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Common Best Proximity Point Theorems On Cone B-Metric Spaces Over Banach Algebras

Seyed Masoud AGHAYAN[1,1], Ahmad ZIREH[2], Ali EBADIAN[1]

[1]*Department of Mathematic, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran*

[2]*Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran*

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Abstract

In this paper, we obtain the existence of some common best proximity point theorems for generalized Lipschitz contractive mappings on cone b-metric spaces over Banach algebras without assumption of normality. Our results generalize the corresponding result by Xu and Radenović (Fixed Point Theory and Appl. 2014, 2014: 102) and by Huang and Radenović (J. Computational Anal. and Appl. 2016, 20(3)). Further, we give an example to illustrate that our works are never equivalent with the counterparts in the literature.

1. INTRODUCTION AND PRELIMINARIES

Cone metric spaces were introduced by Huang and Zhang as a generalization of metric spaces in [8]. The distance $d(x,y)$ of two elements x and y in cone metric space X is defined to be a vector in an ordered Banach spaces $E,$ a mapping $T: X \to X$ is said to be contractive if there is a constant $k \in [0,1)$ such that

$$d(T\,x, Ty) \le kd(x,y), \ \ x, y \in X.$$

In [8], the authors proved that there exists a unique fixed point for contractive mappings in complete cone metric spaces then many authors have focused on fixed point theorems in such spaces. But recently, scholars obtained that cone metric fixed point results are equivalent to the usual metric ones. Because of these reasons, people have not worked on studying fixed point theorems in cone metric spaces. However, the current situation changed, since very recently, Liu and Xu [15] introduced cone metric space over Banach algebras and defined generalized Lipschitz mappings where the contractive coefficient is vector instead of usual real constant.

Moreover, they gave an example to illustrate that the non-equivalence of fixed point theorems between cone metric spaces over Banach algebras and usual metric spaces. Subsequently, Xu and Radenović [17] omitted the normality of cones by using c-sequence. The main purpose of this paper is to obtain some common best point theorems, which is a kind of generalization of common fixed point theorem using extra

[1] Corresponding author, e-mail: *masoud.aghayan64@gmail.com*

condition, in such framework without assumption of normal cones by using a kind of new Lipschitz condition. Also we generalize some fixed results in [17] and [8]. Finally, by an example, we support our results and obtain the non-equivalence of common best proximity results between b-metric spaces over Banach algebras and usual b-metric spaces. Let E be a real Banach algebra. That is, E is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in E,\ \alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;

2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

4. $\|xy\| \leq \|x\|\|y\|$.

Throughout this paper, we shall assume E be a Banach algebra with a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in E$ and 0 the zero element of E. An element $x \in E$ is said to be invertible if there is an inverse element $y \in E$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Proposition 1 [16]. Let E be a Banach algebra with a unit e, and $x \in E$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Now let E be a Banach algebra. A subset P of E is called a cone of E if

P is non-empty closed and $\{0,\ e\} \subset P$;

$\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;

$P\ 2 = PP \subset P$;

$P \cap (-P) = \{0\}$.

For a given cone $P \subset E$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int}P$, where $\operatorname{int}P$ denotes the interior of P. If $\operatorname{int}P \neq \emptyset$ then P is called a solid cone. The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in E$,

$$0 \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of P. In the following we always assume that P is a cone in E with $\operatorname{int}P \neq \emptyset$ and "$\preceq$" is the partial ordering with respect to P .

Definition 1 [15]. Let X be a nonempty set and E be a Banach algebra. Suppose that a mapping $d: X \times X \to E$ satisfies for all $x, y, z \in X$,

1. $0 \preceq d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;

2. $d(x,y)=d(y,x)$;

3. $d(x,z) \preceq d(x,\ y)+d(y,z)$.

Then d is called a cone metric on X and (X,d) is called a cone metric space over Banach algebra.

Definition 2 Let X be a nonempty set, $s \geq 1$ be a constant and E be a Banach algebra. Suppose that a mapping $d: X\times X \to E$ satisfies for all $x,y,z \in X$,

1. $0 \preceq d(x,y)$ and $d(x,y)=0$ if and only if $x=y$;

2. $d(x,y)=d(y,x)$;

3. $d(x,z) \preceq s[d(x,\ y)+d(y,z)]$.

Then d is called a cone b-metric on X and (X,d) is called a cone b-metric space over Banach algebra.

Note that a cone metric space over Banach algebra must be a cone b-metric space over Banach algebra. Conversely, it is not true.

Example 1 Let $E = C^1_{\mathbb{R}}\ [a,b]$ be the set of continuous functions on the interval $[a,b]$ with the supremum norm. Define multiplication in the usual way. Then E is a Banach algebra with a unit 1. Set $P=\{x\in E:\ x(t)\ \geq\ 0, t\in[a,b]\}$ and $X=\mathbb{R}$. Define a mapping $d: X\times X \to E$ by $d(x,\ y)(t)=|x-y|^2 e^t$ for all $x,\ y\in X$. So (X,d) is a cone b-metric space over Banach algebra E with the coefficient $s\ =\ 2$, but it is not a cone metric space over Banach algebra since the triangle inequality is not satisfied.

