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DNA CODES FROM REVERSIBLE GROUP CODES BY A VIRUS OPTIMISATION ALGORITHM

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Abstract. In this paper, we employ group rings and some known results on group codes to study reversible group DNA codes. We define and study reversible cyclic DNA codes from a group ring point of view and we also introduce the notion for self-reciprocal group ring elements. Moreover, we search for reversible group DNA codes with the use of a virus optimisation algorithm. We obtain many good DNA codes that satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints.

The interest in studying and designing DNA codes has been started with Adleman when he solved a computationally difficult mathematical problem by introducing an algorithm using DNA strands and molecular biology tools [1] and it is still an ongoing research area. Some known methods for designing DNA codes that satisfy certain conditions include the study of reversible codes [15], reversible self-dual codes over $GF(4)$ [9], the study of cyclic and extended cyclic constructions or the study of linear constructions [7].

Recently in [4], linear codes derived from group ring elements are considered to construct reversible DNA codes that satisfy the Hamming distance constraint. This suggests that the study of group rings is an interesting research direction that may have some useful applications to DNA coding. In this work, we employ group rings and a matrix construction given in [4] to study reversible cyclic DNA codes. We also use group rings to define a self-reciprocal group ring element. Moreover, we construct reversible group codes of different lengths over the finite commutative Frobenius ring R, that satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints.

The paper is organised as follows. In Section 2, we give the basic definitions and results on linear codes, DNA codes, group rings, group codes and reversible group codes. In Section 3, we define and study reversible cyclic DNA codes from a group ring point of view. In this section, we also define a self-reciprocal group ring element. In Section 4, we present two generator matrices for reversible group codes which we then use to search for DNA codes that satisfy the Hamming distance,

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the reverse, the reverse-complement and the fixed GC-content constraints. In our search scheme, we employ a virus optimisation algorithm which allows us to obtain numerical results in a reasonable quick time. We finish with concluding remarks and directions for possible future research.

1. Preliminaries

1.1. **Linear Codes and DNA Codes.** In this section, we recall basic definitions on linear codes, DNA codes and DNA constraints.

A linear code of length n over $\mathbb{F}_4$ is a subspace of $\mathbb{F}_4^n$, and we also call an element of a linear code a codeword. The Hamming distance $d(\mathbf{x},\mathbf{y})$ between two codewords is the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ are distinct. The minimum Hamming distance d_H of a linear code $\mathcal{C}$ is defined as

$$min\{d_H(\mathbf{x},\mathbf{y}) \mid \mathbf{x}\neq\mathbf{y},\ \forall\ \mathbf{x},\mathbf{y}\in\mathcal{C}\}.$$

Let $S_{D_4}=\{A,C,G,T\}$ represents the four nucleotides in DNA, which are adenine (A), cytosine (C), guanine (G) and thymine (T) and let $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, where $x_i\in S_{D_4}$. A DNA code $\mathcal{D}$ of length n is defined as a set of codewords $(x_1,x_2,\ldots,x_n)$ where $x_i\in S_{D_4}=\{A,T,C,G\}$. We use a hat to denote the Watson-Crick complement of a nucleotide, $\hat{A}=T,\hat{T}=A,\hat{C}=G$ and $\hat{G}=C$. Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in S_{D_4}$, then $\mathbf{x}^r=(x_n,x_{n-1},\ldots,x_2,x_1)$ $\mathbf{x}^c=(x_1^c,x_2^c,\ldots,x_n^c)$ and $\mathbf{x}^{rc}=(x_n^c,x_{n-1}^c,\ldots,x_2^c,x_1^c)$ denote the reverse of a DNA codeword, the complement of a DNA codeword and the reverse complement of a DNA codeword respectively. In this paper, the fixed GC-content is simply half the length of the DNA code D.

A good DNA code $\mathcal{D}$ of length n is defined as a set of codewords $(x_1,x_2,\ldots,x_n)$ where $x_i\in S_{D_4}=\{A,T,C,G\}$, such that $\mathcal{D}$ satisfies some or all of the following constraints [2]:

(i) The Hamming distance constraint (HD):

$$min\{d_H(\mathbf{x},\mathbf{y}):\ \ \forall\ \mathbf{x},\mathbf{y}\in\mathcal{D}\text{ and }x\neq y\}$$

(ii) The reverse constraint (RV):

$$min\{d_H(\mathbf{x}^r,\mathbf{y}):\forall\ \mathbf{x},\mathbf{y}\in\mathcal{D}\text{ and }\mathbf{x}^r\neq\mathbf{y}\ \}$$

(iii) The reverse-complement constraint (RC):

$$min\{d_H(\mathbf{x}^{rc},\mathbf{y}):\ \forall\ \mathbf{x},\mathbf{y}\in\mathcal{D}\text{ and }\mathbf{x}^{rc}\neq\mathbf{y}\}$$

(iv) The fixed GC-content constraint (GC): The set of codewords with length n, distance d and GC weight w, where w is the total number of Gs and Cs present in the DNA strand:

$$w_{\mathbf{x}_{DNA}}=|\{x_i:\mathbf{x}_{DNA}=(x_i),x_i\in\{C,G\}\}|.$$

A DNA code can be identified with a code over $\mathbb{F}_4=\{0,1,\omega,\omega^2\}$ by employing the standard bijective correspondence between $\mathbb{F}_4$ and the DNA alphabet $S_{D_4}=\{A,T,C,G\}$ given by

$$\eta:\mathbb{F}_4\to S_{D_4},$$

with $\eta(0)=A$, $\eta(1)=T$, $\eta(\omega)=C$ and $\eta(\omega^2)=G$. The same correspondence has already been used in the literature, for example, please see [9]. We extend the bijection η so that $\eta(\mathcal{C})$ is regarded as a DNA code for some code $\mathcal{C}$ over $\mathbb{F}_4$.

