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SINGULAR PERTURBATION PROBLEMS USING INDEFINITE LYAPUNOV FUNCTIONS

Ayhan ALBOSTAN[1], Ahmet ÖZEK[1]

ABSTRACT: *With a dynamic system of differential equations consisting of many variables the singular perturbation methods are of great importance. If we know that some variables decrease faster than the others, we can reduce the number of the equations. The reduction makes their solving much more easy. For example, if we use some numerical methods, the time and storage necessary for the calculations are reduced.*

In this paper, a singular perturbation problem, the existence and separation of the small, and large solutions of a differential equation are considered, but instead of the usual way an indefinite Lyapunov function is used for the investigation.

KEYWORDS: *indefinite Lyapunov function, singular perturbation, two-time-scale system, nonlinear systems*

SİNGÜLER PERTÜRBASYON PROBLEMLERİNDE TANIMSIZ LYAPUNOV FONKSİYONLARININ KULLANILMASI

ÖZET: *Çok değişkenli dinamik bir sistemin türevsel denkleminin çözümünde singüler pertürbasyon yöntemi büyük önem taşımaktadır. Eğer sistemdeki bir kısım değişkenlerin azalması, diğerlerine göre daha hızlı ise denklem sistemini indirgeyebiliriz. Bu indirgeme, denklem sisteminin çözümünü oldukça kolaylaştırmaktadır. Örneğin sayısal yöntemler ile çözüm yapılırsa zaman ve işlem bakımından orijinal sisteme göre daha iyi sonuç elde edilir. Bu makalede bir singüler pertürbasyon problemi olan, hızlı ve yavaş çözümleri içeren bir türevsel denklem gözönünde tutulmuştur, sözkonusu problemin incelenmesinde alışılagelmiş bir yöntem yerine tanımsız Lyapunov fonksiyonu kullanılmıştır.*

ANAHTAR KELIMELER: *tanımsız Lyapunov fonksiyonu, singüler pertürbasyon, iki zaman skalalı sistemler, doğrusal olmayan sistemler*

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I. INTRODUCTION

Exact closed-form analytic solutions of nonlinear differential equations are possible only for a limited number of special classes of differential equations. In general, we have to resort to approximate methods [1], [2].

The methods presented here can be classified into three categories: Describing function methods, Numerical solution methods and Singular perturbation methods [2].

The singular perturbation model of a dynamical system is a state-space model in which the derivatives of some of the state are multiplied by a small positive parameter □; that is

$$x = f(t,x,z,\varepsilon), x \in R^{n} \tag{1}$$

$$\dot{z} = g(t,x,z,\varepsilon), z \quad R^{m'} \tag{2}$$

We assume that the functions f and g are continously differentiable in their arguments for $(t,x,z,\varepsilon) \in [o,t,]xD_1xD_2x[0,\varepsilon_o]$, where $D_1 \subset R^n$ and $D_2 \subset R^m$ are open connected sets. When we set $\varepsilon=0$ in (1)-(2), the dimension of the state equation reduces from $n+m$ to n because the differential equation (2) degenerates into the equation

$$0 = g(t,x,z,o) \tag{3}$$

we shall say that the nodel (1)-(2) is in standard form if and only if (3) has $k \geq 1$ isolated real roots.

$$z=h_i(t,x),\ i=1,2,\ldots\ldots,\ k \tag{4}$$

for each $(t,x) \in [o,t,]xD_1$. This assumption ensures that a well-defined n-dimensional reduced model will correspond to each root of (3). To obtain the i the reduced model, we substitute (4) into (1) at $\varepsilon=0$ to obtain

$$\dot{x} = f(t,x,h(t,x),o) \tag{5}$$

where we have dropped the subscript i from h. It will be clear from the context which root of (4) we are using. This model is sometimes called a quasi-steady-state model because z, whose velocity $\dot{z} = g/\varepsilon$ can be large when ε is small and $g \neq 0$, may rapidly converge to a root of (3) which is the equilibrium of (2). The model (5) is also known as the slow model [1].