Definition 3 [8]. Let (X,d) be a cone b-metric space over a Banach algebra E, $x\in X$ and let $\{x_n\}$ be a sequence in X. Then:

1. $\{x_n\}$ converges to x whenever for each $c\in E$ with $0 \ll c$, there is a natural number N such that $d(x_n,x) \ll c$ for all $n \geq N$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

2. $\{x_n\}$ is a Cauchy sequence whenever for each $c\in A$ with $0 \ll c$ there is a natural number N such that $d(x_n,\ x_m) \ll c$ for all $n,m \geq N$.

3. (X,d) is a complete cone b-metric space if every Cauchy sequence is convergent.

In the following we have some lemmas that we use them in main theorems.

Lemma 1 [16]. If $r(k)<1$ then $\|k_n\| \to 0\ \ (n\to\infty)$.

Lemma 2 [16]. Let E be a Banach algebra with a unit e and $u,v\in E$. If u commutes with v, then $r(u+v)\leq r(u)+r(v)$, $r(uv)\leq r(u)r(v)$.

Lemma 3 [17]. Let E be a Banach algebra with a unit e and let a be a vector in E. If $r(a)<1$, then

$$r\left((e-a)^{-1}\right)<\frac{1}{1-r(a)} .$$

Lemma 4 [17]. Let E be a Banach algebra with a unit e and P be a cone (not necessarily normal) in E. The following properties hold for each $u, v \in E$,

1. If $0 \preceq u \ll c$ for each $c \in \mathrm{int}P$, then $u = 0$.

2. If $u \preceq v$ and $v \ll w$, then $u \ll w$.

3. If $u \in P$ and $r(u) < 1$, then $\|u^n\| \to 0 \ (n \to \infty)$.

4. If $u \preceq ku$, where $u, k \in P$ and $r(k) < 1$, then $u = 0$.

5. If $c \in \mathrm{int}P$ and $u_n \to 0 \ (n \to \infty)$, then there exists N such that, for all $n > N$, one has $u_n \ll c$.

Let (X, d) be a cone b-metric space over Banach algebra E and A a nonempty subset of X. We say that A is bounded whenever there exists $c \in P$ such that $d(x, y) \preceq c$ for all $x, y \in A$.

Definition 4 Let (X,d) be a cone b-metric space over Banach algebra E and A and B nonempty subsets of X. An element $p \in P$ is said to be a lower bounded for $A \times B$ whenever $p \preceq d(a, b)$ for all $(a, b) \in A \times B$. Moreover, if $p \succeq q$ for all lower bounded q for $A \times B$, then p is called the greatest lower bounded for $A \times B$. In this case, we denote it by $\mathrm{dis}(A, B)$. Clearly in above definition, $\mathrm{dis}(A, B)$ is a unique element in P. Also, 0 is always a lower bounded for $A \times B$.

Let (X, d) be a cone b-metric space over Banach algebra E and A and B nonempty subsets of X. We define

$$A_0 = \{x \in A : \ d(x, \ y) = \mathrm{dis}(A, B) \ \text{for some } y \in B\},$$

$$B_0 = \{y \in B : \ d(x, y) = \mathrm{dis}(A, B) \ \text{for some } x \in A\}.$$

From the above definition, it is clear that for every $x \in A_0$ there exists $y \in B_0$ such that $d(x, y) = \mathrm{dis}(A, B)$ and conversely, for every $y \in B_0$ there exists $x \in A_0$ such that $d(x, y) = \mathrm{dis}(A, B)$.

Definition 5 Let (X, d) be a cone b-metric space over Banach algebra E and A and B nonempty subsets of X. $S : A \to B$ and $T : A \to B$ be non-self mappings, an element $x \in A$ is called a common best proximity point of the mappings if they satisfy the condition that

$$d(x, Sx) = \mathrm{dis}(A, B) = d(x, Tx).$$

Definition 6 Let (A, B) be a pair of nonempty subsets of a cone b-metric space (X, d) over Banach algebra E with $A_0 \neq \varnothing$. Then that pair (A, B) is said to have the weak P-property if and only if

$$\begin{cases} d(x_1, y_1) = \mathrm{dis}(A,B) \\ d(x_2, y_2) = \mathrm{dis}(A,B) \end{cases} \Rightarrow d(x_1, \ x_2) \preceq d(y_1, \ y_2), \tag{1}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Definition 7 Let A and B be two non-empty subsets of a cone b-metric space (X,d) over Banach algebra $E.$ The mappings $S:A\to B$ and $T:A\to B$ are said to be commute proximally if they satisfy the condition that $\left[d(u,Sx)=d(v,Tx)=\operatorname{dis}(A,B)\right]\Rightarrow\ Sv=Tu.$

2. COMMON BEST PROXIMITY POINT THEOREMS FOR GENERALIZED LIPSCHITZ MAPPINGS OF THE FIRST KIND

In this section, by omitting the assumption of normality of the cones, we shall prove some common best proximity point theorems of generalized Lipschitz mappings of the first kind in the setting of cone metric spaces over Banach algebras. We begin this section with reviewing some facts on c-sequence theory.