We denote the complete weight enumerator of a code $\mathcal{C}$ over $\mathbb{F}_4$ by

$$CWE_{\mathcal{C}}(a,b,c,d) = \sum_{c\in\mathcal{C}} a^{n_0(c)}b^{n_1(c)}c^{n_\omega(c)}d^{n_{\omega^2}(c)},$$

where $n_s(c)$ denotes the number of occurrences of s in a codeword c. We identify the complete weight enumerator of a DNA code $\mathcal{D}$ with that of a code $\mathcal{C}$ over $\mathbb{F}_4$, where $\mathcal{D} = \eta(\mathcal{C})$. The GC-weight of a codeword $c \in \mathcal{C}$ is the sum of $n_\omega(c)$ and $n_{\omega^2}(c)$. Therefore, if we let

$$GCW_{\mathcal{C}}(a,b) = CWE_{\mathcal{C}}(a,a,b,b),$$

then $GCW_{\mathcal{C}}(a,b)$ is the GC-weight enumerator of a code $\mathcal{C}$, where the coefficient of b^i is the same as the number of codewords with GC-weight i.

Let $A_4^R(n,d)$ denote the maximum cardinality of a DNA code for a given distance d and length n that satisfies the Hamming distance and reverse constraints. Let $A_4^{RC}(n,d)$ be the maximum size of a DNA code of length n satisfying the HD and RC constraints for a given d, $A_4^{GC}(n,d,w)$ be the maximum size of a DNA code of length n satisfying the HD constraint for a given d with a constant GC-weight w, and $A_4^{RC,GC}(n,d,w)$ the maximum size of a DNA code of length n satisfying the HD and RC constraints for a given d with a constant GC-weight w. In [12], for an even n, the following equality is given;

$$A_4^{RC}(n,d) = A_4^R(n,d). \tag{1.1}$$

1.2. **Group Rings and Group Codes.** We shall now give the standard definition of group rings. Let G be a finite group of order n and let R be a finite ring. Then any element in RG is of the form $v = \sum_{i=1}^n \alpha_{g_i} g_i$, $\alpha_{g_i} \in R$, $g_i \in G$. Addition in RG is done by coordinate addition, namely

$$\sum_{i=1}^n \alpha_{g_i} g_i + \sum_{i=1}^n \beta_i g_i = \sum_{i=1}^n (\alpha_{g_i} + \beta_i) g_i.$$

The product of two elements in RG is given by

$$\left(\sum_{i=1}^n \alpha_{g_i} g_i\right)\left(\sum_{j=1}^n \beta_j g_j\right) = \sum_{i,j} \alpha_{g_i}\beta_j g_i g_j.$$

This gives that the coefficient of g_k in the product is $\sum_{g_i g_j = g_k} \alpha_{g_i}\beta_j$.

The following matrix construction was used to study group codes over Frobenius rings in [6]. Let R be a finite commutative Frobenius ring and let $G = \{g_1, g_2, \dots, g_n\}$ be a group of order n and let $v = \sum_{i=1}^n \alpha_{g_i} \in RG$. Define the matrix $\sigma(v) \in M_n(R)$ to be

$$\sigma(v) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \dots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \dots & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \alpha_{g_n^{-1}g_3} & \dots & \alpha_{g_n^{-1}g_n} \end{pmatrix}. \tag{1.2}$$

We note that the elements $g_1^{-1}, \dots, g_n^{-1}$ are simply the elements of the group G given in some order. This particular order is used because it aids in certain proofs and computations. In [6], the following code construction is given:

$$\mathcal{C}(v) = \langle \sigma(v) \rangle. \tag{1.3}$$

The code is formed by taking the row space of $\sigma(v)$ over the ring R. Such codes are refereed to as group codes or, for simplicity, G-codes. Moreover, in [6], it is shown that this matrix construction of G-codes corresponds to an ideal in the group ring RG and thus the resulting group code has the group G as a subgroup of its automorphism group. Please see [6] for more details on group codes generated from group rings. From now on, every time we refer to G-codes, we mean codes constructed as given above.

1.3. **Reversible Group Codes.** Here, we recall an interesting result from [4] on group codes. Namely, this result shows that for certain groups and for a specific ordering of the group elements, one can construct G-codes that are reversible. We first start with a definition from [4].

Definition 1.1. A code $\mathcal{C}$ is said to be reversible of index α if $\mathbf{a}_i$ is a vector of length α and $\mathbf{c}^\alpha = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{s-1}) \in \mathcal{C}$ implies that $(\mathbf{c}^\alpha)^r = (\mathbf{a}_{s-1}, \mathbf{a}_{s-2}, \ldots, \mathbf{a}_1, \mathbf{a}_0) \in \mathcal{C}$.

Let G be a finite group of order $n = 2l$ and let $H = \{e, h_1, h_2, \ldots, h_{\ell-1}\}$ be a subgroup of index 2 in G. Let $\beta \notin H$ be an element in G, with $\beta^{-1} = \beta$. We list the elements of $G = \{g_1, g_2, \ldots, g_n\}$ as follows:

$$\{e, h_1, \ldots, h_{\ell-1}, \beta h_{\ell-1}, \beta h_{\ell-2}, \beta h_2, \beta h_1, \beta\}. \tag{1.4}$$

The following result was proved in [4].

Theorem 1.2. *Let R be a finite ring. Let G be a finite group of order $n = 2\ell$ and let $H = \{e, h_1, h_2, \ldots, h_{\ell-1}\}$ be a subgroup of index 2 in G. Let $\beta \notin H$ be an element in G with $\beta^{-1} = \beta$. List the elements of G as in (1.4), then any linear G-code in R^n (a left ideal in RG) is a reversible code of index 1.*

In [4], the authors make a connection between reversible G-codes and DNA codes, this is because reversibility is a desirable property for DNA codes.