Halanay investigated a singular perturbation problem in [3]. His example was as follows:

Let us consider the following singular perturbation problem:

$$\begin{pmatrix} \dot{x}_1 \\ \varepsilon \dot{x} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6}$$

where $x_1 \in R^n$, $x_2 \in R^m$, A_{11}, A_{21}, A_{12} and A_{22} are ***nxn***, ***mxn***, ***nxm*** and ***mxm*** matrices, respectively. Generally this equation can be reduced as $\varepsilon \approx 0$, so from equation

$$A_{21}x_1 + A_{22}x_2 = 0 \text{ or } x_2 = -A_{22}^{-1} A_{21} x_1$$

we have:

$$\dot{x} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)x \tag{7}$$

However this reduction is not always correct. In the procedure used by Halanay we define at first a matrix $H(n+m \times n+m)$ as

$$H = \begin{pmatrix} I_1 & 0 \\ T & I_2 \end{pmatrix}, \tag{8}$$

where I_1 and I_2 are the unit matrices of dimension ***nxn*** and ***mxm***, respectively. T is an unknown matrix ***(nxm)***. It can easily be proved, that

$$H^{-1} = \begin{pmatrix} I_1 & 0 \\ -T & I_2 \end{pmatrix} \tag{9}$$

We transform the equation by the matrix H.

The new varable is $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, *i.e.* $y=Hx$

where, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. the new equation for y is:

$$\dot{y} = H \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{pmatrix} H^{-1} y \text{ i.e.}$$

$$\dot{y} = \begin{pmatrix} A_{11} - A_{12}T & A_{12} \\ TA_{11} + \frac{1}{\varepsilon}A_{21} - TA_{12}T - \frac{1}{\varepsilon}A_{22}T & TA_{12} + \frac{1}{\varepsilon}A_{22} \end{pmatrix} y \tag{10}$$

Assume that there exists a T which satisfies:

$$\varepsilon T(A_{11}-A_{12}T)+(A_{21}-A_{22}T)=0 \tag{11}$$

Then the approximation of the differential equation is

$$\dot{y}_1=(A_{11}-A_{12}T)y_1+A_{12}y_2 \tag{12}$$

where $T\approx A_{22}^{-1}A_{21}$, if ε is small.

From equation (11) we can determine matrix T. Now let us see investigate the properties of equation (12) using matrix T. If the eigenvalues of matrix $A_{22}+\varepsilon TA_{12}$ have negative real parts, then y_2 tends to zero. It is actually so if A_{22} is a stable matrix and ε is small enough. Morever, the smaller ε is, the faster y_2 tends to zero. However, if A_{22} is not stable the singular perturbation technique is not justified. If one does not consider the sufficient condition for applying the singular perturbation technique a wrong result can be obtained instead of a good approximation.

Note that in this case solutions y tend to the subspace of the variable y_1. (See Fig.1) $y_1(t)$ can be called as large solution and $y_2(t)$ as small solition, expressing the fact that $y_2(t)$ tends to zero faster than $y_1(t)$.

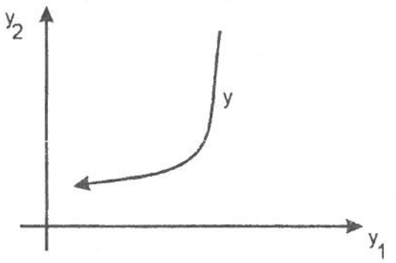

Fig.1

In this paper, an explicite third order differential equation is investigated. We give sufficient conditions of the case in which a second order equation can be approached. It was not intended to choose parameter ε but we studied the basic problem, the existence and the separation of the small (fast) and large (slow) solutions. For the investigation an indefinite Lyapunov function has been used.

