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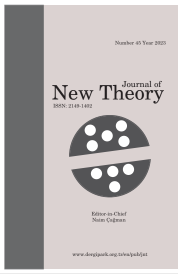

Analyzing Stability and Data Dependence Notions by a Novel Jungck-Type Iteration Method

Yunus Atalan[1], **Esra Erbaş**[2]

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Abstract − Finding the ideal circumstances for a mapping to have a fixed point is the fundamental goal of fixed point theory. These criteria can also be used for the structure under investigation. One of this theory's most well-known theorems, Banach's fixed point theorem, has been expanded adopting various methods, making it possible to conduct numerous research studies. Thanks to the Jungck-Contraction Theorem, which has been proven through commutative mappings, many fixed point theorems have been obtained using classical fixed point iteration methods and newly defined methods. This study aims to investigate the convergence, stability, convergence rate, and data dependency of the new four-step fixed-point iteration method. Nontrivial examples are also included to support some of the results herein.

Keywords *Jungck-contraction principle, fixed point, iteration method, stability, data dependence*

Mathematics Subject Classification (2020) 47H09, 47H10

1. Introduction

The solutions of some problems in mathematics can be reduced to finding the solution of an equation that can be written as $f(x)-x=0$ for a function f satisfying the appropriate conditions. The points x, which are the solutions of equations of this type, are called the fixed points of the f function. With its extensive range of applications in fields such as differential and integral equations [1], approximation theory and game theory [2], fixed point theory has emerged as a captivating and fundamental subject within nonlinear analysis. Moreover, this theory yields fruitful outcomes across various domains, including optimization [3], physics [4], economics [5], and medicine [6]. Consequently, fixed point theory has remained a dynamic research area, drawing significant attention from researchers in the past fifty years, due to its foundation in analysis and topology, and continues to generate a vibrant body of literature.

Geometrically, the definition of a fixed point means the point on the $y=x$ line. The theorems formulated to establish the existence and uniqueness of a fixed point are commonly referred to as fixed-point theorems. One of the most famous existence and uniqueness theorems is the theorem, which was proved by Banach [7] in 1922 and called the Banach Contraction Principle. While this theorem states that a contraction mapping defined on itself in complete metric spaces will have a unique fixed point, it also offers a method called iteration in order to reach this unique fixed point.

[1]yunusatalan@aksaray.edu.tr (Corresponding Author); [2]esraa_erbas@hotmail.com

[1,2]Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, Aksaray, Türkiye

The main idea in the studies on the iterations mentioned above is to determine under which conditions the sequences obtained from these algorithms, which are formed by using certain mapping classes, converge to the fixed point, the equivalence of the convergence behavior with other methods. Furthermore, testing the convergence speed, analysis of the data dependency, and stability of the iteration methods are considered one of the main targets of these studies.

Since the Picard iteration used in the Banach Contraction Principle cannot converge to the fixed point of non-expansive mappings, this problem has been tried to be overcome by defining new iteration methods. As a result of this approach, many iteration methods have been brought to the literature and studies on the definition of new iterations have continued to maintain their popularity today.

While the iterative sequence converges to the fixed point of a certain mapping class, it may not converge to the fixed point of another mapping class. This problem has revealed the concept of equivalence of convergence for iteration methods, and whether the iteration methods in the literature and the newly defined iteration methods are equivalent in terms of convergence have been examined in various spaces [8, 9]. A large literature has been created as a result of trying to determine which of the two iteration methods, which are shown to be equivalent in terms of convergence, converges to the fixed point of the relevant mapping more rapidly [10, 11].

After showing that the iterative sequence converges to the fixed point of the used mapping, it can be shown that the new sequence to be obtained by using another mapping called the approximation operator for this iteration method is also convergent to the fixed point of the approximate operator. In such a case, the questions of how close the fixed points of both mappings are to each other and how to calculate this distance bring up the concept of data dependency. There are many studies on different kinds of constructs on whether fixed point iteration methods are data dependent [12–15].

Mathematically, the concept of stability can be thought of as the fact that small changes to be applied to the structure studied cannot disrupt the functioning of it. In this context, many studies have been carried out on the stability of fixed-point iteration methods. The approach here is; instead of the sequence to be obtained from the iteration method used, calculation errors, rounding errors, etc., it can be characterized as the convergence of the new sequence to the fixed point of the mapping, although another sequence is obtained for various reasons [16, 17].

Because the mapping used in the Banach Contraction Principle is contraction, researchers have sought to obtain various generalizations of this theorem for different types of mappings [18–20]. One of the notable generalizations of this theorem was made by Jungck [21] in 1976 using commutative mappings.

In this paper, a Jungck-type four-step iteration method is introduced and the convergence and stability of the sequence obtained from this method, which is constructed using a certain type of mapping, under favorable conditions are investigated. Moreover, the convergence behavior of the new iterative sequence is compared with other Jungck-type iterative sequences in the literature. In addition, the concept of data dependence is analyzed and some of the results mentioned here are supported by numerical examples.

2. Preliminaries

Jungck [21] expressed one of the noteworthy generalizations of the Banach Contraction Principle using commutative mappings as follows:

Theorem 2.1. Let $f_1, f_2 : \mathfrak{B} \to \mathfrak{B}$ be two functions satisfy in the following conditions, for all $b_1, b_2 \in \mathfrak{B}$:

i. (f_1, f_2) is a commutative pair of map *ii.* f_2 is continuous

iii. $f_1(\mathfrak{B}) \subsetneq f_2(\mathfrak{B})$

iv. $\wp(f_1b_1, f_1b_2) \leq t\wp(f_2b_1, f_2b_2)$ such that $t \in [0,1]$

in which $\mathfrak{B}$ is complete metric space with respect to metric function $\wp$. In this case f_1 and f_2 have a unique common fixed point $p \in \mathfrak{B}$.