1.4. **Virus Optimization Algorithm.** A new bio-inspired optimization technique called as virus optimization algorithm (VOA) is proposed in [5] for difficult and complex mathematical and engineering problems. The VOA is a meta-heuristic optimization technique based on population and it mimics the behavior of viruses assaulting a living cell. In each replication step, the number of the viruses increases then antivirus applied to virus population to avoid the plosive growing of the virus population. Thus, the number of the virus in the population is controlled with help of the antivirus. In the VOA, the viruses in the population are separated into two groups as common and strong. In the initialization phase of the VOA, there are two steps; parameter setting and the generation of initial viruses. Parameter setting is a key for an effective search process in the search space. After the parameters have been set, the initial virus population is randomly produced and the viruses are classified. In the replication procedure, new viruses are produced by using strong and common viruses in the initial population. When the new viruses are generated by the replication procedure, the corresponding objective function values are evaluated. Then, the old and new viruses are then combined together. If the performance of the virus population is not improved, the antivirus procedure is applied to the population and it is followed by the verification of the termination criterion. If the termination criteria has not been met, the replication is repeated.For more details on this approach see [10].

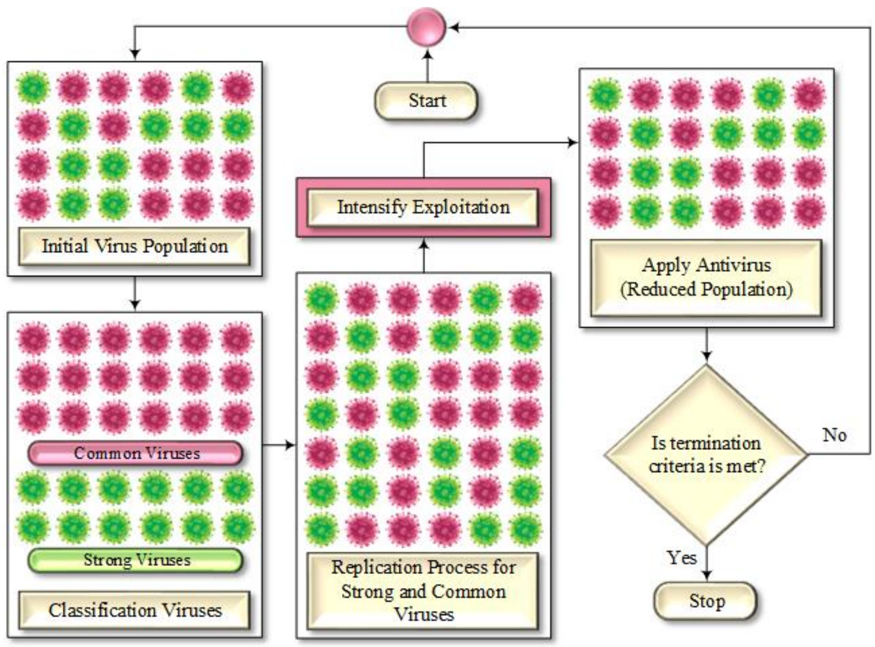

FIGURE 1. Flowchart of VOA

2. Reversible Group Codes and DNA Codes

In this section, we use the results from Section 1.3 to define and study reversible cyclic DNA codes. We also define self-reciprocal group ring elements.

2.1. Reversible Cyclic DNA Codes as Ideal in Group Rings. Cyclic codes have a canonical algebraic description as ideals in the polynomial ring $R[x]/\langle x^n - 1\rangle$, where R is a Frobenius ring and n is the length of the code. An alternate view of cyclic codes is to see them as ideals in the group ring RC_n where C_n is the cyclic group of order n.

For a cyclic code C, there exists a relationship between reversible codes and self-reciprocal polynomials. More precisely, in Theorem 1 in [13], the following is proven. The cyclic code over $\mathbb{F}_q$, generated by the monic polynomial $g(x)$, is reversible if and only if $g(x)$ is self-reciprocal. Therefore, in this setting the search for reversible codes coincides with the search for self-reciprocal polynomials that divide $x^n - 1$ over the field $\mathbb{F}_q$.

Often in the literature, reversible cyclic codes are studied over polynomial rings due to the fact that polynomial rings have a rich algebraic description. In this section, we intend to study reversible cyclic codes in a different setting - from a group ring point of view. We begin with a definition.

Definition 2.1. Let C_n be the cyclic group of order n and let

$$\{e = c^0, c, c^2, \dots, c^{n-1}\}$$

be a fixed listing of the elements of C_n. Let $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$. The reciprocal of v is defined as

$$v^* = c^{n-1}(\sum_{i=0}^{n-1} \alpha_{c^i}(c^i)^{-1}) = \sum_{i=0}^{n-1} \alpha_{c^i} c^{n-(i+1)}.$$

We call the group ring element v self-reciprocal if and only if $v^* = v$.

