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A New Weighted Exponential Distribution and its Application to the Complete and Censored Data

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Abstract

The class of weighted exponential (WE) distribution was introduced in the seminal paper by Gupta and Kundu (2009) and have received a great deal of attention in recent years. In the present paper, we define a flexible extension of the weighted exponential distribution called new weighted exponential (NWE) distribution. Various structural properties including statistical and reliability measures of the new distribution are derived. The method of maximum likelihood is used to estimate the parameters of the distribution in complete and censored data setting. A simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators. Finally, two real data sets have been analyzed for illustrative purposes and it is observed that in both cases the proposed model fits better than Weibull, gamma, weighted exponential, two-parameter weighted exponential, log-logistic , generalized exponential and generalized Weibull distributions.

1. INTRODUCTION

In last three decades or so, an extensive research works have appeared in the literature on the theory of statistical distributions (see, for example, Kharazmi (2016), Kharazmi and Saadatinik (2016)). The class of weighted exponential (WE) distribution was introduced in the seminal paper by Gupta and Kundu (2009) and have received a great deal of attention in recent years. The proposed model has some interesting stochastic representations, especially it can be obtained by implementing Azzalini's method (1985) to the exponential distribution. The well-known Azzalini's method in generalizing family of distribution stated as: Let U and V be two continuous independent random variables with densities f and g, and cumulative distribution functions (CDF) F and G, respectively. Then for any $\alpha \in R$, the conditional distribution of U given $V < \alpha U$ is

$$f_{U|V<\alpha U}(u) = \frac{f(u)G(\alpha u)}{\mathrm{P}(V<\alpha U)}. \quad (1)$$

With the above general result, the weighted exponential distribution denoted by $WE(\lambda, \alpha)$, is obtained when U and V follow exponential distributions with mean $1/\lambda$ and $\alpha > 0$. Its probability density function (PDF) is given as

$$f_X(x, \alpha, \lambda) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1 - e^{-\lambda\alpha x}\right), \quad (2)$$

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where $x > 0$, $\alpha > 0$ and $\lambda > 0$. Here α and λ are the shape and scale parameters, respectively. The main properties and different interpretations of this density are established by authors. Recently have been shown attention to WE distribution and its applications in the literature. For example, Shakhatreh (2012) generalized the WE distribution to the two-parameter weighted exponential distributions (TWE). Kharazmi et al. (2015) extended weighted exponential distribution to the generalized weighted exponential (GWE) distribution and studied its different properties. Several interesting properties of GWE distribution have been established by authors. The GWE distribution contains the above mentioned distributions as its sub-models. It was observed that the GWE distribution can provide a better fit for survival time data relative to other common distributions such as weighted exponential (WE), two parameter weighted exponential (TWE), gamma, weibull and generalized exponential (GE) distribution. Ghitany et al. (2016) proposed weighted half exponential power (WHEP) distribution, which can be used to model negative or positive skewed data.

In the present paper, we introduce a new weighted exponential distribution and provide a comprehensive description of some mathematical properties with the hope that it will attract wider applications in reliability, engineering and in other areas of research. The interesting NWE distribution has several desirable properties, and provides more flexibility to fitting censored and uncensored survival data in the real applications. For illustrative purposes we use two real data sets, and it is observed that NWE provides better fit than WE model and other common statistical distributions.

The paper is organized as follows. In Section 2 we define the proposed NWE distribution and provide different interesting stochastic representations for construction of NWE distribution. Section 3 presents some statistical and reliability properties of the proposed model. Section 4 gives some results about asymptotic distribution of order statistics, stochastic ordering, Renyi entropy and an extention model with four parameters of NWE distribution. We discuss MLE procedure of unknown parameters for both censored and complete data in Section 5. In Section 6, a Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators for each parameter. The analysis of two real data sets have been presented in Section 7. Finally in Section 8 we conclude the paper.

2. DEFINITION AND STOCHASTIC REPRESENTATIONS

In this section, we introduce the definition of the new weighted exponential distribution denoted by $NWE(\alpha,\beta,\lambda)$ and four stochastic representations are given here also.

