).N(y,z)$ i.e. the condition (IFM-10) is not provided.

Proposition 7. Let X and Y be two sets of real numbers, such that $x + y \in Y$ for each $x, y \in X$. Let $f: Y \to (0,\frac{1}{2})$ be a function. Define the functions M and N by

$$M(x,y) = \begin{cases} 1 & x = y \\ f(x+y) & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1 - f(x+y) & x \neq y \end{cases} \tag{10}$$

and denote $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. Then $(M, N,*,\lozenge)$ is a stationary intuitionistic fuzzy metric on X.

Proof. We only show the condition (IFM-10). (Remember the function $F(x,y) = f(x + y)$ is a symmetric function on $X \times X$.)

(IFM-10) We show that $N(x,z) \leq N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\}$ for all $x, y, z \in X$. For this we examine the following five cases. Take $x, y, z \in X$.

Case 1.If $x = y = z$; $N(x,z) = 0$ and

$$N(x,y) \lozenge N(y,z) = \min\{1, N(x,y) + N(y,z)\}$$
$$= min\{1,0 + 0\} = 0$$

so $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 2.If $x = y \neq z$; $N(x,z) = 1 - f(x + z) \in (\frac{1}{2}, 1)$ and

$$N(x,y) \lozenge N(y,z) = \min\{1, N(x,y) + N(y,z)\}$$
$$= \min\{1,0 + 1 - f(y + z)\} = 1 - f(y + z).$$

Since $x = y$, $f(x + z) = f(y + z)$ and so $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 3.If $x \neq y = z$; $N(x,z) = 1 - f(x + z) \in (\frac{1}{2}, 1)$ and

$$N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\}$$
$$= min\{1,1 - f(x + y) + 0\} = 1 - f(x + y).$$

Since $y = z$, $1 - f(x + z) = 1 - f(x + y)$ and so $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 4 .If $x = z \neq y$; $N(x,z) = 0$ and

$$N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\}$$
$$= min\{1,1 - f(x+y) + 1 - f(y+z)\} = 1.$$

(Remember that $f(x+y), f(y+z) \in (0,\frac{1}{2})$ for all $x,y,z \in X$ and so $1 < 2 - f(x,y) - f(y,z) < 2$)

Case 5.If $x \neq y \neq z \neq x$; $N(x,z) = 1 - f(x+z) \in (\frac{1}{2},1)$ and

$$N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\}$$
$$= min\{1,1 - f(x+y) + 1 - f(y+z)\} = 1.$$

Example 12. As a particular case, if we take $X = (\frac{\pi}{6},\frac{\pi}{4})$ and $f(x) = cos\, x$ then (10) becomes

$$M(x,y) = \begin{cases} 1 & x = y \\ cos(x+y) & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1 - cos(x+y) & x \neq y \end{cases}$$

so, $(M, N,*,\lozenge)$ is a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. But in general $(M, N,*,\lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$.

It is clear that $F(x,y) = f(x+y)$ is a symmetric function for all $x, y \in X = (\frac{\pi}{6},\frac{\pi}{4})$ and the function F provides the above proposition and if we take $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$ then $(M, N,*,\lozenge)$ be a stationary intuitionistic fuzzy metric on X. But, if we take $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$ then $(M, N,*,\lozenge)$ is not an intuitionistic fuzzy metric on X. So indeed, if we choose $x = \frac{11\pi}{45}, y = \frac{31\pi}{180}$ $z = \frac{43\pi}{180}$;

$$\mathrm{N(x,z)} \cong 0{,}95$$

and

$$N(x,y) + N(y,z) - N(x,y).N(y,z) \cong 0{,}93.$$

So $N(x,z) > N(x,y) \lozenge N(y,z) = N(x,y) + N(y,z) - N(x,y).N(y,z)$ i.e. the condition (IFM-10) is not provided.

Proposition 8. Let X and Y be two sets of real numbers such that $x - y \in Y$ for each $x, y \in X$. Let $f: Y \to (0,\frac{1}{2})$ be a function such that $f(z) = f(-z)$ for each $z \in Y$. Define the functions M and N by

$$M(x,y) = \begin{cases} 1 & x = y \\ f(x-y) & x \neq y \end{cases}, \; N(x,y) = \begin{cases} 0 & x = y \\ 1 - f(x-y) & x \neq y \end{cases} \tag{11}$$

and denote $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. Then $(M, N,*,\lozenge)$ is a stationary intuitionistic fuzzy metric on X.

Proof. It is proved in a similar way with Proposition 7.

Example 13. As a particular case if we take $X = (0,\frac{\sqrt{2}}{2})$ and $f(x) = x^2$ then (11) becomes

$$M(x,y) = \begin{cases} 1 & x = y \\ (x-y)^2 & x \neq y \end{cases}, \; N(x,y) = \begin{cases} 0 & x = y \\ 1 - (x-y)^2 & x \neq y \end{cases}$$

so, $(M, N,*,\lozenge)$ is a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. But in general $(M, N,*,\lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$.