The condition specified by *iv* in this theorem is known as the Jungck Contraction mapping, and when taking f_2 as a unit function, it corresponds to the classical Banach Contraction Principle. Building upon this theorem, Jungck introduced the following iteration method:

Assume that $\mathfrak{B}$ be a Banach space, $\mathcal{C}$ any set, and $S,T:\mathfrak{B}\to\mathcal{C}$ satisfy $T(\mathcal{C}) \subseteq S(\mathcal{C})$.

$$Sx_{n+1} = Tx_n \tag{1}$$

This is referred to as the Jungck iteration method. If $S = I$ and $\mathcal{C} = \mathfrak{B}$ in Equation 1, the classical Picard iteration method [22] is obtained. Many researchers have worked on this method introduced by Jungck and have obtained many fixed point theorems by rewriting the classical iteration methods in Jungck type. Some of the works done with this approach are as follows for $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\mu_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty} \subseteq [0,1]$:

Jungck-SP iteration method [23] is defined as under:

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sy_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Sz_n + \beta_n Tz_n \\ Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{2}$$

Jungck-CR iteration is defined by [24]:

$$\begin{cases} Su_{n+1} = (1-\alpha_n)Sv_n + \alpha_n Tv_n \\ Sv_n = (1-\beta_n)Tu_n + \beta_n Tw_n \\ Sw_n = (1-\gamma_n)Su_n + \gamma_n Tu_n \end{cases} \tag{3}$$

Furthermore, if $\{\alpha_n\}_{n=0}^{\infty} = 0$ in Equation 3, the following Jungck-type Agarwal iteration method is obtained [25]:

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Sx_n + \beta_n Tx_n \end{cases} \tag{4}$$

If $\{\alpha_n\}_{n=0}^{\infty} = 0$ and $\{\beta_n\}_{n=0}^{\infty} = 1$ in Equation 3, the following Jungck-type Sahu iteration method is obtained [25]:

$$\begin{cases} Sx_{n+1} = Ty_n \\ Sy_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{5}$$

The Jungck-Khan iteration method is defined as follows [26]:

$$\begin{cases} Su_{n+1} = (1-\alpha_n-\beta_n)Su_n + \alpha_n Tv_n + \beta_n Tu_n \\ Sv_n = (1-b_n-c_n)Su_n + b_n Tw_n + c_n Tu_n \\ Sw_n = (1-a_n)Su_n + a_n Tu_n \end{cases} \tag{6}$$

The new four-step iteration method that we have defined inspired by the literature on the iteration methods given above is as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, y_n + \alpha_n T y_n \\ y_n = (1-\beta_n)\, T x_n + \beta_n T z_n \\ z_n = (1-\gamma_n)\, w_n + \gamma_n T w_n \\ w_n = (1-\mu_n)\, x_n + \mu_n T x_n \end{cases} \tag{7}$$

The following iteration method is obtained by rewriting the iteration method given by Equation 7 in Jungck-type:

$$\begin{cases} S x_{n+1} = (1-\alpha_n)\, S y_n + \alpha_n T y_n \\ S y_n = (1-\beta_n)\, T x_n + \beta_n T z_n \\ S z_n = (1-\gamma_n)\, S w_n + \gamma_n T w_n \\ S w_n = (1-\mu_n)\, S x_n + \mu_n T x_n \end{cases} \tag{8}$$

The following statements hold for the Jungck-type iteration methods given above for $n \in \{0, 1, 2...\}$, taking $S = I$ and $\mathcal{C} = \mathfrak{B}$:

Remark 2.2. *i.* The classical SP iteration method [27] can be obtained from the iteration method provided by Equation 2;

ii. The classical CR iteration method [28] can be obtained from the iteration method provided by Equation 3;

iii. The classical Agarwal-S [29] and classical Sahu [30] iteration methods can be obtained from the iteration methods provided by Equation 4 and Equation 5, respectively.

iv. If $\mu_n = 0$ is chosen in the iteration method provided by Equation 7, the classical CR iteration [28] is obtained.

v. If $\mu_n = 0$ is chosen in the iteration method provided by Equation 8, the Jungck-CR iteration method provided by Equation 3 is obtained.

Some auxiliary theorems and definitions have been given to obtain the main results in the following:

Definition 2.3. [24] Suppose that $\mathfrak{B} \neq \emptyset$ and $S, T : \mathfrak{B} \to \mathfrak{B}$ are mappings.

i. $b \in \mathfrak{B}$ is referred to as the common fixed point of T and S if $b = Tb = Sb$

ii. $c \in \mathfrak{B}$ is referred to as the coincidence point of T and S if $c = Tb = Sb$

iii. The pair of maps (S, T) is referred to as commuting if $TSb = STb$ for all $b \in \mathfrak{B}$

iv. The pair of maps (S, T) is referred to as weakly compatible if $TSb = STb$ whenever $Tb = Sb$ for some $b \in \mathfrak{B}$.

Definition 2.4. [31] Let $\left\{\Theta_n^{(i)}\right\}_{n=0}^{\infty}$ be two sequences with $\lim_{n\to\infty} \Theta_n^{(i)} = \Theta_i$, $i \in \{1, 2\}$. Then, it is said that $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{\left\|\Theta_n^{(1)} - \Theta_1\right\|}{\left\|\Theta_n^{(2)} - \Theta_2\right\|} = 0$$

Definition 2.5. [31] Assume that $\left\{\Theta_n^{(i)}\right\}_{n=0}^{\infty}$ and $\left\{\Pi_n^{(i)}\right\}_{n=0}^{\infty}$ are four sequences for $i \in \{1, 2\}$ such that $\Pi_n^{(i)} \geq 0$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} \Theta_n^{(i)} = \Theta^*$, and $\lim_{n\to\infty} \Pi_n^{(i)} = 0$. Suppose that the following error estimates are available:

$$(\forall n \in \mathbb{N}) \quad \left\|\Theta_n^{(i)} - \Theta^*\right\| \leq \Pi_n^{(i)} \quad i \in \{1, 2\}$$

If $\left\{\Pi_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{\Pi_n^{(2)}\right\}_{n=0}^{\infty}$ (in the sense of Definition 2.4), then it is said that $\left\{\Theta_n^{(1)}\right\}_{n=0}^{\infty}$ converges to Θ^* faster than $\left\{\Theta_n^{(2)}\right\}_{n=0}^{\infty}$.

Definition 2.6. [32] Assume that $S, T : \mathcal{C} \to \mathfrak{B}$ are mappings satisfy $T(\mathcal{C}) \subseteq S(\mathcal{C})$ and $p = Tb = Sb$. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ attained by $Sx_{n+1} = f(T, x_n)$ converges to p for any $x_0 \in \mathcal{C}$. Let $\{Sy_n\}_{n=0}^{\infty} \subsetneq \mathfrak{B}$ be an arbitrary sequence and set $\epsilon_n = d\left(Sy_{n+1}, f\left(T, y_n\right)\right),\ n \in \{0, 1, 2, \dots\}$. Then $f\left(T, x_n\right)$ will be called (S, T)-stable if and only if $\lim\limits_{n\to\infty} \epsilon_n = 0$ implies that $\lim\limits_{n\to\infty} Sy_n = p$.