Definition 1. A random variable X is said to have a new weighted exponential distribution $NWE(\alpha,\beta,\lambda)$, with shape parameters $\alpha > 0$, $\beta > 0$ and scale parameter $\lambda > 0$, if the PDF of X is given as following

$$f_X(x,\alpha,\beta,\lambda) = Ce^{-\lambda x}\left[1 - e^{-\lambda\alpha x} - \frac{1}{\beta+1}\left(1 - e^{-\lambda\alpha(\beta+1)x}\right)\right],\ \ x > 0, \tag{3}$$

where $C = \frac{\lambda(\alpha+1)(\alpha(\beta+1)+1)}{\beta\alpha^2}$.

The following theorem explores the shape of the PDF (3).

Theorem 1. The PDF of the $NWE(\alpha,\beta,\lambda)$ distribution is unimodal.

Proof : The derivative of $f(x)$ can be written as

$$f' = Ce^{-\lambda x}g(x),$$

where

$$g(x) = -\left(\frac{1}{\beta+1} + \alpha\right)e^{-\alpha\lambda(\beta+1)x} + (1+\alpha)e^{-\alpha\lambda x} - \frac{\beta}{\beta+1}.$$

Now we have

$$g'(x) = e^{-\alpha\lambda x}\, h(x),$$

where

$$h(x) = \big(1+\alpha(\beta+1)\big)\lambda\alpha e^{-\alpha\beta\lambda x} - \alpha\lambda(1+\alpha).$$

The function $h(x)$ is decreasing because $h'(x) < 0$ and since $h(0) = \alpha\beta$ and $h(\infty) = -\alpha\lambda(1+\alpha)$, it follows that $g'(x)$ changes sing from positive to negative. Finally, since $g(x) = 0$ and $g(\infty) = -\frac{\beta}{\beta+1}$, it follows that $g(x)$ changes sing from positive to negative. This implies that $f(x)$ is unimodal . ■

Figure 1 shows the PDF of the NWE distribution for fixed scale parameter $\lambda = 1$ and selected shape parameters.

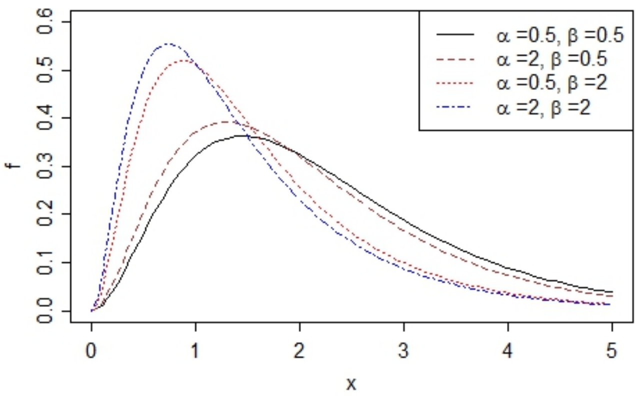

Fig. 1. *Plots of the NWE density function for fixed scale parameter* $\lambda = 1$ *and some selected shape parameters.*

It is easy to show that if $\alpha \to +\infty$, then (3) converges to $\exp(\lambda)$ and if $\beta \to +\infty$ then (3) converges to $WE(\alpha,\lambda)$ and if $\alpha \to 0$ then (3) converges to $gamma(3,\lambda)$.

Representation 1.

Proposed distribution can be obtained by implementing Azzalini's method. Suppose X_1 be a random variable having the exponential distribution with mean $1/\lambda$ and X_2 be a random variable having the weighted exponential with parameters (β,λ). It can be easily observed that for any $\alpha > 0$, the random variable $X = X_1 | X_2 < \alpha X_1$ has the density function (3).

Representation 2.

The NWE distribution can be obtained as hidden truncation model proposed by Arnold and Beaver (2000). Suppose Z and Y are two dependent random variables with the joint density function

$$f_{Z,Y}(z,y) = \frac{\lambda^2\beta}{\beta+1} z\left(e^{-\lambda z(y+1)} - e^{-\lambda z\left((\beta+1)y+1\right)} \right) \quad z>0,\ y>0.$$

It can be shown that the conditionally random variable $Z|Y \le \alpha$ has the NWE distribution.

Representation 3.