Definition 8 [12]. Let P be a solid cone in a Banach space $E.$ A sequence $\{u_n\}\subset P$ is a c-sequence if for each $c\gg 0$ there exists $n_0\in N$ such that $u_n\ll c$ for $n\geq n_0.$

Lemma 5 [12]. Let P be a solid cone in a Banach algebra $E,$ $\{u_n\}$ and $\{v_n\}$ be two c-sequences in $P.$ If $\alpha,\beta\in P$ are two arbitrarily given vectors, then $\{\alpha u_n+\beta v_n\}$ is a c-sequence.

Definition 9 Let A and B be two non-empty subsets of a cone b-metric space (X,d) over Banach algebra E with $s\geq 1$ and P be a solid not necessarily normal cone of $E.$ Non-self mappings $S,T:A\ \to\ B$ are said to satisfy generalized Lipschitz condition of the first kind if there exist $\alpha_i\in P$ where $i=1,...,4,$ $s\left(r(\alpha_1)+r(\alpha_2)\right)+r(\alpha_3)+s(s+\ 1)r(\alpha_4)<1$ and α_i $(i\ =\ 1,...,4)$ commute with each other, then for each $x,y\in A,$

$$d(Sx,Sy)\preceq\alpha_1 d(Tx,Ty)+\alpha_2 d(Tx,Sx)+\alpha_3 d(Ty,Sy)\ +\alpha_4\left[d(Ty,Sx)+d(Sy,Tx)\right].$$

Theorem 1 Let (X,d) be a complete cone b-metric space over Banach algebra E with the coefficient $s\geq 1$ and P be a solid cone of $E.$ Let A and B be two non-empty subsets of X and the pair (A,B) satisfies the weak P-property. Moreover, assume that A_0 is non-empty and closed and $S,T:A\to B$ are two non-self mappings satisfying the following conditions:

(a) S and T commute proximally;

(b) T is continuous;

(c) $S(A_0)\subseteq B_0$ and $S(A_0)\subseteq T(A_0)$;

(d) S and T satisfy the generalized Lipschitz condition of the first kind. Then, there exists a unique point $x\in A$ such that

$$d(x,Tx)=\operatorname{dis}(A,B)=d(x,Sx).$$

Proof Let x_0 be a fixed element in $A_0.$ Since $S(A_0)\subseteq T(A_0),$ then there exists an element $x_1\in A_0$ such that $Sx_0=Tx_1.$ Then by continuing this process we can choose $x_n\in A_0$ such that there exists $x_{n+1}\in A_0$ satisfying

$$Sx_n=Tx_{n+1}\ \text{for each}\ n\in N. \tag{2}$$

Since $S(A_0)\subseteq B_0$ there exists an element $u_n\in A_0$ such that

$d(Sx_n, u_n) = dis(A, B)$ for each $n \in N$.
(3)

Further, it follows from the choice x_n and u_n that $d(Sx_n, u_n) = dis(A, B) = d(Sx_{n+1}, u_{n+1})$,

By using the weak P-property and generalized Lipschitz condition of the first kind, we have

$$d(u_n, u_{n+1}) \preceq d(Sx_n, Sx_{n+1}) \preceq \alpha_1 d(T\,x_n, T\,x_{n+1}) + \alpha_2 d(T\,x_n, Sx_n) + \alpha 3 d(T\,x_{n+1}, Sx_{n+1})$$

$$+\alpha_4 \left[d(T\,x_{n+1}, Sx_n) + d(Sx_{n+1}, Tx_n) \right]$$

$$\preceq \alpha_1 d(Sx_{n-1}, Sx_n) + \alpha_2 d(Sx_{n-1}, Sx_n) + \alpha_3 d(Sx_n, Sx_{n+1}) + s\alpha_4 \left[d(Sx_{n-1}, Sx_n) + d(Sx_n, Sx_{n+1}) \right]. \quad (4)$$