Definition 2.7. [33] Assume that (X, d) is a metric space and the maps $S, T : X \to X$ satisfy the following conditions for all $x, y \in X$:

i. $T(X) \subseteq S(X)$

ii. for non-negative λ and μ satisfying the condition $\lambda + \mu < 1$,

$$d\left(Tx, Ty\right) \leq \lambda d\left(Sx, Sy\right) + \mu\left(\frac{d\left(Sx, Tx\right).d\left(Sy, Ty\right)}{1 + d\left(Sx, Sy\right)}\right) \tag{9}$$

iii. $S\left(X\right)$ is complete sub-space of X

Then, the mappings S and T have a coincidence point. In addition, if S and T are weakly compatible, these mappings have a unique common fixed point.

Lemma 2.8. [34] Suppose that $\left\{\rho_n^{(k)}\right\}_{n=0}^{\infty}$ are two sequences such that $\rho_n^{(k)} \geq 0$, for each $n \in \mathbb{N}$ and for $k \in \{1, 2\}$. Assume that $\lim\limits_{n\to\infty} \rho_n^{(2)} = 0$ and $\mu \in (0, 1)$. If $\rho_{n+1}^{(1)} \leq \mu\rho_n^{(1)} + \rho_n^{(2)}$, then $\lim\limits_{n\to\infty} \rho_n^{(1)} = 0$.

Lemma 2.9. [35] Assume that $\{a_n\}_{n=1}^{\infty}$ is a non negative real sequence and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ satisfying the following condition:

$$a_{n+1} \leq (1 - \mu_n)a_n + \mu_n\eta_n$$

where $\mu_n \in$ (0,1) such that $\sum\limits_{n=1}^{\infty} \mu_n = \infty$ and $\eta_n \geq 0$. Then, the following inequality holds:

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup \eta_n$$

Definition 2.10. [36] Suppose that $(\mathfrak{B}, d)$ is a metric space and $A_1 : \mathfrak{B} \to \mathfrak{B}$ is operator with fixed point p and there exist a fixed point iteration method that converges to p. $A_2 : \mathfrak{B} \to \mathfrak{B}$ is referred to as approximate operator of A_1 for a suitable $\mu > 0$ if $d\left(A_1x, A_2x\right) \leq \mu$, for each $x \in \mathfrak{B}$.

3. Main Results

In this part of the study, the concept of convergence is analyzed using the new iteration method. It is also shown that this result can be obtained independently of the condition applied to the control sequences. In addition, the theorems such as stability, convergence speed, and data dependence are proved.

Theorem 3.1. Assume that X is a Banach space, Y an arbitrary set and $S, T : Y \to X$ satisfy the condition given by Inequality 9 with $p = Tx_p = Sx_p$. Suppose that $S(Y)$ is a complete subset of X such that $T(Y) \subseteq S(Y)$ and $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence given by Equation 8 with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. If $Y = X$ and S and T are weakly compatible then, p is a unique common fixed point of S and T.

Proof.

By using Equation 8, Inequality 9, and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\mu_n\}_{n=0}^{\infty} \subseteq [0, 1]$, in the following inequalities are obtained:

$$\begin{aligned}
\|Sx_{n+1} - p\| &= \|(1-\alpha_n) Sy_n + \alpha_n Ty_n - p\| \\
&\leq (1-\alpha_n) \|Sy_n - Tx_p\| + \alpha_n \|Ty_n - Tx_p\| \\
&\leq (1-\alpha_n) \|Sy_n - Sx_p\| \\
&\quad + \alpha_n \left\{ \lambda \|Sy_n - Sx_p\| + \mu \left(\frac{\|Sy_n - Ty_n\| . \|Sx_p - Tx_p\|}{1 + \|Sy_n - Sx_p\|} \right) \right\} \\
&\leq (1-\alpha_n) \|Sy_n - Sx_p\| + \lambda\alpha_n \|Sy_n - Sx_p\| \\
&= [1 - \alpha_n (1-\lambda)] \|Sy_n - Sx_p\|
\end{aligned}$$

and

$$\begin{aligned}
\|Sy_n - p\| &= \|(1-\beta_n) Tx_n + \beta_n Tz_n - p\| \\
&\leq (1-\beta_n) \|Tx_n - Tx_p\| + \beta_n \|Tz_n - Tx_p\| \\
&\leq (1-\beta_n) \left\{ \lambda \|Sx_n - Sx_p\| + \mu \left(\frac{\|Sx_n - Tx_n\| . \|Sx_p - Tx_p\|}{1 + \|Sx_n - Sx_p\|} \right) \right\} \\
&\quad + \beta_n \left\{ \lambda \|Sz_n - Sx_p\| + \mu \left(\frac{\|Sz_n - Tz_n\| . \|Sx_p - Tx_p\|}{1 + \|Sz_n - Sx_p\|} \right) \right\} \\
&= \lambda (1-\beta_n) \|Sx_n - Sx_p\| + \lambda\beta_n \|Sz_n - Sx_p\|
\end{aligned}$$

Similarly,

$$\|Sz_n - p\| \leq [1 - \gamma_n (1-\lambda)] \|Sw_n - Sx_p\|$$

and

$$\|Sw_n - p\| \leq [1 - \mu_n (1-\lambda)] \|Sx_n - Sx_p\|$$

If these inequalities are nested and necessary simplifications are made considering that $[1-\gamma_n (1-\lambda)] \leq 1$ and $[1-\mu_n (1-\gamma)] \leq 1$, then it is attained that

$$\|Sx_{n+1} - p\| \leq \lambda[1 - \alpha_n (1-\lambda)] \|Sx_n - p\| \tag{10}$$

If induction is applied to the last inequality, then

$$\|Sx_{n+1} - p\| \leq \lambda^{n+1} \prod_{i=0}^{n} [1 - \alpha_i(1-\lambda)] \|Sx_0 - p\| \tag{11}$$

By using $1 - x \leq e^{-x}$, for all $x \in [0,1]$, it is obtained in the following inequality:

$$\begin{aligned}
\|Sx_{n+1} - p\| &\leq \lambda^{n+1} \|Sx_0 - p\| \prod_{i=0}^{n} e^{-(1-\lambda)\alpha_i} \\
&= \lambda^{n+1} \|Sx_0 - p\| e^{-(1-\lambda) \sum\limits_{i=0}^{n} \alpha_i}
\end{aligned}$$