Using the moment generating function (MGF), it can be seen if $X \sim NWE(\alpha,\beta,\lambda)$ then,

$$X = U + V + T,$$

where $U \sim \exp(\lambda)$, $V \sim \exp(\lambda(\alpha+1))$ and $T \sim \exp(\lambda(1+\alpha(\beta+1)))$ and independent.

Representation 4.

The NWE distribution can be stated as a mixtures of weighted exponential distributions as following

$$f_X(x,\alpha,\beta,\lambda) = w f_{X_1}(x,\alpha,\beta,\lambda) + (1-w) f_{X_2}(x,\alpha,\beta,\lambda) \tag{4}$$

where $w = \frac{\alpha(\beta+1)+1}{\alpha\beta}$ and $X_1 \sim WE(\alpha,\lambda)$, $X_2 \sim WE(\alpha(\beta+1),\lambda)$.

Remark. All the above four stochastic representations can be used to generating random numbers from NWE distribution. Note that the simplest way to generate NWE random number is to use the stochastic representation 3.

In the next, we obtain the CDF of NWE distribution based on the representation 4 as

$$F_X(x,\alpha,\beta,\lambda) = w F_1(x,\alpha,\lambda) + (1-w) F_2(x,\alpha,\beta,\lambda),$$

where $F_1(x,\alpha,\lambda)$ denote the CDF of the $WE(\alpha,\lambda)$ as

$$F_1(x,\alpha,\lambda) = \frac{\alpha+1}{\alpha}\left[1 - e^{-\lambda x} - \frac{1}{\alpha+1}\left(1 - e^{-\lambda(\alpha+1)x}\right)\right],$$

and $F_2(x,\alpha,\beta,\lambda)$ is CDF of the $WE(\alpha(\beta+1),\lambda)$ as

$$F_2(x,\alpha,\beta,\lambda) = \frac{\alpha(\beta+1)+1}{\alpha(\beta+1)}\left[1 - e^{-\lambda x} - \frac{\left(1 - e^{-\lambda(\alpha(\beta+1)+1)x}\right)}{\alpha(\beta+1)+1}\right].$$

3. STATISTICAL AND RELIABILITY PROPERTIES

In this section, we derive some main properties of NWE distribution.

If $X \sim NWE(\alpha,\beta,\lambda)$, then the moment generate function (MGF) of X for any $t < \lambda$ is given by

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)\left(\frac{\lambda(1+\alpha)}{\lambda(1+\alpha) - t}\right)\left(\frac{\lambda\left(1+\alpha\left(\beta+1\right)\right)}{\lambda\left(1+\alpha\left(\beta+1\right)\right) - t}\right). \tag{5}$$

Therefore differentiating $M_X(t)$ and having $t = 0$, we obtain

$$\mu = E\left[X\right] = \frac{1}{\lambda} + \frac{1}{\lambda(\alpha+1)} + \frac{1}{\lambda(1+\alpha(\beta+1))}, \tag{6}$$

and

$$\sigma^2 = V\left(X\right) = \frac{1}{\lambda^2} + \frac{1}{\lambda^2(\alpha+1)^2} + \frac{1}{\lambda^2(1+\alpha(\beta+1))^2}. \tag{7}$$

Another measures such as coefficient of variation (CV), skewness and Kurtosis for the NWE distribution respectively are given as

$$CV=\sqrt{1-2\frac{\left(3+\alpha(\beta+2)\right)\left(1+\alpha(\beta+1)\right)\,(1+\alpha)}{\left(\alpha^{2}(\beta+1)+2\alpha(\beta+2)+3\right)^{2}}}, \tag{8}$$

$$Sk=\frac{E(x-\mu)^{3}}{\sigma^{3}}=\frac{2\left(1+\frac{1}{(1+\alpha)^{3}}+\frac{1}{(1+\alpha(\beta+1))^{3}}\right)}{\sqrt[2]{\left(1+\frac{1}{(1+\alpha)^{2}}+\frac{1}{(1+\alpha(\beta+1))^{2}}\right)^{3}}}, \tag{9}$$

$$k=\frac{E(x-\mu)^{4}}{\sigma^{4}}=\frac{6\left[1+\frac{1}{(1+\alpha)^{4}}+\frac{1}{(1+\alpha(\beta+1))^{4}}\right]}{\left[1+\frac{1}{(1+\alpha)^{2}}+\frac{1}{(1+\alpha(\beta+1))^{2}}\right]^{2}}+3. \tag{10}$$

Figure 2 shows the skewness and kurtosis of the NWE distribution as a function of α for selected values of parameter β and fixed scale parameter $\lambda=1$.