II. SOME DEFINITIONS

II.1. Definitions about attractivity

Consider the system

$$\dot{x} = g(t, x) \tag{13}$$

where $g \in P[R^+ x \Omega, R^n]$, $R+=[t_0, \infty)$ $\Omega \subset R^n$, connected and open set. Denote $d(S,R)$ the distance between sets S an R.

Definition 1: $M \subset R^+ xR^n$ is stable set, if $\forall \varepsilon>0$, $\alpha>0$ and $\forall t_1 \geq t_0$ $\exists \delta>0$ such that $d[y_0, M(t_1)]<\delta$ and $|y_0|<\alpha$ implies $d[y(t,t_1,y_0), M(t)]<\varepsilon$ for $\forall t \geq t_1$. Where $y(t,t_1,y_0)$ is the solution of (13) with the intial value $y_0=y(t,t_1,y_0)$.

Definition 2: M is uniformly stable set of (13) if it is stable and δ does not depend on t_1.

Definition 3: M is uniformly asymptotically stable set of (13) if it is uniformly stable and for $\forall \varepsilon>0$, $\forall \alpha>0$ there exist $t_1>0$ end $\delta_0>0$, which does not depend on ε, such that if $d[y_0, M(t_1)]<\delta_0$ and if $|y_0| \leq \alpha$ then $d[y(t,t_1, y_0), M(t)]<\varepsilon$ for all $t \geq t_1$.

Definition 4: M is invariant set of (13) if for $\forall(t_1, y_0)$:M the solution $y(t, t_1, y_0)$ exists if $t \geq t_1$ and $y(t, t_1, y_0) \in M$ for all $t \geq t_1$.

Definition 5: $\overline{M}$ is uniformly attractive, if it is uniformly asymptotically stable (consequently invariant) set. The domain of the attractivity of M is the set

$$A = \left\{(\tau, s) \in R^+ x\Omega : \lim_{t \to \infty} d[t, \tau, s], \overline{M}(t) = 0\right\}$$

II.2. Definite and indefinite Lyapunov functions

Lyapunov functions are often for stability investigations of differential equations. They generally defined by

$V: R^+ x\Omega \to R$, $\Omega \subset R^n$ and the origin is in Ω.

In stability investigations these functions are generally positive definite ones. Considering the behaviour of this Lyapunov functions, we can be informed about the trajectories [4],[5],[6],[9]. In [7],[8] *V.* Kerte'sz is using an indefinite Lyapunov function. Using this function we can separate "fast" and "slow" solutions. This separation is the basis of our investigation in the singular perturbation problem.

III. THE MAIN THEOREM AND ITS PROOF

Let P be a symmetric constant nxn matrix, such that the quadratic form $x^T Px$, $x \in R^n$ is indefinite, $A(x,t)$ is continous nxn matrix function, $x \in R^n$.

Consider equation

$$\dot{x} = A(x,t)x \tag{14}$$

Instead of $A(x,t)$, we write simply A.

We define an indefinite Lyapunov function and its derivatives

$$\hat{V}(x) = x^T Px, \quad \dot{\hat{V}}(x) = x^T \left(A^T P + PA\right)x \tag{15}$$

and another function denoted by ρ,

$$\rho(x,t) = \frac{\dot{\hat{V}}(x)}{\hat{V}(x)} = \frac{x^T \left(A^T P + PA\right)x}{x^T Px} \tag{16}$$

and the sets

$$Q_1 = \{x \in R^n : \hat{V}(x) > 0$$

$$Q_2 = \{x \in R^n : \hat{V}(x) < 0 \tag{17}$$

Now we have two theorems:

Theorem 1: Supposing $x^T(A^TP+PA)x>0$ if $\hat{V}(x)=0$, $x \neq 0$ and $t \geq t_1$, then Q_1 is an invariant set of the solutions of (14). In this case if there exists a $\rho_0(t)$ such that $\rho_0(t) \leq \rho(x,t)$ in Q_1, then

$$V(t) \geq V(t_1)\exp\left(\int_{t_1}^{t} \rho_0(\tau)d\tau\right) \tag{18}$$

where $V(t) = \hat{V}(x(t))$ and $x(t)$ is a solution of (14) which satisfies $x(t_1) \in Q_1$.