If the limit for the last inequality as $n \to \infty$ is taken, it can be observed that $Sx_n \to p$. It will be demonstrated that S and T have a unique common fixed point like p. Suppose the pair (S,T) has another coincidence point, say q. Therefore,

$$\begin{aligned}
0 \leq \|p - q\| = \|Tx_p - Tx_q\| &\leq \lambda (\|Sx_p - Sx_q\|) + \mu \left(\frac{\|Sx_p - Tx_p\| . \|Sx_q - Tx_q\|}{1 + \|Sx_p - Sx_q\|} \right) \\
&= \lambda \|Sx_p - Sx_q\|
\end{aligned}$$

which implies that $p = q$, that is S and T have a unique coincidence point. Since S and T are weakly compatible and $Sx_p = Tx_p = p$, then $TTp = TTx_p = TSx_p = STx_p$ signifies $Tp = Sp$. Thus, Tp is the unique coincidence point of (S, T), then $Tp = p$. As a result, the (S, T) pair of maps have a unique common fixed point. $\square$

In the next theorem, it is proven that the result of Theorem 3.1 can be derived without the $\sum_{n=0}^{\infty} \alpha_n = \infty$ condition:

Theorem 3.2. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Then $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Moreover, if $Y = X$ and S and T are weakly compatible, then p is a unique common fixed point of S and T.

Proof.
Since $[1 - \alpha_n (1 - \lambda)] \leq 1$, from Inequality 10, it is attained the following inequality

$$\|Sx_{n+1} - p\| \leq \lambda^{n+1} \|Sx_0 - p\|$$

Given that $\lambda < 1$ and taking the limit in the last inequality, one can obtain $Sx_n \to p$ as $n \to \infty$. It can be observed from Theorem 3.1 that p is the unique common fixed point of the T and S. $\square$

Theorem 3.3. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Suppose that iterative sequence $\{Sx_n\}_{n=0}^{\infty}$ given by Equation 8 converges to p with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, it is (S, T)-stable.

Proof.
Assume that $\varepsilon_n = \|Sa_{n+1} - f(T, a_n)\|$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Besides, $\{Sa_n\}_{n=0}^{\infty} \subsetneq X$ is any sequence obtained from the following equation:

$$\begin{cases} Sa_{n+1} = (1 - \alpha_n) Sb_n + \alpha_n Tb_n \\ Sb_n = (1 - \beta_n) Ta_n + \beta_n Tc_n \\ Sc_n = (1 - \gamma_n) Sd_n + \gamma_n Td_n \\ Sd_n = (1 - \mu_n) Sa_n + \mu_n Ta_n \end{cases} \tag{12}$$

It will be shown that $\lim\limits_{n\to\infty} Sa_n = p$. By using Inequality 9 and Equation 12, the following inequalities are obtained:

$$\begin{aligned} \|Sd_n - p\| &= \|(1 - \mu_n) Sa_n + \mu_n Ta_n - p\| \\ &\leq (1 - \mu_n) \|Sa_n - p\| + \mu_n \|Ta_n - Tx_p\| \\ &\leq (1 - \mu_n) \|Sa_n - p\| + \mu_n \left\{ \lambda \|Sa_n - Sx_p\| + \mu \left(\frac{\|Sa_n - Ta_n\| . \|Sx_p - Tx_p\|}{1 + \|Sa_n - Sx_p\|} \right) \right\} \\ &\leq [1 - \mu_n (1 - \lambda)] \|Sa_n - p\| \end{aligned} \tag{13}$$

and

$$\begin{aligned} \|Sc_n - p\| &= \|(1 - \gamma_n) Sd_n + \gamma_n Td_n - p\| \\ &\leq (1 - \gamma_n) \|Sd_n - p\| + \gamma_n \|Td_n - Tx_p\| \\ &\leq (1 - \gamma_n) \|Sd_n - p\| + \gamma_n \left\{ \lambda \|Sd_n - p\| + \mu \left(\frac{\|Sd_n - Td_n\| . \|Sx_p - Tx_p\|}{1 + \|Sd_n - Sx_p\|} \right) \right\} \\ &\leq [1 - \gamma_n (1 - \lambda)] \|Sd_n - p\| \end{aligned} \tag{14}$$

Similarly,

$$\|Sb_n - p\| \leq (1-\beta_n)\,\lambda\,\|Sa_n - p\| + \beta_n\lambda\,\|Sc_n - p\| \tag{15}$$

Substituting Inequality 13 in Inequality 14 and Inequality 14 in Inequality 15, and making the necessary simplifications considering that $[1-\mu_n(1-\lambda)] \leq 1$, $[1-\gamma_n(1-\lambda)] \leq 1$, and

$$\|Sb_n - p\| \leq \lambda\,\|Sa_n - p\| \tag{16}$$

In addition,

$$\begin{aligned}\|Sa_{n+1} - p\| &\leq \|Sa_{n+1} - f(T,a_n)\| + \|f(T,a_n) - p\| \\ &\leq \varepsilon_n + \|Sa_{n+1} - p\| \\ &\leq \varepsilon_n + (1-\alpha_n)\,\|Sb_n - p\| + \alpha_n\,\|Tb_n - p\| \\ &\leq \varepsilon_n + (1-\alpha_n)\,\|Sb_n - p\| + \alpha_n\left\{\lambda\,\|Sb_n - p\| + \mu\left(\frac{\|Sb_n - Tb_n\|\,.\,\|Sx_p - Tx_p\|}{1+\|Sb_n - Sx_p\|}\right)\right\} \\ &= \varepsilon_n + [1-\alpha_n(1-\lambda)]\,\|Sb_n - p\|\end{aligned} \tag{17}$$

Substituting Inequality 16 in Inequality 17,

$$\|Sa_{n+1} - p\| \leq \varepsilon_n + \lambda[1-\alpha_n(1-\lambda)]\,\|Sa_n - p\|$$

Hence, from Lemma 2.8, it is obtained that $\lim\limits_{n\to\infty} Sa_n = p$.