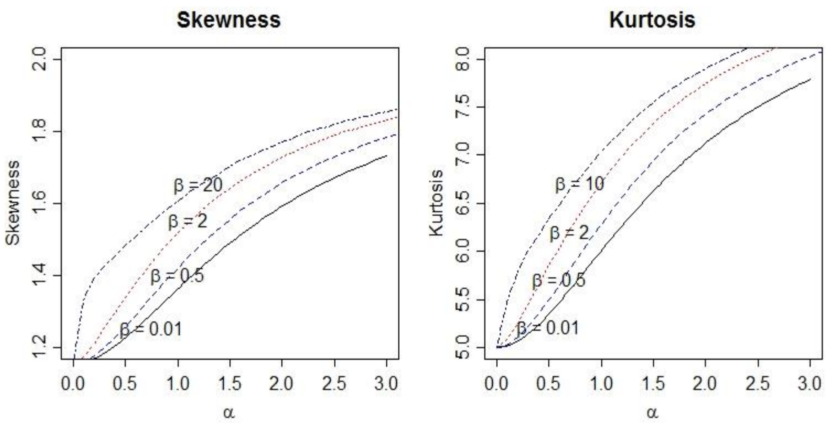

Fig. 2. *Plots of skewness (left) and kurtosis (right) for NWE distribution.*

The survival function of the NWE distribution is given by

$$S(x)=wS_{1}(x)+(1-w)S_{2}(x), \tag{11}$$

where

$$S_{1}(x)=\frac{\alpha+1}{\alpha}e^{-\lambda x}\left(1-\frac{1}{(\alpha+1)}e^{-\lambda\alpha x}\right),$$

and

$$S_{2}(x)=\frac{\alpha(\beta+1)+1}{\alpha(\beta+1)}e^{-\lambda x}\left(1-\frac{1}{(\alpha(\beta+1)+1)}e^{-\lambda\alpha(\beta+1)x}\right).$$

The hazard rate function (HRF) of X can be written as

$$h(x) = P(x)h_1(x) + (1 - P(x))h_2(x), \tag{12}$$

where $P(x) = \dfrac{wS_1(x)}{wS_1(x) + (1-w)S_2(x)}$ and $h_1(x)$ is the HRF of the $WE(\alpha, \lambda)$, is given by

$$h_1(x) = \frac{(\alpha+1)\lambda\left(1 - e^{-\alpha\lambda x}\right)}{\left(1 + \alpha - e^{-\lambda\alpha x}\right)},$$

and $h_2(x)$ is the HRF of $WE(\alpha(1+\beta), \lambda)$ is given by

$$h_2(x) = \frac{(\alpha(\beta+1)+1)\lambda\left(1 - e^{-\alpha(\beta+1)\lambda x}\right)}{\left(1 + \alpha(\beta+1) - e^{-\alpha\lambda(\beta+1)x}\right)}.$$

The following lemma provides sufficient conditions for the shape of the HRF of any PDF on $(0, \infty)$.

Lemma 1. [Glaser (1980)] Let X be a continuous random variable on $(0, \infty)$ with twice differentiable PDF $f(x)$ and HRF $h(x)$. Define $\eta(x) = -(\ln f(x))'$. If $\eta(x)$ is increasing, then $h(x)$ is increasing.

Theorem 2. The NWE distribution has increasing HRF for $\alpha, \beta, \lambda > 0$.

Proof : By using (3), we have

$$\eta'(x) = -\frac{d^2}{dx^2}\ln\big(f(x)\big) = \frac{\alpha\beta\lambda e^{-\alpha\lambda x}}{(e^{-\alpha\lambda(\beta+1)x} - (\beta+1)e^{-\alpha\lambda x} + \beta)^2}\, g(x),$$

where $g(x) = \beta e^{-\alpha\lambda(\beta+1)x} - (\beta+1)e^{-\alpha\beta\lambda x} + 1$. The function $g(x)$ is increasing because $g'(x) > 0$ and since $g(0) = 0$, it follows that $g(x) > 0$. Therefore $\eta'(x) \geq 0$ and it follows that $\eta(x)$ is increasing. ■

In figure 3, we plotted the hazard rate function of the NWE distribution for selected values of the shape parameters and fixed scale parameter $\lambda = 1$. Since the HRF is increasing, this is suitable for modeling lifetime data in engineering context when wear-out is present.