(If $\int_{t_1}^{\infty} \rho_0(t)dt > -\infty$,then $\lim_{t\to\infty} |x(t)| > 0$

Theorem 2: Supposing that, $x^T(A^TP+PA)x<0$ if $\hat{V}(x)=0$, $x \neq 0$ and $t \geq t_1$, then Q_2 is an invariant set of the solutions. In this case if there exists a $\rho_0(t)$, that $\rho_0(t) \leq V(x,t)$ in Q_2, then

$$V(t) \leq V(t) \exp\left(\int_{t_1}^{t} \rho_0(\tau)d\tau\right) \tag{19}$$

Where $V(t)=\hat{V}(x)(t))$, and $x(t)$ is a solution of (14) which satisfies $x(t_1)\in Q_2$.

(If $\int_{t_1}^{\infty}\rho_0(t)dt > -\infty$,then $\lim_{t\to\infty}|x(t)| > 0$)

The following lemma [7] will be used for the proof of these theorems.

Lemma 1: Let consider matrix A in part 3, and set Q_1. Then for every solution of (14) which satisfies $V(t)\neq 0$ if $t_1\leq t\leq t_2$ for some t_2 $(t_1\leq t_2<\infty)$, the next equation hold

$$V(t)=V(t_1)\exp\left(\int_{t_1}^{t}\frac{x^T(\tau)\left(A^T(\tau,x(\tau))P+PA(\tau,x(\tau))\right)x(\tau)}{V(\tau)}d(\tau)\right) \tag{20}$$

$t_1\leq t\leq t_2$ where $V(t)=x^T(t)Px(t)$.

Proof: As we know

$$\frac{d}{dt}\ln V(t)=\frac{\frac{d}{dt}V(t)}{V(t)} \tag{21}$$

Noticing that $\frac{d}{dt}V(t)=x^T(t)(A^T(t,x(t)P+PA(t,x(t)))x(t)$ we integrate both sides of the equation, and the lemma is proved.

From it $|V(t)|=|V(t_1)|\exp\int_{t_1}^{t}\rho d\tau, V(t)=V(t_1)\exp\int_{t_1}^{t}\rho d\tau$, (where, $\rho=\rho(x(\tau),\tau)$).

The proof of the Theorem 1: Let $x(t)$ be the solution of (14) for which $x(t_1)\in Q_1$. So inequality $\hat{V}(x(t_1))>0$ is satisfied. As we have considered the expression

$$f(x,t)=x^T(A^TP+PA)x \tag{22}$$

this is positive if $\hat{V}(x)=0$ and $x\neq 0$, because $f(x,t)$ is continous in its variables. (P is constant, A consists of continous functions). Boundary set $\hat{V}(x)=0$ has an open neigh bourhood $K(t)$, where

$f(x,t)\big|_{x\in K(t)} > 0$ Obviously, $K(t)\cap Q_1\neq\emptyset$.

Solution $x(t)$ is continous and so $V(t)=\hat{V}(x(t))$ is continous too. Assume that a solution $x(t)$ of (14) satisfying the conditions of Theorem 1., does not remain in set Q_1. Because of the continuity of $V(t)$, there exists a $t^*>t_1$, for which $\lim_{t\to t^*-0} V(t)=0$, *i.e.* for $\forall\varepsilon>0\exists\delta$:

$0<t^*-t<\delta\Rightarrow 0<V(t)<\varepsilon$. moreover $V(t^*)=0$ and $\exists\, t_3<t^*$, such that

$$\forall t\in(t_3,t^*): f(x,t)>0 \tag{23}$$

Let us denote $2\varepsilon=V(t_3)$ $(t_3>0)$. We want to find a δ, for for which if $t\in(t^*-\delta,<t^*)$ then

$$V(t)<\varepsilon \tag{24}$$

and $t_3\leq t^*-\delta$.