Since α_3 commute with α_4, so $r(\alpha_3 + s\alpha_4) \le r(\alpha_3) + sr(\alpha_4) < 1$, by Proposition 1 we have that $e - (\alpha_3 + s\alpha_4)$ is invertible. Hence by (4), we deduce that

$$d(u_n, u_{n+1}) \preceq hd(Sx_{n-1}, Sx_n) \preceq \ldots \preceq h^n d(Sx_0, Sx_1), \quad (5)$$

where $h = (e - (\alpha_3 + s\alpha_4))^{-1} (\alpha_1 + \alpha_2 + s\alpha_4)$. By Lemma 2, Lemma 3 and the fact that $\alpha_i\ (i = 1, \ldots, 4)$ commute, we speculate that

$$r(h) = r\left((e - (\alpha_3 + s\alpha_4))^{-1} (\alpha_1 + \alpha_2 + s\alpha_4) \right) \le \frac{r(\alpha_1 + \alpha_2 + s\alpha_4)}{1 - r(\alpha_3 + s\alpha_4)} \le \frac{r(\alpha_1) + r(\alpha_2) + sr(\alpha_4)}{1 - r(\alpha_3) - sr(\alpha_4)} < 1. \quad (6)$$

(note that by assumption we have $s(r(\alpha_1) + r(\alpha_2)) + r(\alpha_3) + s(s+1) r(\alpha_4) < 1$). Let $m, n \in N$ and $m > n$, we have

$$d(u_n, u_m) \preceq s\left[d(u_n, u_{n+1}) + d(u_{n+1}, u_m) \right] \preceq s\, d(u_n, u_{n+1}) + s^2 \left[d(u_{n+1}, u_{n+2}) + d(u_{n+2}, u_m) \right]$$

$$\preceq s\, d(u_n, u_{n+1}) + s^2 d(u_{n+1}, u_{n+2}) + \ldots + s^{m-n-1} \left[d(u_{m-2}, u_{m-1}) + d(u_{m-1}, u_m) \right]$$

$$\preceq s\, d(u_n, u_{n+1}) + s^2 d(u_{n+1}, u_{n+2}) + \ldots + s^{m-n-1} d(u_{m-2}, u_{m-1}) + s^{m-n} d(u_{m-1}, u_m).$$

Since $r(sh) = sr(h) < 1$ so $(e - sh)$ is invertible and by (5), we imply that

$$d(u_n, u_m) \preceq (sh^n + s^2 h^{n+1} + \ldots + s^{m-n} h^{m-1}) d(Sx_0, Sx_1)$$

$$= sh^n \left(e + sh + \ldots + (sh)^{m-n-1} \right) d(Sx_0, Sx_1) \preceq sh^n (e - sh)^{-1} d(Sx_0, Sx_1).$$

Because $r(h) < 1$ it verifies that $\|h^n\| \to 0$ as $n \to \infty$. Moreover, $\{h_n\}$ is a c-sequence. Therefore, we obtain from Lemma 5 that $\left\{ sh^n (e - sh)^{-1} d(Sx_0, Sx_1) \right\}$ is a c-sequence. Therefore, $\{u_n\}$ is a Cauchy sequence. Since (X, d) is complete, $\{u_n\} \subseteq A_0$ and A_0 is closed. There exists $u \in A_0$ such that $u_n \to u$ as $n \to \infty$. Also, we have that

$d(Sx_n, u_n) = dis(A, B) = d(T\,x_n, u_{n-1})$. Since S and T commute proximally we get that

$$Tu_n = Su_{n-1}. \quad (7)$$

By continuity of T we have $Tu_n \to Tu$ as $n \to \infty$. Also (7) implies that $Su_n \to Tu$ as $n \to \infty$. It means that $\{d(u_n, u)\}$, $\{d(Tu_n, Tu)\}$ and $\{d(Su_n, Tu)\}$ are c-sequence. We claim that $Tu = Su$ because

$$d(Su,Tu) \preceq s[d(Su,Su_n) + d(Su_n,Tu)] \preceq sd(Su,Su_n)$$

$$\preceq s\alpha_1 d(Tu,Tu_n) + s\alpha_2 d(Su,Tu) + s\alpha_3 d(Tu_n,Su_n) + s\alpha_4 [d(Tu_n,Su) + d(Su_n + Tu)]$$

$$\preceq s\alpha_2 d(Su,Tu) + s\alpha_4 [sd(Tu_n,Tu) + sd(Tu,Su)] = (s\alpha_2 + s^2\alpha_4) d(Su,Tu).$$

Put $k = s\alpha_2 + s^2\alpha_4$, since $r(k) < 1$ then by part 4 of Lemma 4 we obtain $d(Su,Tu) = 0$. Since $Su \in S(A_0) \subseteq B_0$, there exists $x \in A_0$ such that

$$d(x,Su) = dis(A,B) = d(x,Tu). \tag{8}$$

Therefore, $Tx = Sx$ because S and T commute proximally. Since $Sx \in S(A_0) \subseteq B_0$, there exists $z \in A_0$, it implies that

$$d(z,Sx) = dis(A,B) = d(z,T\,x). \tag{9}$$

By generalized Lipschitz condition of the first kind, we get that

$$d(Su,Sx) \preceq \alpha_1 d(Tu,Tx) + \alpha_2 d(Su,Tu) + \alpha_3 d(Sx,Tx) + \alpha_4 [d(Su,Tx) + d(Sx,Tu)]$$
$$= (\alpha_1 + 2\alpha_4) d(Su,Sx). \tag{10}$$