For the cyclic group C_n, the matrices $\sigma(v)$ and $\sigma(v^*)$ can be written as follows:

$$\sigma(v) = \begin{pmatrix} \alpha_e & \alpha_c & \alpha_{c^2} & \cdots & \alpha_{c^{n-1}} \\ \alpha_{c^{n-1}} & \alpha_e & \alpha_c & \cdots & \alpha_{c^{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_c & \alpha_{c^2} & \alpha_{c^3} & \cdots & \alpha_e \end{pmatrix},$$

$$\sigma(v^*) = \begin{pmatrix} \alpha_{c^{n-1}} & \alpha_{c^{n-2}} & \cdots & \alpha_c & \alpha_e \\ \alpha_e & \alpha_{c^{n-1}} & \cdots & \alpha_{c^2} & \alpha_c \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^{n-2}} & \alpha_{c^{n-3}} & \cdots & \alpha_e & \alpha_{c^{n-1}} \end{pmatrix}.$$

Theorem 2.2. *The cyclic code $C(v) = \langle \sigma(v) \rangle$ where $v \in RC_n$, is reversible of index 1 if and only if v is self-reciprocal.*

Proof. The proof follows from the fact that v is self-reciprocal if and only if $\sigma(v) = \sigma(v^*)$. The index 1 follows from the construction of the matrix $\sigma(v)$. □

We illustrate this theorem with an example.

Example 2.3. Let $v_1 = 1 + 2c + 2c^3 + c^4 \in \mathbb{Z}_3 C_5$, where $C_5 = \{e, c, c^2, c^3, c^4\}$. Here, $\alpha_e = 1$, $\alpha_c = 2$, $\alpha_{c^2} = 0$, $\alpha_{c^3} = 2$ and $\alpha_{c^4} = 1$. Then

$$\sigma(v_1) = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

Now, $v_1^* = c^4(1 + 2c^4 + 2c^2 + c) = 1 + 2c + 2c^3 + c^4 = v_1$, and

$$\sigma(v_1^*) = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

Thus $\sigma(v_1) = \sigma(v_1^*)$. Also, $\sigma(v) = \sigma(v^*)$ can be written as

$$\sigma(v_1) = \sigma(v_1^*) = \begin{pmatrix} 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

Clearly, $C(v_1) = C(v_1^*) = \langle \sigma(v_1) \rangle = \langle \sigma(v_1^*) \rangle$ is the $[5, 4, 2]$ cyclic code. We also see that the code $C(v_1)$ is reversible since in the generator matrix the reverse of each row of $C(v_1)$ is also in $C(v_1)$.

We now give the group ring analogue of the notion of lifted polynomials which is defined in [16].

Definition 2.4. Let $C_n = \{e, c, \ldots, c^{n-1}\}$ be a cyclic group of order n and $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in \mathbb{F}_p C_n$ be a self-reciprocal element. A lifted element of v denoted by $\ell(v) \in \mathbb{F}_{p^s} C_n$ is defined as follows:

(1) if n is odd then

$$\ell(v) = \sum_{i=0}^{(n-1)/2} \theta_i;\ \theta_i = \begin{cases} \beta_i c^i + \beta_i c^{n-i}, & \alpha_{c^i} \neq 0, \\ 0, & \alpha_{c^i} = 0, \end{cases}$$

(2) if n is even then

$$\ell(v) = \sum_{i=0}^{n/2} \theta_i;\ \theta_i = \begin{cases} \beta_i c^i + \beta_i c^{n-i}, & \alpha_{c^i} \neq 0,\ i \neq \frac{n}{2}, \\ 0, & \alpha_{c^i} = 0, \\ \beta_{n/2} c^{n/2}, & \alpha_{c^i} \neq 0,\ i = \frac{n}{2}, \end{cases}$$

where $\beta_i \in \mathbb{F}_{p^s}^*$.

Lemma 2.5. *If the element v is self-reciprocal then $\ell(v)$ is self-reciprocal.*

Proof. The result follows from the definitions. □

Example 2.6. Let $v = 1 + 2c + 2c^3 + c^4 \in \mathbb{Z}_3 C_5$, where $C_5 = \{e, c, c^2, c^3, c^4\}$. Then, for $\beta_i \in \mathbb{F}_{3^4} = \mathbb{F}_{81}$,

$$\ell(v) = \sum_{i=0}^{2} \theta_i = \theta_0 + \theta_1 + \theta_2 = (\beta_0 1 + \beta_0 c^4) + (\beta_1 c + \beta_1 c^3) + 0,$$

$$\ell(v) = \beta_0 + \beta_1 c + \beta_1 c^3 + \beta_0 c^4.$$

For $\beta_0 = \alpha^4, \beta_1 = \alpha^6$, we have $\ell(v) = \alpha^4 + \alpha^6 c + \alpha^6 c^3 + \alpha^4 c^4$. Now,

$$\ell(v^*) = \alpha^4 + \alpha^6 c + \alpha^6 c^3 + \alpha^4 c^4 = \ell(v),$$

which gives that $\ell(v)$ is self-reciprocal.

Theorem 2.7. *Let R be a finite commutative Frobenius ring and let C_n be the cyclic group of order n. Let $\ell(v)$ be a lifted element of a self-reciprocal element of group ring RC_n. Then the cyclic code $C(\ell(v))$ is reversible.*

Proof. Follows from Theorem 2.2. □

The following definition is the group ring analogue of the notion of the co-term polynomial which is defined in [8].

Definition 2.8. Let C_n be the cyclic group of order n and let

$$\{e, c, c^2, \ldots, c^{n-1}\}$$

be a fixed listing of C_n. Let $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$. Then v is called a co-term element if $\alpha_{c^i} = \alpha_{c^{n-i}}$ for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Moreover, $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$ is the co-term element if and only if $(\alpha_{c^1}, \alpha_{c^2} \ldots, \alpha_{c^{n-1}})$ is self-reversible.

Example 2.9. Consider the element v_1 from Example 2.3. We saw there that $v_1 = v_1^*$ and therefore v_1 is self-reciprocal. The element v_1 is not a co-term element since for instance, $\alpha_{c^1} \neq \alpha_{c^4}$, i.e., $\alpha_{c^1} = 2$ and $\alpha_{c^4} = 1$.