We examine function $V(t)=\hat{V}(x(t))$ in domain Q_1. Here $V(t)>0$. We may use Lemma 1. $(t_3<t<t^*)$, $V(t)=V(t_3)\exp\int_{t_3}^{t}\rho d\tau$, from (24) we get

$$2\varepsilon \exp\int_{t_3}^{t}\rho d\tau<\varepsilon, \text{ if } t_3<t^*-\delta<t<t^*$$

$$\exp\int_{t_3}^{t}\rho d\tau<\frac{1}{2} \tag{25}$$

But we know that, $\rho(t)>0$ if $t\in[t_3, t^*]$,(see (23) and the definition of K) *i.e.*

$$\int_{t_3}^{t}\rho d\tau>0 \tag{26}$$

so inequality (25) can not be true, so a δ satisfying the definition of the times does not exist. The other statement of the theorem can be derived from the lemma.

Remark: The proof of Theorem 2 is similar to the proof of Theorem 1.

IV. THE APPLICATION OF AN INDEFINITE LYAPUNOV FUNCTION IN A SINGULAR PERTURBATION PROBLEM OF A NONLINEAR THIRD ORDER SYSTEM

IV.1. Theorem 3:

Let $\beta>0$ be a constant and let

$$\begin{aligned} &a_0(t,x_1,x_2,x_3)\\ &a_1(t_1,x_1,x_2,x_3)\\ &a_2(t,x_1,x_2,x_3)\end{aligned} \tag{27}$$

be continous functions. Let the matrices used in Theorem 2 be:

$$A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{pmatrix}, \underline{P}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -\beta^2\end{pmatrix} \text{ and } x=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\neq 0 \tag{28}$$

introducing notations $x_1=y$, $x_2=\dot{y}$, $x_3=\ddot{y}$.

If there exists constants p, $p_{ik}(i=0,1;k=1,2)$ such that

$$\begin{gathered}0<p_{01}<a_0<p_{02},\\ 0<p_{11}<a_1<p_{12},\\ 0<p<a_2,\end{gathered} \tag{29}$$

and

$$p>\frac{p_2^2}{p_1^2}+p_2, \tag{30}$$

where $p_2=max\ (p_{02},p_{12})$, $p_1=min(p_{10}, p_{11})$, then there exists $\beta>0$, for which the expression

$$x^T(A^TP+PA)x \tag{31}$$

is greater than zero if

$$x^TPx=0 \tag{32}$$

Proof: we substitute A, P and x into (31) and after multiplication by 1/2 we get:

$$\beta^2a_2x^2_3+(1+\beta^2a_1)x_2x_3+\beta^2a_0x_1x_3+x_1x_2 \tag{33}$$

If $x\neq 0$, we may introduce some new variables:

$$\bar{x}=\frac{x_1}{x_3}, \qquad \bar{y}=\frac{x_2}{x_3} \tag{34}$$

by these (31) is:

$$a_2\beta^2+a_0\beta^2\bar{x}+\left(1+a_1\beta^2\right)\bar{y}+\overline{xy} \tag{35}$$

Denote $\chi=(\bar{x},\bar{y})$ expression (35), and we fix the values of a_0, a_1, a_2 and vary $\bar{x}$ and $\bar{y}$. The problem is whether

$$\chi(\bar{x},\bar{y})>0 \tag{36}$$

if

$$\bar{x}^2+\bar{y}^2=0 \tag{37}$$

(For $x\neq 0$ equation (37) is equivalent to (32))

On the plane $\bar{x}$, $\bar{y}$ expression (36) means a domain bounded by a hyperbola. Its equation is

$$\chi(\bar{x},\bar{y}) = 0 \text{ i.e.}$$

$$\bar{y} = -\beta^2 a_0 + \frac{\beta^2 a_0(1+a_1\beta^2) - \beta^2 a_2}{\bar{x} + (1+a_1\beta^2)} \tag{38}$$