The mean residual life function (MRLF) of the NWE distribution is obtained as

$$m(t) = E(X - t \mid X > t) = P(x)m_1(t) + (1 - P(x))m_2(t), \tag{13}$$

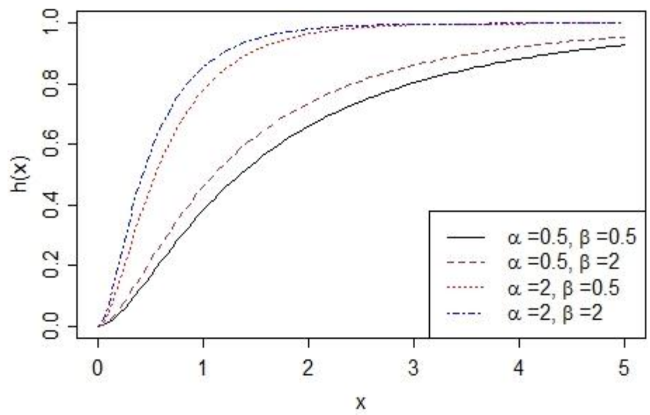

Fig. 3. *Plots of the hazard rate function of the NWE distribution for fixed scale parameter $\lambda = 1$ and some selected shape parameters*

where $P(x) = \dfrac{wS_1(x)}{wS_1(x) + (1-w)S_2(x)}$ and $m_1(t)$ is the MRLF of the $WE(\alpha, \lambda)$, is given by

$$m_1(t) = \frac{\dfrac{1}{\lambda}\left(1 - \dfrac{1}{(1+\alpha)^2} e^{-\lambda\alpha t}\right)}{1 - \dfrac{1}{1+\alpha} e^{-\lambda\alpha t}},$$

and $m_2(t)$ is the MRLF of $WE(\alpha(1+\beta), \lambda)$ is given by

$$m_2(t) = \frac{\dfrac{1}{\lambda}\left(1 - \dfrac{1}{(1+\alpha(\beta+1))^2} e^{-\lambda\alpha(\beta+1)t}\right)}{1 - \dfrac{1}{(1+\alpha(\beta+1))} e^{-\lambda\alpha(\beta+1)t}}.$$

According to theorem 2, NWE distribution has increasing hazard rate function (IFR) and hence decreasing mean residual life (DMRL).

4. ASYMPTOTIC DISTRIBUTION OF ORDER STATISTICS, STOCHASTIC ORDERING, RENYI ENTROPY AND A GENERALIZED MODEL BASED ON NWE DISTRIBUTION

4.1. Asymptotic distribution of order statistics

In this section we provide the asymptotic distribution of the minimum and maximum of a random sample of size n from NWE distribution.

Theorem 3. Let $X_{1:n}$ and $X_{n:n}$ be the minimum and maximum of a random sample $X_1, X_2, ..., X_n$ from $NWE(\alpha, \beta, \lambda)$, respectively, then

$$(a)\ \lim_{n\to\infty} P\left\{\frac{X_{1:n}-a_n^*}{b_n^*}\le x\right\}=1-e^{-x^3},\qquad x>0,$$

$$(b)\ \lim_{n\to\infty} P\left\{\frac{X_{n:n}-a_n}{b_n}\le x\right\}=\exp(-e^{-x})\qquad \mathrm{x}\in \mathrm{R},$$

where $a_n^*=0$, $b_n^*=F^{-1}(\frac{1}{n})$, $a_n=F^{-1}(1-\frac{1}{n})$, $b_n=\frac{1}{nf(a_n)}$ and $F^{-1}(c)$ is the inverse function of CDF $F(c)$.