Since $r(\alpha_1 + 2\alpha_4) < 1$ then $d(Su,Sx) = 0$ and $Su = Sx$. From (8) and (9) we have

$d(x,Su) = dis(A,B) = d(z,Sx)$, the weak P-property of the pair (A,B) implies

$d(x,z) \preceq d(Sx,Su) = 0$. So $x = z$ and

$$d(x,Sx) = dis(A,B) = d(x,Tx). \tag{11}$$

Suppose that x^* is another common best proximity point of the mappings S and T so that

$$d(x^*,Sx^*) = dis(A,B) = d(x^*,T\,x^*). \tag{12}$$

Since S and T commute proximally, then $Sx = Tx$ and $Sx^* = Tx^*$. So we have

$$d(Sx,Sx^*) \preceq \alpha_1 d(Tx,Tx^*) + \alpha_2 d(Tx,Sx) + \alpha_3 d(Tx^*,Sx^*) + \alpha_4 [d(T\,x^*,Sx) + d(Tx,Sx^*)]$$

$$= (\alpha_1 + 2\alpha_4) d(Sx,Sx^*),$$

which $r(\alpha_1 + 2\alpha_4) < 1$ implies that $Sx = Sx^*$. Since the pair (A,B) satisfies weak P-property, from (11) and (12) we have that $d(x,x^*) \preceq d(Sx,Sx^*)$. Eventually, we have that $x = x^*$. Hence S and T have a unique common best proximity point. □

Example 2 Let $E = C^1_{\mathbb{R}}[a,b]$ be the same as in the Example 1. The

$P = \{x \in E : x(t) \geq 0, t \in [a,b]\}$ is a non-normal cone in E. Let $X = \mathbb{R}^2$ and define $d : X \times X \to E$ by $d(x,y) = \left(|x_1 - y_1|^2 + |x_2 - y_2|^2\right)e^t$, for all $t \in [a,b]$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. So (X,d) is a cone b-metric over Banach algebra E with the coefficient $s = 2$. Put A, B be two subsets of X given by $A = \{(x, y) \in \mathbb{R}^2 : x = 0, 0 \leq y \leq 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x = 1, 0 \leq y \leq 1\}$. Let $T, S : A \to B$ be defined as $T(0,y) = (1,y)$ for each $0 \leq y \leq 1$, and $S(0,y) = (1, \frac{y}{4})$ for each $0 \leq y \leq 1$. Then (A,B) is a pair of nonempty closed and bounded subsets of X such that $A_0 = A$, $B_0 = B$ and $dis(A,B) = e^t$. It is verified that the (A,B) satisfies the weak P-property. Also T and S satisfy the properties mentioned in Theorem 1. Hence the conditions of Theorem 1 are satisfied and it is seen that $0 = (0,0)$ is the unique common best proximity point of S and T.

Remark 1 Since in Example 2 the solid cone P in the Banach algebra E is not normal, it shows that the main results without assumption of normality in this paper is meaningful.

If we suppose that S and T are self-mappings on X and $A = B = X$ then Theorem 1 implies the following common fixed point theorem, that generalizes and complements some results in the literature.

Corollary 1 Let (X,d) be a complete cone b-metric space over Banach algebra E with the coefficient $s \geq 1$ and P be a solid not necessarily normal cone of E. Moreover, assume that $S, T : X \to X$ are two self mappings on X, satisfy the following conditions:

(a) S and T commute;

(b) $S(X) \subseteq T(X)$;

(c) T is continuous;

(d) S and T satisfy the generalized Lipschitz condition of the first kind.

Then S and T have a unique common fixed point.

The following Corollary is a generalization of Theorem 2.1 in [15], Theorem 3.2 and Theorem 3.3 in [17] and also results Theorem 2.1 in [8] and shows existence and uniqueness of fixed point theorems in cone b-metric spaces over Banach algebras which the solid cone is not necessarily normal.

Corollary 2 Let (X,d) be a complete cone b-metric space over Banach algebra E with the coefficient $s \geq 1$ and P be a solid not necessarily normal cone of E. Moreover, assume that $S : X \to X$ is a self mappings on X, which satisfies the generalized Lipschitz condition of the first kind. Then S has a unique fixed point.

Proof If we put $T = I$ (I is identity map), then from Corollary 1 we obtain that S has a unique fixed point.