The asymptotes are parallel to the axis $\bar{x}$, and axis $\bar{y}$, and consist of points

$$\bar{x}_0 = -1 - a_1\beta^2 \tag{39}$$

$$\bar{y}_0 = -1 - \beta^2 a_0$$

Expression (37) means a circle around the origin with radius β. (36) and (37) are satisfied if the circle is

a) in the quadrant bounded by the asymptotes, in which where is no point of the hyperbola, and

b) (36) is satisfied between the graph of the hyperbola. (see Fig. 2)

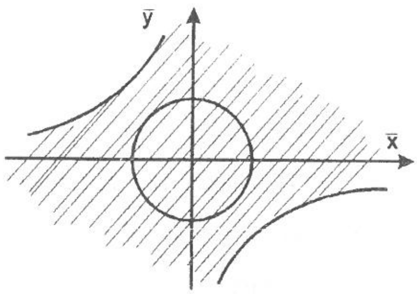

Fig. 2

These conditions are

a) $\chi(0,0) > 0$, $\beta \le |x_0|$, (40)

i.e

$$\beta \le |1 + a_1\beta^2| . \tag{41}$$

The sign of $\bar{x}_0$ is the same of $\bar{y}_0$, and

$$\beta \le |\bar{y}_0|, \qquad \beta \le |a_0|\beta^2 \tag{42}$$

b) $\chi(\bar{x}_0, \bar{y}_0) > 0$, (43)

that is

$$a_2\beta^2 - a_0\beta^2(1+a_1\beta^2) > 0, \tag{44}$$

so

$$a_2 > a_0 (1 + a_1 \beta^2) \tag{45}$$

Because of the conditions of the theorem $a_1, a_0 > 0$, so (43) is equivalent to

$$\frac{a_2 - a_0}{a_1 a_0} > \beta^2. \tag{46}$$

(40) satisfies because of condition $a_2 > 0$. In (39) both $\bar{x}_0$ and $\bar{y}_0$ are negative. Inequality (41) is

$$\beta \le \left|1 + a_1\beta^2\right| = 1 + a_1\beta^2$$

so

$$a_1\beta^2 - \beta + 1 \ge 0 \tag{47}$$

If $a_1 \ge 1/4$ then $\beta \in R$, if $0 < a_1 < 1/4$ then $\beta \in \left(\frac{1 + \sqrt{1 - 4a_1}}{2a_1}, \infty\right)$. So if $a_1 \ge 1/4$ then (42) and (43) hold if

$$\frac{1}{a_0} \le \beta < \sqrt{\frac{a_2 - a_0}{a_1 a_0}}. \tag{48}$$

in case of $0 < a_1 < 1/4$, (42) and (43) are true if

$$max\left(\frac{1}{a_0}, \frac{1 + \sqrt{1 - 4a_1}}{2a_1}\right) < \beta < \sqrt{\frac{a_2 - a_0}{a_1 a_0}}. \tag{49}$$

These intervals are not allowed to be empty by the conditions of the theorem, because

$$\frac{1}{a_0} < \frac{1}{p_1} < \sqrt{\frac{p - p_2}{p_2^2}} < \sqrt{\frac{a_2 - a_0}{a_1 a_0}}.$$

or if $a_1 \in (0, 1/4)$

$$max\left(\frac{1}{a_0}, \frac{1 + \sqrt{1 - 4a_1}}{2a_1}\right) < max\left(\frac{1}{a_0}, \frac{1}{a_1}\right).$$

i.e.

$$max\left(\frac{1}{a_0}, \frac{1 + \sqrt{1 - 4a_1}}{2a_1}\right) < \frac{1}{p_1} < \sqrt{\frac{p - p_2}{p_2^2}} < \sqrt{\frac{a_2 - a_0}{a_1 a_0}}. \tag{50}$$

There exist β with the required properties. Obviously, if a_0, a_1, a_2 are not constants, but satisfy the conditions stated in theorem, (36) remains true.