It is clear that $F(x,y) = f(x-y)$ is a symmetric function for all $x, y \in X = (0,\frac{\sqrt{2}}{2})$ and the function F provides the above proposition and if we take $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$ then $(M, N,*,\lozenge)$ be a stationary intuitionistic fuzzy metric on X. But, if we take $a * b = a.b$,

$a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$ then $(M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric on X. So indeed, if we choose $x = 0{,}01, y = 0.3, z = 0{,}001$;

$$N(x,z) \cong 0{,}9999$$

and

$$N(x,y) + N(y,z) - N(x,y).N(y,z) = 0{,}9924$$

so $N(x,z) > N(x,y) \lozenge N(y,z) = N(x,y) + N(y,z) - N(x,y).N(y,z)$ i.e. the condition (IFM-10) is not provided.

Proposition 9. Let $f: X \to (0, \frac{1}{2})$ be a function and define the functions M and N by

$$M(x,y) = \begin{cases} 1 & x = y \\ f(x) + f(y) & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1 - f(x) - f(y) & x \neq y \end{cases} \tag{12}$$

and denote $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. Then $(M, N, *, \lozenge)$ is a stationary intuitionistic fuzzy metric on X.(Notice that now the range of $F$is $(0,1)$.)

Proof. It is proved in a similar way with the Proposition 7.

Example 14. As a particular case if we take $X = (2, +\infty)$ and $f(x) = \frac{1}{x}$ then (12) becomes

$$M(x,y) = \begin{cases} 1 & x = y \\ \dfrac{1}{x} + \dfrac{1}{y} & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1 - \dfrac{1}{x} - \dfrac{1}{y} & x \neq y \end{cases} \tag{12}$$

so, $(M, N, *, \lozenge)$ is a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. But in general $(M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$.

It is clear that $F(x,y) = \frac{1}{x} + \frac{1}{y}$ is a symmetric function for all $x, y \in X = (2, +\infty)$ and the function F provides the above proposition and if we take $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$ then $(M, N, *, \lozenge)$ be a stationary intuitionistic fuzzy metric on X. But, if we take $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$ then $(M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric on X. So indeed, if we choose $x = 1000, y = 3, z = 10000$;

$$N(x,z) \cong 0{,}9989$$

and

$$N(x,y) + N(y,z) - N(x,y).N(y,z) \cong 0{,}8885.$$

So $N(x,z) > N(x,y) \lozenge N(y,z) = N(x,y) + N(y,z) - N(x,y).N(y,z)$ i.e. the condition (IFM-10) is not provided.

At the same time $x \in X$ and chose $r < \frac{1}{2} - \frac{1}{x}$ then $B_{(M,N)}(x, r, t) = \{x\}$ and so $\tau_{(M,N)}$ is discrete topology.

Example 15. If we take $X = (0, \frac{1}{2})$ and we take f as the identity function on X, then (12) becomes

$$M(x,y) = \begin{cases} 1 & x = y \\ x + y & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1 - x - y & x \neq y \end{cases}$$

so, $(M, N, *, \lozenge)$ is a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. But in general $(M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$.

It is clear that $F(x,y) = x + y$ is a symmetric function for all $x, y \in X = (0, \frac{1}{2})$ and the function F provides the above proposition and if we take $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in$

[0,1] then $(M, N, *, \lozenge)$ be a stationary intuitionistic fuzzy metric on X. But, if we take $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$ then $(M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric on X. So indeed, if we choose $x = 0{,}01, y = 0{,}4, z = 0{,}0001$;

$$N(x,z) = 1 - 0{,}01 - 0{,}0001 \cong 0{,}9899$$

and

$$N(x,y) + N(y,z) - N(x,y).N(y,z) \cong 0{,}8360$$

so $N(x,z) > N(x,y) \lozenge N(y,z) = N(x,y) + N(y,z) - N(x,y).N(y,z)$ i.e. the condition (IFM-10) is not provided.

At the same time if we chose $0 < r < \frac{1}{2} - x$ then $B_{(M,N)}(x,r,t) = \{x\}$ and so, $\tau_{(M,N)}$ is discrete topology.

2.5 Intuitionistic fuzzy metrics deduced from pair of functions

Proposition 10. Let g and h be two functions on X with values in (0,1) such that $sup\{g(x): x \in X\} < inf\{h(x): x \in X\}$ and $(h+g)(x) = c$ (with $0 < c < 2$) for all $x \in X$. Define the functions M and N by

$$M(x,y) = \begin{cases} 1 & x = y \\ h(x) - g(y) & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ 1 - h(x) + g(y) & x \neq y \end{cases} \tag{13}$$

and denote $a * b = max\{0, a + b - 1\}$, $a \lozenge b = min\{1, a + b\}$ for all $a, b \in [0,1]$. Then $(M, N, *, \lozenge)$ is a stationary intuitionistic fuzzy metric on X.