3. COMMON BEST PROXIMITY POINT THEOREMS FOR GENERALIZED LIPSCHITZ MAPPINGS OF THE SECOND KIND

Definition 10 Let A and B be two non-empty subsets of a cone b-metric space (X, d) over Banach algebra E with the coefficient $s \geq 1$ and P be a solid not necessarily normal cone of E. Non-self mappings $S, T : A \to B$ are said to satisfy generalized Lipschitz condition of the second kind if there exists a $\alpha \in P$ which $r(\alpha) < \frac{1}{s}$ such that for all x, y in A,

$$d(Sx, Sy) \preceq \alpha\ \omega_{x,y}, \tag{13}$$

where

$$\omega_{x,y} \in \left\{ d(Tx, Ty), d(Tx, Sx), d(Ty, Sy),\ \frac{d(Tx, Sy) + d(Ty, Sx)}{2s} \right\}.$$

Theorem 2 Let A and B be two non-empty subsets of a cone b-metric space (X, d) over Banach algebra E with the coefficient $s \geq 1$ and P be a solid not necessarily normal cone of E. Assume that the pair (A, B) satisfies the weak P-property. Moreover, A_0 is a non-empty and closed and $S, T : A \to B$ be two non-self mappings, which satisfy the following conditions:

(a) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$;

(b) S and T commute proximally;

(c) T is continuous;

(d) S and T satisfy the generalized Lipschitz condition of the second kind.

Then there exists a unique element $x \in A$ such that

$d(x, Tx) = dis(A, B)$ and $d(x, Sx) = dis(A, B)$.

Proof Let x_0 be a fixed element in A_0. Since $S(A_0) \subseteq T(A_0)$, then there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Then by continuing this process we can choose $x_n \in A_0$ such that there exists $x_{n+1} \in A_0$ satisfying $Sx_n = T\, x_{n+1}$ for each $n \in N$, since $S(A_0) \subseteq B_0$, there exists an element $u_n \in A$ such that

$$d(Sx_n, u_n) = dis(A, B) \text{ for each } n \in N. \tag{14}$$

By choosing x_n and u_n it follows that

$$d(Sx_n, u_n) = dis(A, B) = d(Sx_{n+1}, u_{n+1}). \tag{15}$$

By using the weak P-property and generalized Lipschitz condition of the second kind, we have

$d(u_n, u_{n+1}) \preceq d(Sx_n, Sx_{n+1}) \preceq \alpha\ \omega_{x_n, x_{n+1}}$, where $\alpha \in P$, $r(\alpha) < \frac{1}{s}$ and

$$\omega_{x_n,x_{n+1}} \in \{d(Tx_n,Tx_{n+1}), d(Tx_n,Sx_n),\ d(Tx_{n+1},Sx_{n+1}),\ \frac{d(Tx_n,Sx_{n+1})+d(Tx_{n+1},Sx_n)}{2s}\}$$

$$=\left\{d(Sx_{n-1},Sx_n), d(Sx_n,Sx_{n+1}),\ \frac{d(Sx_{n-1},Sx_{n+1})}{2s}\right\}.$$

We distinguish three cases. We will prove in each cases that $\{u_n\}$ is a Cauchy sequence then, we continue the proof.

Case I. If we have that

$$d(u_n,u_{n+1}) \preceq d(Sx_n,Sx_{n+1}) \preceq \alpha d(Sx_{n-1},Sx_n) \preceq \alpha^n\ d(Sx_0,Sx_1).$$

Since $r(\alpha)<\frac{1}{s}$, it implies that $\|\alpha^n\| \to 0$ and $\{\alpha^n\}$ is a c-sequence. By Lemma 5 we get that $\{\alpha^n\ d(Sx_0,Sx_1)\}$ is a c-sequence as well. Hence $\{u_n\}$ is a Cauchy sequence.

Case II. If we consider that $d(u_n,u_{n+1}) \preceq d(Sx_n,Sx_{n+1}) \preceq \alpha\ d(Sx_n,Sx_{n+1})$. Because $r(\alpha)<1$, it shows that $d(Sx_n,Sx_{n+1})=0$ and also $d(u_n,u_{n+1})=0$. Therefore, $u_n = u_{n+1}$ for all $n \in N$ and clearly $\{u_n\}$ is a Cauchy sequence.

Case III. Assume that

$$d(u_n,u_{n+1}) \preceq d(Sx_n,Sx_{n+1}) \preceq \alpha\ \frac{d(Sx_{n-1},Sx_{n+1})}{2s} \preceq \alpha\ \frac{d(Sx_{n-1},Sx_n)+d(Sx_n,Sx_{n+1})}{2}.$$

Since $r(\alpha)\ <\frac{1}{s}\leq 1$ implies that $(2e-\alpha)$ is invertible, then (16) leads to

$$d(Sx_n,Sx_{n+1}) \preceq (2e-\alpha)^{-1}\ \alpha d(Sx_{n-1},Sx_n).$$

Put $h=(2e-\alpha)^{-1}\alpha$, then $r(h)\ \leq\ \frac{r(\alpha)}{2-r(\alpha)}\ <\frac{1/s}{2-1/s}=\ \frac{1}{2s-1}\leq\frac{1}{s}$, so we have that