IV.2. Applications of indefinite Lyapınov function in appoximations of a third order nonlinear differential equation

Let us see the next nonlinear third order differential equation:

$$\dddot{y}+a_2\ddot{y}+a_1\dot{y}+a_0y=0$$

where a_0, a_1, a_2 are functions of $t, y, \dot{y}, \ddot{y}$. Assume that the conditions of Theorem 3 are satisfied. In our investigations we use 'he notations of Theorem 3.

Let us have a β as in the previous theorem. Because of Theorem 1., if $x^TPx=0$ then $x^T(A^TP+PA)x>0$. By Theorem 1. Q_1 is an invariant set of the solutions. Fig 3., shows the meaning of it in the space of $y, \dot{y}, \ddot{y}$. If β is large enough, set Q_1 is near the plain (x_1, x_2), so plain (x_1, x_2) is approximately an invariant set of the solutions.

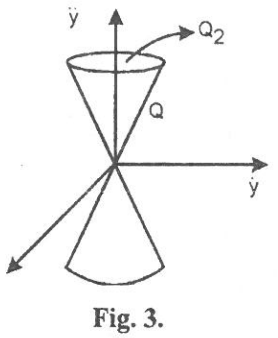

Fig. 3.

IV.3. On the behaviour of the solutions

Let Q_1 and Q_2 be satisfying the definitions used in part 3. From Theorems 1. and 3., applying (Definition 5) on Fig. 4 in the domain between the hyperbolas $\chi>0$, in Q_2 $V<0$, so in the expression of V the power of ρ is negative.

$$\rho=\frac{\chi}{V}<0$$

On the solutions V increases, so as in Fig. 4, the solutions come out of set Q_2. So set Q_1 is an attractor and its region of attractivity is Q_2.

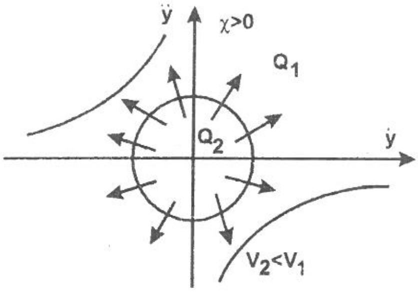

Fig. 4.

The proof is easy because in $\boldsymbol{Q_2}$ Lemma 1. holds, and on the cone Theorem 3 also satisfies. Thus, $\boldsymbol{Q_1}$ is a uniformly attaractive set. If β is large enough, it means that x_3 tends faster to zero than x_2 or x_1. So we encounter a typical singular perturbation problem, solved by indefinite Lyapunov function and by the singular perturbation technique.

V. NUMERICAL EXAMPLE

To illustrate our results, there is a numerical example for the linear equation

$$\dddot{y}+5\ddot{y}+\dot{y}+y=0,$$

the coefficients of which satisfy the conditions of Theorem 3, if $1\leq\beta<2$. The numerical approximation of the roots of its characteristical equation is

$$\lambda 1=-4.8360$$

$$\lambda 2=-0.0820+j0.4473$$

$$\lambda 3=-0.0820-j0.04473$$

We can easily see, that the real part of λ_1 has larger absolute value, than that of the others. So the component of the solutions belonging to λ_1 decreases much faster, than that of the others.

The eigenvectors v_1, v_2, v_3 belonging to eigenvalues λ_1, λ_2, λ_3 are approximately

$$v_1=\begin{pmatrix}0.043\\-0.207\\1\end{pmatrix},\quad v_2=\begin{pmatrix}1\\-0.802\\-0.194\end{pmatrix},\quad v_3=\begin{pmatrix}1\\-0.448\\-0.047\end{pmatrix}$$

Accordingly, components decreasing faster than the others are approximately in direction of basic vector

$$e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The solutions tend to a plane which is close to the plane (x_1, x_2).

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