We denote the vector $\mathbf{v} = (\alpha_{c^0}, \alpha_{c^1}, \ldots, \alpha_{c^{n-1}}) \in R^n$ for $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$. In [8], the following is proven.

Lemma 2.10. *Let $S \subseteq R^n$ be a non empty subset such that $\mathbf{v}^r \in S$ whenever $\mathbf{v} \in S$. Then the code generated by S is a linear reversible code of length n over R.*

Theorem 2.11. *Let $v = \sum_{i=0}^{n-1} \alpha_{c^i} c^i \in RC_n$ be a co-term element and let t be a specified positive integer. Suppose v corresponds to the vector $\mathbf{v} = (\alpha_{c^0}, \alpha_{c^1}, \ldots, \alpha_{c^{n-1}}) \in R^n$. For any length n and even dimension, define the $(2t+2) \times n$ matrix as:*

$$\kappa_t(v) = \begin{pmatrix} \alpha_{c^{n-(t+1)}} & \alpha_{c^{n-t}} & \cdots & \alpha_{c^{n-(t+3)}} & \alpha_{c^{n-(t+2)}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^1} & \alpha_{c^2} & \cdots & \alpha_{c^{n-1}} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \cdots & \alpha_{c^{n-2}} & \alpha_{c^{n-1}} \\ \alpha_{c^{n-1}} & \alpha_{c^0} & \cdots & \alpha_{c^{n-3}} & \alpha_{c^{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^{n-t}} & \alpha_{c^{n-(t-1)}} & \cdots & \alpha_{c^{n-(t+2)}} & \alpha_{c^{n-(t+1)}} \end{pmatrix},$$

and for odd length n and odd dimension, define the $(2t+2) \times n$ matrix as:

$$\kappa_t(v) = \begin{pmatrix} \alpha_{c^{n-(t+1)}} & \alpha_{c^{n-t}} & \cdots & \alpha_{c^{n-(t+3)}} & \alpha_{c^{n-(t+2)}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^1} & \alpha_{c^2} & \cdots & \alpha_{c^{n-1}} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \cdots & \alpha_{c^{n-2}} & \alpha_{c^{n-1}} \\ \alpha_{c^{n-1}} & \alpha_{c^0} & \cdots & \alpha_{c^{n-3}} & \alpha_{c^{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{c^{n-t}} & \alpha_{c^{n-(t-1)}} & \cdots & \alpha_{c^{n-(t+2)}} & \alpha_{c^{n-(t+1)}} \\ \alpha_{c^{n-(n-1)/2}} & \alpha_{c^{n-((n-1)/2-1)}} & \cdots & \alpha_{c^{n-((n-1)/2+2)}} & \alpha_{c^{n-((n-1)/2+1)}} \end{pmatrix},$$

where $t < \lfloor \frac{n}{2} \rfloor$. Then the code $C = \langle \kappa_t(v) \rangle$ is reversible.

Proof. Let $\mathbf{v} = (\alpha_{c^0}, \alpha_{c^1}, \ldots, \alpha_{c^{n-1}})$. Since $v \in RC_n$ is a co-term element, it follows that $(\alpha_{c^1}, \alpha_{c^2}, \ldots, \alpha_{c^{n-1}})$ is self-reversible. Also, since v is a co-term element, for a positive integer $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ the reverse of the i-th row of the matrix $\kappa_t(v)$ equals to the $(n+1) - i$-th row. We also have

$$(\alpha_{c^{n-(n-1)/2}}, \alpha_{c^{n-((n-1)/2-1)}}, \cdots, \alpha_{c^{n-((n-1)/2+2)}}, \alpha_{c^{n-((n-1)/2+1)}})^r =$$
$$(\alpha_{c^{n-(n-1)/2}}, \alpha_{c^{n-((n-1)/2-1)}}, \cdots, \alpha_{c^{n-((n-1)/2+2)}}, \alpha_{c^{n-((n-1)/2+1)}}).$$

In both cases of the theorem, the spanning sets, that is the rows of $\kappa_t(v)$, satisfy the conditions of Lemma 2.10. Thus the code is reversible. □

Example 2.12. Let $v = 1 + \omega c + c^2 + \omega^2 c^3 + \omega^2 c^6 + c^7 + \omega c^8 \in \mathbb{F}_4 C_9$ be a co-term element and

$$\mathbf{v} = (1, \omega, 1, \omega^2, 0, 0, \omega^2, 1, \omega)$$

be the corresponding vector. All the codes $\langle \kappa_t(v) \rangle$ are reversible. For $t = 0$,

$$\kappa_0(v) = \begin{pmatrix} \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} \end{pmatrix}$$

$$= \begin{pmatrix} \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 \\ 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega \end{pmatrix}.$$

For $t = 2$,

$$\kappa_2(v) = \begin{pmatrix} \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} \\ \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} \\ \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} \\ \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} & \alpha_{c^8} \\ \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} & \alpha_{c^7} \\ \alpha_{c^7} & \alpha_{c^8} & \alpha_{c^0} & \alpha_{c^1} & \alpha_{c^2} & \alpha_{c^3} & \alpha_{c^4} & \alpha_{c^5} & \alpha_{c^6} \end{pmatrix}$$

$$= \begin{pmatrix} \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 & \omega & 1 \\ 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 & \omega \\ \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega & 1 \\ 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 & \omega \\ \omega & 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 & 1 \\ 1 & \omega & 1 & \omega & 1 & \omega^2 & 0 & 0 & \omega^2 \end{pmatrix}.$$

It can be easily seen that in each of the above matrices, the reverse of each row is contained in the same matrix.

2.2. **Self-Reciprocal Group Ring Elements.** In this section, we define a self-reciprocal group ring element which is the analogue notion of the notion of a self-reciprocal polynomial.