Proof. We only show the conditions (IFM-9) and (IFM-10).

(IFM-9)

$$\begin{aligned} N(x,y) &= 1 - h(x) + g(y) = 1 - [c - g(x)] + g(y) \\ &= 1 - c + g(y) + g(x) = 1 - h(y) + g(x) = N(y,x) \end{aligned}$$

(IFM-10) We show that $N(x,z) \leq N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\}$ for all $x, y, z \in X$. For this we examine the following five cases. Take $x, y, z \in X$.

Case 1. If $x = y = z$; $N(x,z) = 0$ and

$$N(x,y) \lozenge N(y,z) = \min\{1, N(x,y) + N(y,z)\} = min\{1, 0 + 0\} = 0$$

so $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 2. If $x = y \neq z$; $N(x,z) = 1 - h(x) + g(z)$ and

$$\begin{aligned} N(x,y) \lozenge N(y,z) &= \min\{1, N(x,y) + N(y,z)\} \\ &= \min\{1, 0 + 1 - h(y) + g(z)\} = 1 - h(y) + g(z). \end{aligned}$$

Since $x = y$, $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 3. If $x \neq y = z$; $N(x,z) = 1 - h(x) + g(z)$ and

$$\begin{aligned} N(x,y) \lozenge N(y,z) &= \min\{1, N(x,y) + N(y,z)\} \\ &= \min\{1, 1 - h(x) + g(y) + 0\} = 1 - h(x) + g(y). \end{aligned}$$

Since $y = z$, $N(x,z) = N(x,y) \lozenge N(y,z)$.

Case 4. If $x = z \neq y$; $N(x,z) = 0$ and

$$N(x,y) \lozenge N(y,z) = min\{1, N(x,y) + N(y,z)\} = min\{1, 1 - h(x) + g(y) + 1 - h(y) + g(z)\}.$$

(Remember that $h, g: X \to (0,1)$ for all $x, y, z \in X$ and so $0 < 2 - h(x) - h(y) + g(y) - g(z) < 4$.) Then $N(x,z) < N(x,y) \lozenge N(y,z)$.

Case 5. If $x \neq y \neq z \neq x$; $N(x,z) = 1 - h(x) + g(z)$ and

$$\begin{aligned} N(x,y) \lozenge N(y,z) &= \min\{1, N(x,y) + N(y,z)\} \\ &= \min\{1, 1 - h(x) + g(y) + 1 - h(y) + g(z)\} \end{aligned}$$

then

$$\begin{aligned} N(x,z) &= 1-h(x)+g(z) = [1-h(x)+g(y)]+h(y)-c+g(z)\pm 2h(y) \\ &= [1-h(x)+g(y)]+[1-h(y)+g(z)]-c+2h(y)-1 \\ &\leq [1-h(x)+g(y)]+[1-h(y)+g(z)] \\ &= min\{1, N(x,y)+N(y,z)\} = N(x,y) \lozenge N(y,z). \end{aligned}$$

Example 16. If we take $X = (0, \frac{1}{2})$, $h(x) = 1 - x$ and $g(x) = x$ then (13) becomes

$$M(x,y) = \begin{cases} 1 & x = y \\ 1-x-y & x \neq y \end{cases}, N(x,y) = \begin{cases} 0 & x = y \\ x+y & x \neq y \end{cases}$$

so, $(M, N, *, \lozenge)$ is a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = max\{0, a+b-1\}$, $a \lozenge b = min\{1, a+b\}$ for all $a, b \in [0,1]$. But in general $(M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$.

So indeed, if we choose $x = 0{,}4, y = 0{,}01, z = 0{,}45$;

$$N(x,z) = x + z = 0{,}85$$

and

$$N(x,y) + N(y,z) - N(x,y).N(y,z) = 0{,}6814$$

so $N(x,z) > N(x,y) \lozenge N(y,z) = N(x,y) + N(y,z) - N(x,y).N(y,z)$ i.e. the condition (IFM-10) is not provided. Also,

$$\begin{aligned} B_{(M,N)}(x,r) &= \{y \in X: M(x,y) > 1-r, N(x,y) < r\} \\ &= \{y \in X: 1-x-y > 1-r, x+y < r\} \\ &= \{y \in X: 0 < y < r-x\} \end{aligned}$$

If $r < x$, it is clear that $B_{(M,N)}(x,r) = \{x\}$ and if $r > x$, $B_{(M,N)}(x,r) = (0, r-x)$ then $B_{(M,N)}(x,r) = \{x\} \cup (0, r-x)$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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