Proof : (a) for the $NWE(\alpha,\beta,\lambda)$, $F^{-1}(0)=0$ and by three times using L' Hospital rule, we have

$$\lim_{\varepsilon\to 0^+}\frac{F\left(F^{-1}(0)+\varepsilon x\right)}{F\left(F^{-1}(0)+\varepsilon\right)}=\lim_{x\to 0^+} x^3\,\frac{e^{-\alpha\beta\lambda x\varepsilon}}{e^{-\alpha\beta\lambda\varepsilon}}=x^3.$$

Therefore, by theorem 8.3.6 (ii) of Arnold et al. (1992), the minimal domain of attraction of the NWE distribution is the standard Weibull distribution with shape parameter 3, proving part (a).

(b) for the $NWE(\alpha,\beta,\gamma)$, $F^{-1}(1)=+\infty$ and it is easy to show that

$$\lim_{x\to+\infty}\frac{d}{dx}\left(\frac{1}{h(x)}\right)=\lim_{x\to+\infty}\frac{d}{dx}\left(\frac{1-F(x)}{f(x)}\right)=\lim_{x\to+\infty}\left(-1+\frac{S(x)\eta(x)}{f(x)}\right)=0,$$

where $\eta(x)=-\left(\ln f(x)\right)'$ and $S(x)=1-F(x)$.

Therefore, by theorem 8.3.3 of Arnold et al. (1992), the maximal domain of attraction of the NWE distribution is the standard Gumbel distribution, proving part (b). ■

Now, we use theorem 3 to find the asymptotic distribution of any order statistic.

Theorem 4. Letfrom $X_1,X_2,...,X_n$ be the order statistics of a random sample $X_{1:n},X_{2:n},...,X_{n:n}$, $i\ge 1$. Then for any fixed $NWE(\alpha,\beta,\gamma)$

(a) $$\lim_{n\to\infty} P\left\{\frac{X_{i:n}-a_n^*}{b_n^*}\le x\right\}=1-\sum_{r=0}^{i-1}\frac{e^{-x^3}x^{3r}}{r!}\qquad x>0,$$

(b) $$\lim_{n\to\infty} P\left\{\frac{X_{n-i+1:n}-a_n}{b_n}\le x\right\}=\sum_{r=0}^{i-1}e^{(-e^{-x})}\,\frac{e^{-rx}}{r!}\qquad x\in R.$$

Proof : The theorem proofs from equation (8.4.2) and (8.4.3) of Arnold et al. (1992). ■

4.2. Stochastic ordering

In this section, we are comparing $WE(\alpha,\lambda)$ and $NWE(\alpha,\beta,\gamma)$ with respect stochastic ordering information. See Shaked and Shanthikumar (2007). Suppose X and Y be two random variables with PDFs $f_X(x)$ and $f_Y(y)$ also CDFs $F_X(x)$ and $F_Y(y)$ respectively.

A random variable X is said to be smaller than Y in the

a) stochastic order $(X \le_{st} Y)$ if $F_Y(x) \le F_X(x)$ for all x.

b) hazard rate order $(X \le_{hr} Y)$ if $h_Y(x) \le h_X(x)$ for all x.

c) mean residual life order $(X \le_{mrl} Y)$ if $m_X \le m_Y$ for all x.

d) likelihood ratio order $(X \le_{Lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ is decreasing in x.

We have the following chain of implications among the various partial orderings discussed above:

$$(X \le_{lr} Y) \Rightarrow (X \le_{hr} Y) \Rightarrow \begin{cases} (X \le_{mrL} Y) \\ (X \le_{st} Y) \end{cases}$$

Theorem 5. If $X \sim NWE(\alpha,\beta,\lambda)$ and $Y \sim WE(\alpha,\lambda)$, then $Y \le_{Lr} X$ an hence $(Y \le_{hr} X)$, $(Y \le_{mrl} X)$ and $(Y \le_{st} X)$.