Definition 2.13. Let G be a finite group and let $\{g_1, g_2, \ldots, g_n\}$ be a fixed listing of the elements of G. Also, let $v = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$. The reciprocal of v is defined as

$$v^* = \sum_{i=1}^{n} \alpha_{g_i} g_{n-(i-1)}.$$

We call v self-reciprocal if and only if $v^* = v$.

Lemma 2.14. *Let* $v = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$. *Then* $(v^*)^* = v$.

Proof. We have that $v^* = \sum_{i=1}^{n} \alpha_{g_i} g_{n-(i-1)}$ by definition. Applying the reciprocal definition to the element v^* again:

$$(v^*)^* = \sum_{i=1}^{n} \alpha_{g_i} g_{n-[n-(i-1)-1]} = \sum_{i=1}^{n} \alpha_{g_i} g_i = v.$$

This gives the result. $\square$

It is well known that a group ring is isomorphic to a well defined ring of matrices and thus every group ring element has an associated matrix. We now generalise the matrix representation of a reciprocal cyclic group ring element to a more general group ring.

The matrix representation of a reciprocal group ring element is as follows:

$$\sigma(v^*) = \begin{pmatrix} \alpha_{g_1g_n} & \alpha_{g_1g_{n-1}} & \alpha_{g_1g_{n-2}} & \cdots & \alpha_{g_1g_1} \\ \alpha_{g_2g_n} & \alpha_{g_2g_{n-1}} & \alpha_{g_2g_{n-2}} & \cdots & \alpha_{g_2g_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_ng_n} & \alpha_{g_ng_{n-1}} & \alpha_{g_ng_{n-2}} & \cdots & \alpha_{g_ng_1} \end{pmatrix}.$$

We now look at an example in which we give the matrix representations of a dihedral group ring element and its reciprocal.

Example 2.15. Consider $\mathbb{Z}_3D_8$ where $\{e, a, a^2, a^3, ba^3, ba^2, ba, b\}$ is the fixed listing of elements of D_8. Let $v = 2 + a^2 + ba + 2ba^2 + ba^3 \in \mathbb{Z}_3D_8$. Then

$$\sigma(v) = \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{pmatrix},$$

and $\sigma(v)$ can be written as following:

$$\sigma(v) = \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Clearly $C(v) = \langle \sigma(v) \rangle$ is the $[8, 4, 4]$ code. It is also clear that $C(v)$ is reversible, that is, the reverse of each codeword of $C(v)$ is also in $C(v)$.

Next,

$$\sigma(v^*) = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

and $\sigma(v^*)$ can be written as follows

$$\sigma(v^*) = \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

We note here that although $\sigma(v) \neq \sigma(v^*)$, the codes $C(v)$ and $C(v^*)$ are the same.

An element v is said to be self-reversible if $v = v^*$.

Theorem 2.16. *Let $v = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$ be a self-reversible element and let t be a specified positive integer where $t < \frac{n}{2}$. Suppose v corresponds to the vector $\mathbf{v} = (\alpha_{g_1}, \alpha_{g_2}, \alpha_{g_3}, \ldots, \alpha_{g_n}) \in R^n$. For any length n and even dimension, we define the $(2t+2) \times n$ matrix as*

$$\kappa_t(v) = \begin{pmatrix} \alpha_{g_{t+2}g_1} & \alpha_{g_{t+2}g_2} & \cdots & \alpha_{g_{t+2}g_{n-1}} & \alpha_{t+2g_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_3g_1} & \alpha_{g_3g_2} & \cdots & \alpha_{g_3g_{n-1}} & \alpha_{g_3g_n} \\ \alpha_{g_2g_1} & \alpha_{g_2g_2} & \cdots & \alpha_{g_2g_{n-1}} & \alpha_{g_2g_n} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \cdots & \alpha_{g_1^{-1}g_{n-1}} & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \cdots & \alpha_{g_2^{-1}g_{n-1}} & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_{t+1}^{-1}g_1} & \alpha_{g_{t+1}^{-1}g_2} & \cdots & \alpha_{g_{t+1}^{-1}g_{n-1}} & \alpha_{g_{t+1}^{-1}g_n} \end{pmatrix}.$$

Then the code $C = \langle \kappa_t(v) \rangle$ is reversible.

Proof. Since v is a self-reversible element, for a positive integer $i \in \{1, 2 \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ the reverse of the i-th row of the matrix $\kappa_t(v)$ equals to the $(n+1) - i$-th row. Therefore, the spanning set, that is the rows of the matrix, satisfy Lemma 2.10. This completes the proof. $\square$

Example 2.17. Let $v = 1 + ab \in \mathbb{F}_2 V_4$, be a self-reversible element, where $V_4 = \{1, b, ab, a\}$ is a Klein-4-group. We have that $\mathbf{v} = (1, 0, 1, 0)$. Then for $t = 0$

$$\kappa_0(v) = \begin{pmatrix} \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} \end{pmatrix},$$

$$\kappa_0(v) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

so $\langle \kappa_0(v) \rangle$ is reversible. Also, for $t = 1$

$$\kappa_1(v) = \begin{pmatrix} \alpha_{g_3g_1} & \alpha_{g_3g_2} & \alpha_{g_3g_3} & \alpha_{g_3g_4} \\ \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \alpha_{g_2^{-1}g_4} \end{pmatrix},$$

$$\kappa_1(v) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

which is clear that $\langle \kappa_1(v) \rangle$ is reversible.

Example 2.18. For the quaternion group Q_8, the fixed listing of elements is

$$\{1, i, j, k, -k, -j, -i, -1\} = \{g_1, g_2, \ldots, g_8\}.$$

Let

$$v = 1 + 2j + k - k - 2j - 1 \in \mathbb{F}_3 Q_8,$$

then $\mathbf{v} = (1, 0, 2, 1, 1, 2, 0, 1)$ is the corresponding vector.