Conversely, assume that $\lim\limits_{n\to\infty} Sa_n = p$. It will be shown that $\lim\limits_{n\to\infty} \varepsilon_n = 0$:

$$\begin{aligned}\varepsilon_n &= \|Sa_{n+1} - f(T,a_n)\| \\ &\leq \|Sa_{n+1} - p\| + \|f(T,a_n) - p\| \\ &\leq \|Sa_{n+1} - p\| + (1-\alpha_n)\,\|Sb_n - p\| + \alpha_n\,\|Tb_n - p\|\end{aligned} \tag{18}$$

By using similar operations in Inequalities 13-17, from Inequality 18,

$$\varepsilon_n \leq \|Sa_{n+1} - p\| + \lambda[1-\alpha_n(1-\lambda)]\,\|Sa_n - p\|$$

If the limit for the above inequality is taken, then it is obtained that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. □

Example 3.4. Assume that $X = \mathbb{R}$ is Banach space, $Y = [0,1]$, and $S, T : Y \to X$ are defined by $Sx = \frac{1}{5}\sin 2x$ and $Tx = \frac{1}{10}\sin^2 x$ respectively. It can be observed that S and T are pairs of maps satisfying Inequality 9 and having unique common fixed point $p = 0$. If the iteration method given by Equation 8 is rewritten for S and T with $\alpha_n = \beta_n = \gamma_n = \mu_n = \frac{1}{n+1}$:

$$\begin{cases} x_{n+1} = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2y_n + \frac{1}{2(n+1)}\sin^2 y_n\right] \\ y_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{2(n+1)}\right)\sin^2 x_n + \frac{1}{2(n+1)}\sin^2 z_n\right] \\ z_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2w_n + \frac{1}{2(n+1)}\sin^2 w_n\right] \\ w_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2x_n + \frac{1}{2(n+1)}\sin^2 x_n\right] \end{cases}$$

It can be observed from Theorem 3.1 that the $\{Sx_n\}_{n=0}^{\infty}$ sequence to be obtained from the above equation converges to $p = 0$. If the sequence $\{Sa_n\}_{n=0}^{\infty}$ is chosen as $Sa_n = (\frac{1}{n+5})$, then $\lim\limits_{n\to\infty} |Sx_n - Sa_n| = 0$. Hence, $\{Sa_n\}_{n=0}^{\infty}$ is approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$. If the iteration method given by Equation

12 is rewritten using S and T:

$$\begin{cases} a_{n+1} = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2b_n + \frac{1}{2(n+1)}\sin^2 b_n\right] \\ b_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{2(n+1)}\right)\sin^2 a_n + \frac{1}{2(n+1)}\sin^2 c_n\right] \\ c_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2d_n + \frac{1}{2(n+1)}\sin^2 d_n\right] \\ d_n = \frac{1}{2}\sin^{-1}\left[\left(\frac{n}{n+1}\right)\sin 2a_n + \frac{1}{2(n+1)}\sin^2 a_n\right] \end{cases}$$

From the above equality, it is obtained that

$$a_{n+1} = \frac{1}{2}\sin^{-1}\left[\begin{array}{c} \frac{1}{2}\left(\frac{n}{n+1}\right)^2\sin^2 a_n + \frac{n}{2(n+1)^2}\sin^2\left\{\frac{1}{2}\sin^{-1}\left(\frac{n}{n+1}\right)u_1 + \frac{1}{2(n+1)}\sin^2\left(\frac{1}{2}\sin^{-1}u_1\right)\right\} \\ +\frac{1}{2(n+1)}\sin^2\left\{\frac{1}{2}\sin^{-1}\left\{\frac{n}{2(n+1)}\sin^2 a_n + \frac{1}{2(n+1)}\sin^2\left(\frac{1}{2}\sin^{-1}u_2\right)\right\}\right\} \end{array}\right]$$

in which $u_1 = \left(\frac{n}{n+1}\right)\sin 2a_n + \frac{1}{2(n+1)}\sin^2 a_n$ and $u_2 = \left(\frac{n}{n+1}\right)\mathrm{u}_1 + \frac{1}{2(n+1)}\sin^2\left(\frac{1}{2}\sin^{-1}\mathrm{u}_1\right)$. If $\varepsilon_n = |Sa_{n+1} - f(T, a_n)|$, then $\lim_{n\to\infty}\left|\left(\frac{1}{n+6}\right) - f(T, a_n)\right| = 0$. As a result, $\lim\limits_{n\to\infty}\varepsilon_n = 0$.

Theorem 3.5. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Consider the sequence $\{Sx_n\}_{n=0}^{\infty}$ obtained from the iteration method given by Equation 8 and the sequence $\{Su_n\}_{n=0}^{\infty}$ obtained from the Jungck-CR iteration method given by Equation 3 under the condition $\alpha_1 < \alpha_n \leq 1$, where $x_0 = u_0 \in Y$. In this case, $\{Sx_n\}_{n=0}^{\infty}$ has a better convergence rate with respect to $\{Su_n\}_{n=0}^{\infty}$.

Proof.

From Inequality 11, it is attained that

$$\|Sx_{n+1} - p\| \leq \lambda^{n+1}\prod_{i=0}^{n}[1 - \alpha_i(1-\lambda)]\,\|Sx_0 - p\| \tag{19}$$

In addition, if similar steps are taken as in the proof of Theorem 3.1 for the Jungck-CR iteration method, then

$$\|Su_{n+1} - p\| \leq [1 - \alpha_n(1-\lambda)]\,\|Su_n - p\|$$

If induction is applied to the above inequality, then

$$\|Su_{n+1} - p\| \leq \prod_{i=0}^{n}[1 - \alpha_i(1-\lambda)]\,\|Su_0 - p\| \tag{20}$$

If the assumption $\alpha_1 < \alpha_n \leq 1$ is applied to Inequalities 19 and 20, then

$$\|Sx_{n+1} - p\| \leq \lambda^{n+1}[1 - \alpha_1(1-\lambda)]^{n+1}\,\|Sx_0 - p\|$$

and

$$\|Su_{n+1} - p\| \leq [1 - \alpha_1(1-\lambda)]^{n+1}\,\|Su_0 - p\|$$

Denote

$$a_n = \lambda^{n+1}[1 - \alpha_1(1-\lambda)]^{n+1}$$

and

$$b_n = [1 - \alpha_1(1-\lambda)]^{n+1}$$

Then,

$$\begin{aligned}\psi_n &= \frac{a_n}{b_n}\\ &= \frac{\lambda^{n+1}[1-\alpha_1(1-\lambda)]^{n+1}}{[1-\alpha_1(1-\lambda)]^{n+1}}\\ &= \lambda^{n+1}\end{aligned}$$

Since $\lambda^{n+1} < 1$, it is obtained that $\lim_{n\to\infty} \psi_n = 0$. From Definition 2.5, $\{Sx_n\}_{n=0}^{\infty}$ has a better convergence speed than $\{Su_n\}_{n=0}^{\infty}$. $\square$

The following example shows that iteration method given by Equation 8 has a higher convergence speed under favorable conditions than the other Jungck-type methods presented in this paper:

Example 3.6. Assume that $X = \mathbb{R}$ is Banacah space, $Y = [0.5, 1.5]$, and $S, T : [0.5, 1.5] \to [1, 81]$ are defined by $Sx = 16x^4$ and $Tx = x^8 + 24x^3 - 44x^2 + 35$, respectively. It can be observed that $T1 = S1 = 16$ and $T([0.5, 1.5]) \subseteq S([0.5, 1.5])$, and (S, T) are pairs of maps satisfying Inequality 9 with $\lambda = 0.4$ and $\mu = 0.2$. The convergence of the Jungck-type iteration methods provided by Equations 2-6 and Equation 8 to the $p = T1 = S1 = 16$ with the control sequences $\alpha_n = \beta_n = \gamma_n = \mu_n = a_n = b_n = c_n = \frac{3}{20}$, for the initial condition $x_0 = 0.75$, are shown in Tables 1 and 2. The following conclusions can be obtained from these tables:

- While newly defined iteration method given by Equation 8 reaches the fixed point at the 16th step,
- the Jungck-SP iteration method given by Equation 2 reaches the fixed point at the 72nd step,
- the Jungck-CR iteration method given by Equation 3 reaches the fixed point at the 17th step,
- the Jungck-Agarwal iteration method given by Equation 4 reaches the fixed point at the 17th step,
- the Jungck-Sahu iteration method given by Equation 5 reaches the fixed point at the 17th step, and
- the Jungck-Khan iteration method given by Equation 6 reaches the fixed point at the 101st.

Table 1. Convergence of some iteration methods for the initial point $x_0 = 0.75$

x_n	New Jungck Type	Jungck-CR	Jungck-Agarwal
x_1	0.75	0.75	0.75
x_2	1.05295512496838	1.05414377123399	1.06137648831351
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{11}	0.99999999994918	0.99999999994220	0.99999999978115
x_{12}	1.00000000000533	1.00000000000615	1.00000000002691
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{15}	0.99999999999999	1.00000000000007	1.00000000000041
x_{16}	1.00000000000000	0.99999999999999	0.99999999999995
x_{17}	$\vdots$	1.00000000000000	1.00000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2. Convergence of some iteration methods for the initial point $x_0 = 0.75$

x_n	Jungck-Sahu	Jungck-SP	Jungck-Khan
x_1	0.75	0.75	0.75
x_2	1.04466023384367	0.87897378710221	0.85315042838595
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{11}	0.99999999993357	0.99820409366943	0.999542288145375
x_{12}	1.00000000000718	0.99883957379660	0.999202265784102
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{16}	0.99999999999999	0.99979688882405	0.999951335812014
x_{17}	1.00000000000000	0.99986856789305	0.99998589613924
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{72}	$\vdots$	1.00000000000000	$\vdots$
x_{101}	$\vdots$	$\vdots$	1.00000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 3.7. Assume that X, Y and the mappings S and T are defined as in Theorem 3.1 with $p = Tx_p = Sx_p$. Suppose that $S_1, T_1 : Y \to X$ are the approximation operators of S and T, respectively, satisfying the conditions $T_1x_p = S_1x_p = q$, $\|Tx - T_1x\| \leq \varepsilon_1$, and $\|Sx - S_1x\| \leq \varepsilon_2$, for ε_1 and ε_2 and for each $x \in Y$. Consider the sequence $\{Sx_n\}_{n=0}^{\infty}$ obtained from the iteration method given by Equation 8 with the condition $\frac{1}{2} \leq \alpha_n$. Moreover, suppose that $\{S_1e_n\}_{n=0}^{\infty}$ is any sequence obtained from the following equation:

$$\begin{cases} S_1e_{n+1} = (1-\alpha_n)\, S_1f_n + \alpha_n T_1 f_n \\ S_1f_n = (1-\beta_n)\, T_1e_n + \beta_n T_1 g_n \\ S_1g_n = (1-\gamma_n)\, S_1h_n + \gamma_n T_1 h_n \\ S_1h_n = (1-\mu_n)\, S_1e_n + \mu_n T_1 e_n \end{cases} \tag{21}$$

If $\{S_1e_n\}_{n=0}^{\infty} \to q$ as $n \to \infty$, then

$$\|p-q\| \leq \frac{7\varepsilon_1 + 11\varepsilon_2}{1-\lambda}$$

Proof.
By using Equation 8 and Inequalities 9 and 21,

$$\begin{aligned} \|Sw_n - S_1h_n\| &= \|(1-\mu_n)\, Sx_n + \mu_n Tx_n - (1-\mu_n)\, S_1e_n - \mu_n T_1e_n\| \\ &\leq (1-\mu_n)\, \|Sx_n - S_1e_n\| + \mu_n \|Tx_n - T_1e_n\| \\ &\leq (1-\mu_n)\, \|Sx_n - Se_n\| + (1-\mu_n)\, \|Se_n - S_1e_n\| \\ &\quad + \mu_n \|Tx_n - Te_n\| + \mu_n \|Te_n - T_1e_n\| \\ &\leq (1-\mu_n)\, \|Sx_n - Se_n\| + (1-\mu_n)\, \varepsilon_2 \\ &\quad + \mu_n \|Tx_n - Te_n\| + \mu_n \varepsilon_1 \end{aligned} \tag{22}$$

Moreover,

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - Se_n\| + \mu \left(\frac{\|Sx_n - Tx_n\| \, . \, \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$$

Suppose that $D_1 = \left(\frac{\|Sx_n - Tx_n\| . \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$. Then, it is attained that

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - Se_n\| + \mu D_1 \tag{23}$$

Substituting Inequality 23 in Inequality 22,

$$\|Sw_n - S_1h_n\| \leq [1 - \mu_n (1 - \lambda)] \|Sx_n - Se_n\| + \mu_n \mu D_1 + (1 - \mu_n) \varepsilon_2 + \mu_n \varepsilon_1 \tag{24}$$

Similarly,

$$\|Sz_n - S_1g_n\| \leq (1 - \gamma_n) \|Sw_n - Sh_n\| + (1 - \gamma_n) \varepsilon_2 + \gamma_n \|Tw_n - Th_n\| + \gamma_n \varepsilon_1$$

and

$$\|Tw_n - Th_n\| \leq \lambda \|Sw_n - Sh_n\| + \mu \left(\frac{\|Sw_n - Tw_n\| \, . \, \|Sh_n - Th_n\|}{1 + \|Sw_n - Sh_n\|} \right)$$

Suppose that $D_2 = \left(\frac{\|Sw_n - Tw_n\| . \|Sh_n - Th_n\|}{1 + \|Sw_n - Sh_n\|} \right)$. Then, it is obtained that

$$\|Sz_n - S_1g_n\| \leq [1 - \gamma_n (1 - \lambda)] \|Sw_n - Sh_n\| + \gamma_n \mu D_2 + (1 - \gamma_n) \varepsilon_2 + \gamma_n \varepsilon_1 \tag{25}$$