$$d(u_n,u_{n+1}) \preceq d(Sx_n,Sx_{n+1}) \preceq h^n d(Sx_0,Sx_1).$$

Like Case I., because of $r(h)<1$, it implies that $\{u_n\}$ is a Cauchy sequence. In all cases we proved that $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}\subseteq A_0$ and A_0 is closed, there exists $u \in A_0$ such that $u_n \to u$. By hypothesis, mappings S and T are commuting proximally and by (15) we have that $Tu_n = Su_{n-1}$, for every $n \in N$. Since T and S are continuous it implies that $Tu=\lim_{n\to\infty} Tu_n =\ \lim_{n\to\infty} Su_{n-1}$. We show that $Su=Tu$ because

$$d(Su,Tu) \preceq s\left[d(Su,Su_n)+d(Su_n,Tu)\right]\ = sd(Su,Su_n) \preceq s\alpha\ \omega_{u,u_n}, \quad (17)$$

where

$$\omega_{u,u_n} \in \{d(Tu,Tu_n), d(Tu,Su), d(Tu_n,Su_n), \frac{d(Tu,Su_n)+d(Tu_n,Su)}{2s}\} = \left\{d(Tu,Su), \frac{d(Tu_n,Su)}{2s}\right\}. \quad (18)$$

If we assume that $\omega_{u,u_n} = d(Su,Tu)$ then $d(Su,Tu) \preceq s\alpha\, d(Tu,Su)$. Since $r(s\alpha) = sr(\alpha) < 1$ hence $d(Su,Tu) = 0$. If we suppose that $\omega_{u,u_n} = \dfrac{d(Tu_n,Su)}{2s}$ then

$$d(Su,Tu) \preceq s\alpha\, \frac{d(Tu_n,Su)}{2s} \preceq s\alpha\left[\frac{d(Tu_n,Tu)}{2} + \frac{d(Tu,Su)}{2}\right] \preceq s\alpha\, \frac{d(Tu,Su)}{2}.$$

Because $r(\frac{s\alpha}{2}) = \frac{s}{2} r(\alpha) < \frac{1}{2} < 1$, it shows that $d(Su,Tu) = 0$ and $Tu = Su$. As $Su \in S(A_0) \subseteq B_0$, there exists an $x \in A_0$ such that

$$d(x,Su) = dis(A,B) = d(x,Tu). \quad (19)$$

Since S and T commute proximally, $Sx = Tx$. Also, $Sx \in S(A_0) \subseteq B_0$, there exists a $z \in A_0$ such that

$$d(z,Sx) = dis(A,B) = d(z,T\,x). \quad (20)$$

By generalized Lipschitz condition of the second kind, we get that $d(Su,Sx) \preceq \alpha\omega_{u,x}$, where

$$\omega_{u,x} \in \{d(Tu,Tx), d(Tu,Su), d(Tx,Sx), \frac{d(Tu,Sx) + d(Tx,Su)}{2s}\}.$$

Since $Sx = Tx$ and $Su = Tu$ it implies that $\omega_{u,x} = \{d(Su,Sx)\}$. Consequently, $d(Su,Sx) \preceq \alpha d(Su,Sx)$. Because of $r(\alpha) < 1$ we have that $Su = Sx$. Since the pair (A,B) satisfies the weak P-property from (19) and (20), we can conclude that $d(x,z) \preceq d(Su,Sx) = 0$. It follows that $x = z$. Therefore we have that

$$d(x,Sx) = dis(A,B) = d(x,Tx). \quad (21)$$

We now show that S and T have unique common best proximity point. For this, assume that x^* in A is a second common best proximity point of S and T, then

$$d(x^*,Sx^*) = dis(A,B) = d(x^*,Tx^*). \quad (22)$$

Since S and T commute proximally, from (21) and (22) we obtain that $Sx = Tx$ and $Sx^* = Tx^*$. By generalized Lipschitz condition of the second kind, we get that

$$d(Sx,Sx^*) \preceq \alpha\omega_{x,x^*},$$

where

$$\omega_{x,x^*} \in \{d(Tx,Tx^*), d(Tx,Sx), d(Tx^*,Sx^*), \frac{d(Tx,Sx^*)+d(Tx^*,Sx)}{2s}\} = \left\{d(Sx,Sx^*), \frac{d(Sx,Sx^*)}{s}\right\}.$$

It is easy to show that $d(Sx,Sx^*)=0$ and $Sx=Sx^*$. By weak P-property, from (21) and (22), we have that $d(x,x^*) \preceq d(Sx,Sx^*)=0$. Consequently, $x=x^*$. Hence S and T have a unique common best proximity point. □