Proof : It is sufficient to show $\frac{f_X(x)}{f_Y(x)}$ is an increasing function of x

$$\frac{d}{dx}\left(\frac{f_X(x)}{f_Y(x)}\right) = \alpha\lambda e^{-\alpha\lambda x} g(x),$$

where $g(x) = \beta e^{-\alpha\lambda(\beta+1)x} - (\beta+1)e^{-\alpha\lambda\beta x} + 1$. The function $g(x)$ is increasing because $g'(x) > 0$ and since $g(0) = 0$ and $g(+\infty) = 1$, it follows that $g(x) > 0$. Therefore $\frac{d}{dx}\left(\frac{f_X(x)}{f_Y(x)}\right) > 0.$ ■

4.3. Entropy measure

Shannon entropy (1948) is a central concept of information theory for expressing the uncertainty about a random variable. Renyi (1961) defined a generalization of Shannon entropy which depends on a parameter ν. Renyi entropy defined by

$$H_\nu(f(x)) = \frac{1}{1-\nu}\log\left\{\int f^\nu(x)\, dx\right\},$$

Where $\nu > 0$ and $\nu \ne 1$. Renyi entropy tends to Shannon entropy as $\nu \to 1$.

For $NWE(\alpha,\beta,\lambda)$, note that

$$H_\nu(f(x)) = \frac{1}{\nu-1}\log\left[\left(\frac{C\beta}{\beta+1}\right)^\nu \int_0^\infty e^{-\lambda\nu x}\left(1+\left(\frac{\beta+1}{\beta}\right)\left(\frac{1}{\beta+1}e^{-\alpha\lambda(\beta+1)x}-e^{-\alpha\lambda x}\right)\right)^\nu dx\right],$$

where $C = \frac{\lambda(\alpha+1)(\alpha(\beta+1)+1)}{\beta\alpha^2}$. It is easy to show that

$$\left|\frac{\beta+1}{\beta}\left(\frac{1}{\beta+1}e^{-\alpha\lambda(\beta+1)x}-e^{-\alpha\lambda x}\right)\right|<1.$$

And we know $(1+z)^\nu = \sum_{k=0}^{\infty}\binom{\nu}{k}z^\nu$, for $|z|<1$, then

$$H_\nu(f(x)) = \frac{1}{\nu-1}\log\left[\left(\frac{C\beta}{\beta+1}\right)^\nu \sum_{k=0}^{\infty}\binom{\nu}{k}\left(\frac{\beta+1}{\beta}\right)^k \int_0^\infty e^{-\lambda(\nu+\alpha k)x}\left(\frac{e^{-\alpha\lambda\beta x}}{\beta+1}-1\right)^k dx\right].$$

By using the binomial expansion $(a+b)^k = \sum_{j=0}^{k}\binom{k}{j}a^j b^{k-j}$ so that

$$H_\nu(f(x)) = \frac{1}{\nu-1}\log\left[\left(\frac{C\beta}{\beta+1}\right)^\nu \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k-j}\binom{\nu}{k}\binom{k}{j}(\beta+1)^{k-j}}{\lambda(\nu+\alpha k+\alpha\beta j)\beta^k}\right].$$

4.4. An extention of NWE distribution

Using the representation 3, one may easily develop a new four -parameter distribution like

$$X = U + V + T + Z,$$

where $U \sim \exp(\lambda)$, $V \sim \exp(\lambda(\alpha+1))$, $T \sim \exp(\lambda(1+\alpha(\beta+1)))$ and$Z \sim \exp(\lambda(1+\alpha(\beta(\tau+1)+1)))$, and independent. The random variable X with above stochastic representation have PDF as

$f(x,\alpha,\beta,\lambda,\tau) = Ce^{-\lambda x} \times$

$$\left[1-e^{-\lambda\alpha(\beta(1+\tau)+1)x}-\frac{\beta(1+\tau)+1}{\beta(1+\tau)}\left(1-e^{-\lambda\alpha\beta(1+\tau)x}\right)-\frac{1}{\beta+1}[\left(1-e^{-\lambda\alpha(\beta(1+\tau)+1)x}\right) - \frac{\beta(1+\tau)+1}{\beta\tau}\left(1-e^{-\lambda\alpha\beta\tau x}\right)]\right],$$

where $C = \frac{\lambda(\alpha+1)(\alpha(\beta+1)+1)(1+\alpha(\beta(1+\tau)+1))}{\alpha(\beta(1+\tau)+1)}$. More work is needed in this direction.

5. MAXIMUM LIKELIHOOD ESTIMATION

5.1. Complete maximum likelihood

In this section, we obtain the equations for finding the maximum likelihood estimators (MLEs) of parameters in complete data setting.