Moreover,

$$\|Sw_n - Sh_n\| \leq \|Sw_n - S_1h_n\| + \varepsilon_2 \tag{26}$$

Substituting Inequality 26 in Inequality 25,

$$\|Sz_n - S_1g_n\| \leq [1 - \gamma_n (1 - \lambda)] \|Sw_n - S_1h_n\| + [1 - \gamma_n (1 - \lambda)]\varepsilon_2 + \gamma_n \mu D_2 + (1 - \gamma_n) \varepsilon_2 + \gamma_n \varepsilon_1 \tag{27}$$

Substituting Inequality 24 in Inequality 27,

$$\begin{aligned} \|Sz_n - S_1g_n\| \leq{} & [1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)] \|Sx_n - S_1e_n\| + [1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)]\varepsilon_2 \\ & + [1 - \gamma_n (1 - \lambda)]\mu_n \mu D_1 + [1 - \gamma_n (1 - \lambda)] (1 - \mu_n) \varepsilon_2 + [1 - \gamma_n (1 - \lambda)]\mu_n \varepsilon_1 \\ & + [1 - \gamma_n (1 - \lambda)]\varepsilon_2 + (1 - \gamma_n) \varepsilon_2 + \gamma_n \varepsilon_1 + \gamma_n \mu D_2 \end{aligned}$$

Similarly,

$$\|Sy_n - S_1f_n\| \leq (1 - \beta_n) \|Tx_n - Te_n\| + (1 - \beta_n) \varepsilon_1 + \beta_n \|Tz_n - Tg_n\| + \beta_n \varepsilon_1 \tag{28}$$

and

$$\|Tz_n - Tg_n\| \leq \lambda \|Sz_n - Sg_n\| + \mu \left(\frac{\|Sz_n - Tz_n\| \, . \, \|Sg_n - Tg_n\|}{1 + \|Sz_n - Sg_n\|} \right)$$

Suppose that $D_3 = \left(\frac{\|Sz_n - Tz_n\| . \|Sg_n - Tg_n\|}{1 + \|Sz_n - Sg_n\|} \right)$. Then, it is obtained that

$$\|Tz_n - Tg_n\| \leq \lambda \|Sz_n - S_1g_n\| + \lambda \varepsilon_2 + \mu D_3$$

Therefore,

$$\begin{aligned} \|Tz_n - Tg_n\| \leq{} & \lambda[1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)] \|Sx_n - S_1e_n\| \\ & + \lambda[1 - \mu_n (1 - \lambda)][1 - \gamma_n (1 - \lambda)]\varepsilon_2 \\ & + \lambda[1 - \gamma_n (1 - \lambda)]\mu_n \mu D_1 + \lambda[1 - \gamma_n (1 - \lambda)] (1 - \mu_n) \varepsilon_2 \\ & + \lambda[1 - \gamma_n (1 - \lambda)]\mu_n \varepsilon_1 + \lambda[1 - \gamma_n (1 - \lambda)]\varepsilon_2 \\ & + \lambda (1 - \gamma_n) \varepsilon_2 + \gamma_n \lambda \varepsilon_1 + \gamma_n \lambda \mu D_2 + \lambda \varepsilon_2 + \mu D_3 \end{aligned} \tag{29}$$

In addition,

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - Se_n\| + \mu \left(\frac{\|Sx_n - Tx_n\| . \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$$

Suppose that $D_4 = \left(\frac{\|Sx_n - Tx_n\| . \|Se_n - Te_n\|}{1 + \|Sx_n - Se_n\|} \right)$. Then, it is attained that

$$\|Tx_n - Te_n\| \leq \lambda \|Sx_n - S_1 e_n\| + \lambda \varepsilon_2 + \mu D_4 \tag{30}$$

Substituting Inequalities 29 and 30 in Inequality 28,

$$\begin{aligned}
\|Sy_n - S_1 f_n\| \leq{} & (1-\beta_n) \lambda \|Sx_n - S_1 e_n\| + (1-\beta_n) \mu D_4 + (1-\beta_n) \lambda \varepsilon_2 + (1-\beta_n) \varepsilon_1 \\
& + \beta_n \lambda [1 - \mu_n (1-\lambda)][1 - \gamma_n (1-\lambda)] \|Sx_n - S_1 e_n\| \\
& + \beta_n \lambda [1 - \mu_n (1-\lambda)][1 - \gamma_n (1-\lambda)] \varepsilon_2 \\
& + \beta_n \lambda [1 - \gamma_n (1-\lambda)] \mu_n \mu D_1 + \beta_n \gamma_n \lambda \mu D_2 + \beta_n \mu D_3 \\
& + \beta_n \lambda [1 - \gamma_n (1-\lambda)] (1-\mu_n) \varepsilon_2 + \beta_n \lambda [1 - \gamma_n (1-\lambda)] \mu_n \varepsilon_1 \\
& + \beta_n \lambda [1 - \gamma_n (1-\lambda)] \varepsilon_2 + \beta_n \lambda (1-\gamma_n) \varepsilon_2 + \beta_n \gamma_n \lambda \varepsilon_1 + \beta_n \lambda \varepsilon_2 + \beta_n \varepsilon_1
\end{aligned} \tag{31}$$

Moreover,

$$\begin{aligned}
\|Sx_{n+1} - S_1 e_{n+1}\| \leq{} & (1-\alpha_n) \|Sy_n - S_1 f_n\| \\
& + \alpha_n \|Ty_n - Tf_n\| + \alpha_n \varepsilon_1
\end{aligned} \tag{32}$$

and

$$\|Ty_n - Tf_n\| \leq \lambda \|Sy_n - Sf_n\| + \mu \left(\frac{\|Sy_n - Ty_n\| . \|Sf_n - Tf_n\|}{1 + \|Sy_n - Sf_n\|} \right)$$

Suppose that $D_5 = \left(\frac{\|Sy_n - Ty_n\| . \|Sf_n - Tf_n\|}{1 + \|Sy_n - Sf_n\|} \right)$. Then,

$$\|Ty_n - Tf_n\| \leq \lambda \|Sy_n - S_1 f_n\| + \lambda \varepsilon_2 + \mu D_5 \tag{33}$$