Example 3 Let $E=\mathbb{R}^2$. For each $(x_1,x_2)\in E$, $\|(x_1,x_2)\|=|x_1|+|x_2|$. The multiplication is defined by $xy=(x_1,x_2)(y_1,y_2)=(x_1y_1,\ x_1y_2+x_2y_1)$. For each $x=(x_1,y_1)$ and $y=(y_1,y_2)$. Then E is a Banach algebra with unit $e=(1,0)$. Let $P=\{(x_1,x_2)\in\mathbb{R}^2 \ | \ x_1,x_2\geq 0\}$. Then P is normal with normal constant $M=1$. Let $X=\mathbb{R}^2$ and the metric d be defined by

$$d(x,y)=d((x_1,x_2),(y_1,y_2))=\left(|x_1-y_1|^2,|x_2-y_2|^2\right)\in P.$$

Then (X,d) is a complete cone b-metric space over a Banach algebra E with $s=2$. Let A and B be two subsets of X given by

$$A=\left\{(x_1,x_2)\in\mathbb{R}^2 \ : \ x_1=-1,\ 0\leq x_2\leq 1\right\}\cup\left\{(x_1,x_2)\in\mathbb{R}^2 \ : \ x_1=1,\ 0\ \leq x_2\leq 1\right\},$$

$$B=\left\{(x_1,x_2)\in\mathbb{R}^2 \ : \ x_1=-2,\ 0\leq x_2\leq 1\right\}\cup\left\{(x_1,x_2)\in\mathbb{R}^2 \ : \ x_1=2,\ 0\leq x_2\leq 1\right\}.$$

Then A and B are closed and bounded subsets of X such that $dis(A,B)=(1,0)$, $A_0=A$, $B_0=B$. Let $T,S:A\to B$ be defined as

$$T(x_1,x_2)=(2|x_1|,x_2) \text{ and } S(x_1,x_2)=(2|x_1|,\frac{x_2}{2}\ x2) \text{ for each } (x_1,x_2)\in A.$$

Therefore T and S satisfy the properties mentioned in Theorem 2. Hence the conditions of Theorem 2 are satisfied and $(1,0)$ is the unique common best proximity point of S and T.

By Theorem 2 we obtain the following results in the fixed point theorem.

Corollary 3 Let (X,d) be a complete cone b-metric space over Banach algebra E with the coefficient $s\geq 1$ and P be a solid not necessarily normal cone of E. Let $S,T:\ X\to X$ be self mappings on X, also T commutes with S and T is continuous. Further, let S and T satisfy $S(X)\subseteq T(X)$ and there exists a constant $r(\alpha)<\frac{1}{s}$ such that for every $x,y\in X$

$$d(Sx,Sy) \preceq \alpha\ \omega_{x,y}, \tag{23}$$

where

$$\omega_{x,y}\in\left\{d(Tx,Ty),d(Tx,Sx),d(Ty,Sy),\ \frac{d(Tx,Sy)+d(Ty,Sx)}{2s}\right\}.$$

Then S and T have a unique common fixed point.

If T is assumed to be identity mapping in Corollary 3, then we have the following result, which is a generalization of Theorem 3.1-3.3 in [17] and results Theorem 2.5 in [8].

Corollary 4 Let (X,d) be a complete cone b-metric space over Banach algebra E with the coefficient $s \geq 1$ and P be a solid not necessarily normal cone of E. Let $S : X \to X$ be self mappings on X and there exists a constant $r(\alpha) < \frac{1}{s}$ such that for every $x, y \in X$

$$d(Sx, Sy) \preceq \alpha\ \omega_{x,y},$$

where

$$\omega_{x,y} \in \left\{ d(x,y), d(x,Sx), d(y,Sy),\ \frac{d(x,Sy)+d(y,Sx)}{2s} \right\}.$$

Then S has a unique fixed point.

Example 4 [8] Let $E = \mathbb{R}^2$. For each $(x_1, x_2) \in E$, $\|(x_1, x_2)\| = |x1| + |x2|$. The multiplication is defined by $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1,\ x_1y_2 + x_2y_1)$. Then (X,d) is a complete cone b-metric space over a Banach algebra E with coefficient $s = 2$. Now define mapping $S : X \to X$ by $S(x_1, x_2) = (\frac{1}{2}\left(\cos\ \frac{x_1}{2} - |\,x_1 - \frac{1}{2}\,|\right),\ \arctan(2 + |\,x_2\,|) + \ln(x_1 + 1))$. Then we have

$$d(S(x_1,x_2), S(y_1,y_2)) \leq (\frac{1}{3}, 2)\left(|x_1 - y_1|^2, |x_2 - y_2|^2\right)\ \leq (\frac{1}{3}, 2) d((x_1,x_2),(y_1,y_2)),$$

where $r((\frac{1}{3},2))\ = \frac{1}{3} < \frac{1}{2}$. In addition, all the other conditions of Corollary 2 are fulfilled (if we put $\alpha_1 = (\frac{1}{3}, 2),\ \alpha_2 = 0,\ \alpha_3 = 0$ and $\alpha_4 = 0$). Therefore, S has a unique fixed point in X.

Remark 2 In Example 4 (see Example 2.7 in [8]), we see the main results in this paper are indeed more different than the standard results of cone b-metric spaces in the literature.

CONFLICT OF INTEREST

The authors declare that they have no competing interests.

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