For $t=0$,

$$\kappa_0(v)=\begin{pmatrix} \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} & \alpha_{g_2g_5} & \alpha_{g_2g_6} & \alpha_{g_2g_7} & \alpha_{g_2g_8} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} & \alpha_{g_1^{-1}g_5} & \alpha_{g_1^{-1}g_6} & \alpha_{g_1^{-1}g_7} & \alpha_{g_1^{-1}g_8} \end{pmatrix},$$

$$\kappa_0(v)=\begin{pmatrix} i & -1 & k & -j & j & -k & 1 & -i \\ 1 & i & j & k & -k & -j & -i & -1 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 \end{pmatrix}.$$

For $t=2$,

$$\kappa_2(v)=\begin{pmatrix} \alpha_{g_4g_1} & \alpha_{g_4g_2} & \alpha_{g_4g_3} & \alpha_{g_4g_4} & \alpha_{g_4g_5} & \alpha_{g_4g_6} & \alpha_{g_4g_7} & \alpha_{g_4g_8} \\ \alpha_{g_3g_1} & \alpha_{g_3g_2} & \alpha_{g_3g_3} & \alpha_{g_3g_4} & \alpha_{g_3g_5} & \alpha_{g_3g_6} & \alpha_{g_3g_7} & \alpha_{g_3g_8} \\ \alpha_{g_2g_1} & \alpha_{g_2g_2} & \alpha_{g_2g_3} & \alpha_{g_2g_4} & \alpha_{g_2g_5} & \alpha_{g_2g_6} & \alpha_{g_2g_7} & \alpha_{g_2g_8} \\ \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} & \alpha_{g_1^{-1}g_5} & \alpha_{g_1^{-1}g_6} & \alpha_{g_1^{-1}g_7} & \alpha_{g_1^{-1}g_8} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \alpha_{g_2^{-1}g_4} & \alpha_{g_2^{-1}g_5} & \alpha_{g_2^{-1}g_6} & \alpha_{g_2^{-1}g_7} & \alpha_{g_2^{-1}g_8} \\ \alpha_{g_3^{-1}g_1} & \alpha_{g_3^{-1}g_2} & \alpha_{g_3^{-1}g_3} & \alpha_{g_3^{-1}g_4} & \alpha_{g_3^{-1}g_5} & \alpha_{g_3^{-1}g_6} & \alpha_{g_3^{-1}g_7} & \alpha_{g_3^{-1}g_8} \end{pmatrix},$$

$$\kappa_2(v)=\begin{pmatrix} k & j & -i & -1 & 1 & i & -j & -k \\ j & -k & -1 & i & -i & 1 & k & -j \\ i & -1 & k & -j & j & -k & 1 & -i \\ 1 & i & j & k & -k & -j & -i & -1 \\ -i & 1 & -k & j & -j & k & -1 & i \\ -j & k & 1 & -i & i & -1 & -k & j \end{pmatrix}=\begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

3. Computational Results

In this section, we define two generator matrices using the map given in Equation (1.2) with a fixed listing of the group elements as given in Equation (1.4). We employ the cyclic group of even order and the dihedral group of order $2n$. We next use these generator matrices to search for DNA codes over $\mathbb{F}_4$. We perform our search in the software package MAGMA ([3]) using a heuristic search scheme called the virus optimization algorithm (VOA). This method, as shown in [10], allows us to obtain the computational results significantly faster then the standard linear search.

We obtain many DNA codes of up to and including length 32. Our DNA codes satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints. We find the lower bounds on $A_4^{RC,GC}(n,d,k)$ by computing the complete weight enumerators of all DNA codes that we found. The generator matrices, weight enumerators, GC-weight enumerators for the codes constructed can be found at [11].

Let $w_1\in R\mathcal{C}_{2n}$, where $\mathcal{C}_{2n}$ is the cyclic group of order $2n$ with its elements being listed as follows:

$$\{1,c^2,c^4,c^6,\ldots,c^{2n-2},c^nc^{2n-2},c^nc^{2n-4},\ldots,c^nc^2,c^n\}. \tag{3.1}$$

Then the generator matrix $\sigma(w_1)$ has the following form:

$$\mathcal{G}_1=\sigma(w_1)=\begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix},$$

where

$$A_1=cir(\alpha_1,\alpha_{c^2},\ldots,\alpha_{c^{2n-2}})$$

is a $n \times n$ circulant matrix,

$$B_1 = rcir(\alpha_{c^n c^{2n-2}}, \alpha_{c^n c^{2n-4}}, \ldots, \alpha_{c^n})$$

is a $n \times n$ reverse circulant matrix. We note that B_2 is a reverse circulant matrix in which the first row is obtained by reversing the last row of the matrix B_1. The matrix A_2 is a circulant matrix in which the first row is obtained by reversing the last row of the matrix A_1. More precisely, $A_2 = cir(\alpha_1, \alpha_{c^{2n-2}}, \ldots, \alpha_{c^4}, \alpha_{c^2})$ is an $n \times n$ circulant matrix and $B_2 = rcir(\alpha_{c^n c^2}, \ldots, \alpha_{c^n c^{2n-4}}, \alpha_{c^n c^{2n-2}}, \alpha_{c^n})$ is an $n \times n$ reverse circulant matrix.