Substituting Inequalities 31 and 33 in Inequality 32,

$$\begin{aligned}
\|Sx_{n+1} - S_1 e_{n+1}\| \leq{} & [1 - \alpha_n (1-\lambda)] (1-\beta_n) \lambda \|Sx_n - S_1 e_n\| + [1 - \alpha_n (1-\lambda)] (1-\beta_n) \lambda \varepsilon_2 \\
& + [1 - \alpha_n (1-\lambda)] (1-\beta_n) \mu D_4 + [1 - \alpha_n (1-\lambda)] (1-\beta_n) \varepsilon_1 \\
& + [1 - \alpha_n (1-\lambda)] \beta_n [1 - \mu_n (1-\lambda)] [1 - \gamma_n (1-\lambda] \|Sx_n - S_1 e_n\| \\
& + [1 - \alpha_n (1-\lambda)] \beta_n \lambda [1 - \mu_n (1-\lambda)] [1 - \gamma_n (1-\lambda] \varepsilon_2 \\
& + [1 - \alpha_n (1-\lambda)] \beta_n \lambda [1 - \gamma_n (1-\lambda)] \mu_n \mu D_1 \\
& + [1 - \alpha_n (1-\lambda)] \beta_n \gamma_n \lambda \, \mu D_2 + [1 - \alpha_n (1-\lambda)] \beta_n \mu D_3 \\
& + [1 - \alpha_n (1-\lambda)] \beta_n \lambda [1 - \gamma_n (1-\lambda)] (1-\mu_n) \varepsilon_2 \\
& + [1 - \alpha_n (1-\lambda)] \beta_n \lambda [1 - \gamma_n (1-\lambda)] \mu_n \varepsilon_1 \\
& + [1 - \alpha_n (1-\lambda)] \; \beta_n \lambda [1 - \gamma_n (1-\lambda)] \varepsilon_2 \\
& + [1 - \alpha_n (1-\lambda)] \beta_n \lambda (1-\gamma_n) \varepsilon_2 + [1 - \alpha_n (1-\lambda)] \beta_n \lambda \gamma_n \varepsilon_1 \\
& + [1 - \alpha_n (1-\lambda)] \beta_n \lambda \varepsilon_2 + [1 - \alpha_n (1-\lambda)] \beta_n \varepsilon_1 + \alpha_n \lambda \varepsilon_2 + \alpha_n \mu D_5 + \alpha_n \varepsilon_1
\end{aligned}$$

For the above inequality , if necessary simplifications are made considering that $\frac{1}{2} \leq \alpha_n$ and $\alpha_n, \beta_n, \gamma_n, \mu_n \in [0,1]$ and $\lambda < 1$, then it is attained that

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq\ & \{[1-\alpha_n(1-\lambda)](1-\beta_n)+[1-\alpha_n(1-\lambda)]\beta_n\}\|Sx_n-S_1e_n\| \\ & +[1-\alpha_n(1-\lambda)]D_1+[1-\alpha_n(1-\lambda)]D_2+[1-\alpha_n(1-\lambda)]D_3 \\ & +[1-\alpha_n(1-\lambda)]D_4+\alpha_nD_5 \\ & +\Big\{[1-\alpha_n(1-\lambda)](1-\beta_n)+[1-\alpha_n(1-\lambda)]\beta_n \\ & \quad +[1-\alpha_n(1-\lambda)]+[1-\alpha_n(1-\lambda)]+\alpha_n\Big\}\varepsilon_1 \\ & +\Big\{[1-\alpha_n(1-\lambda)](1-\beta_n)+[1-\alpha_n(1-\lambda)]\beta_n+[1-\alpha_n(1-\lambda)] \\ & \quad +[1-\alpha_n(1-\lambda)]+[1-\alpha_n(1-\lambda)]+[1-\alpha_n(1-\lambda)]+\alpha_n\Big\}\varepsilon_2\end{aligned}$$

and

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq\ & [1-\alpha_n(1-\lambda)]\|Sx_n-S_1e_n\|+[1-\alpha_n(1-\lambda)](D_1+D_2+D_3+D_4) \\ & +\alpha_nD_5+\{3[1-\alpha_n(1-\lambda)]+\alpha_n\}\varepsilon_1+\{5[1-\alpha_n(1-\lambda)]+\alpha_n\}\varepsilon_2\end{aligned}$$

Therefore,

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq\ & [1-\alpha_n(1-\lambda)]\|Sx_n-S_1e_n\| \\ & +2\alpha_n(D_1+D_2+D_3+D_4+D_5) \\ & +7\alpha_n\varepsilon_1+11\alpha_n\varepsilon_2\end{aligned}$$

Hence,

$$\begin{aligned}\|Sx_{n+1}-S_1e_{n+1}\| \leq\ & [1-\alpha_n(1-\lambda)]\|Sx_n-S_1e_n\| \\ & +\alpha_n(1-\lambda)\left\{\frac{7\varepsilon_1+11\varepsilon_2+2(D_1+D_2+D_3+D_4+D_5)}{1-\lambda}\right\}\end{aligned} \tag{34}$$

It is clear that $\lim\limits_{n\to\infty}(D_1+D_2+D_3+D_4+D_5)=0$. With this in mind, consider the following equalities:

$$a_n=\|Sx_n-S_1e_n\|$$

$$\mu_n=\alpha_n(1-\lambda)\in(0,1)$$

and

$$\eta_n=\left\{\frac{7\varepsilon_1+11\varepsilon_2+2(D_1+D_2+D_3+D_4+D_5)}{1-\lambda}\right\}$$

It can be observed that Inequality 34 satisfies all the conditions of Lemma 2.9. Hence, it follows by its conclusion that

$$0\leq\limsup_{n\to\infty}\|Sx_n-S_1e_n\|\leq\limsup_{n\to\infty}\left\{\frac{7\varepsilon_1+11\varepsilon_2+2(D_1+D_2+D_3+D_4+D_5)}{1-\lambda}\right\}=\frac{7\varepsilon_1+11\varepsilon_2}{1-\lambda}$$

By using $\{S_1e_n\}_{n=0}^{\infty}\to q$ and $\{Sx_n\}_{n=0}^{\infty}\to p$,

$$\|p-q\|\leq\frac{7\varepsilon_1+11\varepsilon_2}{1-\lambda}$$

□

4. Conclusion

This paper introduces a new four-step fixed-point iteration method, which is rewritten with the help of the Jungck Contraction Principle, and some fixed-point theorems for a general class of mappings are investigated. The results show that the new iteration method converges faster than the other methods presented in this paper. This method is stable and can obtain a data dependence result. Numerical examples are given to concretize the stability and convergence speed analysis. In future work, researchers can rewrite the iteration method provided in this paper by considering the Volterra-Fredholm integral equations as an operator in complex-valued Banach spaces with appropriate conditions and study the solution of these integral equations.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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