Let $w_2 \in R\mathcal{D}_{2n}$, where $\mathcal{D}_{2n}$ is the dihedral group of order $2n$ with its elements being listed as follows:

$$\{e, a, a^2, \ldots, a^{n-1}, ba^{n-1}, ba^{n-2}, \ldots, ba, b\}. \tag{3.2}$$

Then the generator matrix $\sigma(w_2)$ has the following form:

$$\mathcal{G}_2 = \sigma(w_2) = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix},$$

where

$$A = cir(\alpha_e, \alpha_a, \ldots, \alpha_{a^{n-1}})$$

is a $n \times n$ circulant matrix and

$$B = cir(\alpha_{ba^{n-1}}, \alpha_{ba^{n-2}}, \ldots, \alpha_b)$$

is a $n \times n$ circulant matrix.

We now present a small example of how we construct the DNA codes using our group ring approach.

Example 3.1. Let D_6 be a dihedral group of order 6 with the ordering of elements $\{e, a, a^2, ba^2, ba, b\}$, $v = w + wa + wa^2 \in \mathbb{F}_4 D_6$ then the generator matrix has the form

$$\sigma(v) = \mathcal{G}_2 = \begin{pmatrix} w & w & w & 0 & 0 & 0 \\ w & w & w & 0 & 0 & 0 \\ w & w & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & w & w \\ 0 & 0 & 0 & w & w & w \\ 0 & 0 & 0 & w & w & w \end{pmatrix}. \tag{3.3}$$

From the above generator matrix, we construct a DNA code $\mathcal{C}$ with 16 codewords satisfying R-constraint with $d = 3$ as follows;

$$\begin{aligned} A_4^R(6,3) = \{&AAAAAA, AAAGGG, AAATTT, GGGAAA, CCCAAA, AAACCC, \\ &TTTAAA, TTTTTT, GGGCCC, CCCCCC, GGGGGG, TTTCCC, \\ &CCCGGG, GGGTTT, TTTGGG, CCCTTT\}. \end{aligned} \tag{3.4}$$

We know by [12] that for an even n:

$$A_4^{RC}(n,d) = A_4^R(n,d). \tag{3.5}$$

Therefore $A_4^{RC}(6,3) = A_4^R(6,3)$. The GC-weight enumerator of $\mathcal{C}$ is

$$GCW(a,b) = 4a^6 + 8a^3b^3 + 4b^6.$$

Thus we construct a DNA code with 8 codewords satisfying hamming distance constraint 3, reversible complement constraint and fixed GC-content constraint with $k = 3$.

We now employ the generator matrices $\mathcal{G}_1$ and $\mathcal{G}_2$, to search for DNA codes that satisfy the Hamming distance, the reverse, the reverse-complement and the fixed GC-content constraints of lengths up to and including 32. We tabulate our findings in Table 2 and Table 1. The results that are equal to or better than the currently known best bounds are written in bold, and new results are also written in bold. Generator matrices, GC-weight enumerators and parameters of codes in Tables 2 and 1 can be found in [11].

TABLE 1. Lower bounds on $A_4^{RC}(n,d)$ and $A_4^{RC,GC}(n,d,k)$ from $\mathcal{G}_2$

n	d	$A_4^{RC}(n,d)$	$A_4^{RC,GC}(n,d,k)$
4	3	**16**	**12**
6	3	**64**	**30**
6	2	**1024**	**480**
8	4	**256**	**176**
8	3	**256**	**152**
8	2	**4096**	**2240**
12	6	**4096**	**1848**
12	4	**16384**	**6144**
14	5	**65536**	**13728**
14	4	**65536**	**13728**
16	6	**65536**	**25880**
16	4	**1048576**	**461824**
16	2	**268435456**	**105431040**
18	4	**4194304**	**1400256**
18	3	**16777216**	**3111680**
18	2	**4294967296**	**1429733376**
20	5	**16777216**	**2956096**
20	4	**1073741824**	**376832000**
20	3	**4294967296**	**756760576**
20	2	**68719476736**	**12108169216**
20	6	**1048576**	**369008**
20	7	**1048576**	**369512**
22	6	**16777216**	**2821728**
22	2	**1099511627776**	**339270959104**
24	4	**68719476736**	**22409117696**
24	3	**68719476736**	**11098587136**
24	2	**17592186044416**	**2841238306816**
24	6	**268435456**	**86739968**

4. Conclusion

In this work, we showed that one can construct good DNA codes from G-codes that are reversible- this is a crucial property for DNA codes. We defined and studied reversible cyclic DNA codes and we also defined self-reciprocal group ring elements. We presented two generator matrices that one can use to search for DNA codes. We employed these generator matrices with the use of only two groups, the cyclic group of even order and the dihedral group of order $2n$, to search for reversible cyclic and dihedral DNA codes that satisfy the Hamming distance, the reverse, the reverse complement and the GC-weight enumerator constraints. Our group ring approach proved to be successful as we constructed many DNA codes. A possible research direction is to consider reversible group ring approach and specifically Theorem 3.10 to construct, possibly, more DNA codes with better parameters.

TABLE 2. Lower bounds on $A_4^{RC}(n,d)$ and $A_4^{RC,GC}(n,d,k)$ from $\mathcal{G}_1$

n	d	$A_4^{RC}(n,d)$	$A_4^{RC,GC}(n,d,k)$
24	4	**4294967296**	**1387323392**
24	3	**68719476736**	**22160015360**
24	2	**17592186044416**	**2835513081856**
26	2	**281474976710656**	**81000264630272**
28	4	**1099511627776**	**328637349888**
28	3	**17592186044416**	**2630898155520**
28	2	**4503599627370496**	**1345974567960576**
30	4	**1125899906842624**	**30497345372160**0
30	3	**4503599627370496**	**650610034606080**
30	2	**1125899906842624**	**162652508651520**
32	2	**1152921504606846976**	**322709486693253120**
32	4	**17592186044416**	**4928618364928**

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This study does not be necessary ethical committee permission or any special permission.

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