Suppose $X_1, \dots, X_n$ be a random sample from $NWE(\alpha, \beta, \lambda)$. The log-likelihood function based on the observed sample $(x_1, \dots, x_n)$ is

$$l(\boldsymbol{\theta}) = \ln L(x_1, \dots, x_n|\boldsymbol{\theta})$$

$$= n\ln\lambda + n\ln(\alpha+1) + n\ln(\alpha(\beta+1)+1) - n\ln\beta - 2n\ln\alpha$$
$$-\lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left(1 - e^{-\alpha\lambda x_i} - \frac{1}{\beta+1}\left(1 - e^{-\alpha\lambda(\beta+1)x_i}\right)\right),$$

where $\boldsymbol{\theta} = (\alpha, \beta, \lambda)$.

To find the MLE estimates for the NWE model parameters, we differentiate the log-likelihood function and equating the resulting to 0 as follows

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha+1} + \frac{n(\beta+1)}{\alpha(\beta+1)+1} - \frac{2n}{\alpha} + \sum_{i=1}^{n} \frac{\lambda x_i e^{-\alpha\lambda x_i} - \lambda x_i e^{-\alpha\lambda(\beta+1)x_i}}{1 - e^{-\alpha\lambda x_i} - \frac{1}{\beta+1}(1 - e^{-\alpha\lambda(\beta+1)x_i})} = 0,$$

$$\frac{\partial L}{\partial \beta} = \frac{n\alpha}{\alpha(\beta+1)+1} - \frac{n}{\beta} + \sum_{i=1}^{n} \frac{-\alpha\lambda(\beta+1)x_i e^{-\alpha\lambda(\beta+1)x_i} - e^{-\alpha\lambda(\beta+1)x_i} + 1}{(\beta+1)^2\left(1 - e^{-\alpha\lambda x_i} - \frac{1}{(\beta+1)}(1 - e^{-\alpha\lambda(\beta+1)x_i})\right)} = 0,$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\alpha x_i e^{-\alpha\lambda x_i} - \alpha x_i e^{-\alpha\lambda(\beta+1)x_i}}{1 - e^{-\alpha\lambda\, x_i} - \frac{1}{\beta+1}(1 - e^{-\alpha\lambda(\beta+1)x_i})} = 0.$$

The MLEs of the unknown parameters cannot be obtained explicitly. They have to be obtained by solving some numerical methods, like Newton-Raphson or Gauss-Newton methods or their variants.

5.2. Censored maximum-likelihood

In real life, sometimes it is hard to get a complete data set. Often with lifetime data, one encounters censoring. There are different forms of censoring: type I, type II, etc. Here, we consider the general case of multi-censored data, the likelihood function is given as

$$L(\underline{\theta}) = \prod_{i=1}^{n_0} f(x_i, \underline{\theta}) \prod_{j=1}^{n_1} F({x_j}^u, \underline{\theta}) \prod_{k=1}^{n_2} \left(1 - F({x_k}^l, \underline{\theta})\right) \prod_{m=1}^{n_3} \left(F({x_m}^u, \underline{\theta}) - F({x_m}^l, \underline{\theta})\right)$$

with x_i the n_0 non-censored observations, ${x_j}^u$ upper values defining the n_1 left-censored observations, ${x_k}^l$ lower values defining the n_1 right-censored observations, $[{x_m}^l, {x_m}^u]$ the intervals defining n_3 interval-censored observations Klein and Moeschberger (2003). Note that $n = n_0 + n_1 + n_2 + n_3$ and that type I and type II censorings are contained as particular cases of multi-censoring. In the case of NWE distribution, the corresponding likelihood equations are complicated, so they are not presented here. The estimation of parameters in this case can be obtained by numerical methods.

6. SIMULATION

In this section, we perform a simulation study to investigate the finite sample properties of MLE estimators described in Section 5. To conduct the experimental study, we generate 5000 synthetic samples of size n = 10, 30, 50, 150, 100 and 200 from NWE with true selected parameters $C_1 = (a = 1, \beta = 2, \lambda = 2)$ and $C_2 = (a = 2, \beta = 2, \lambda = 2)$. To examine the estimation accuracies, the absolute bias and the mean squared error (MSE) are computed. Figures 4-7 show a graphical representation of the absolute bias and the MSE of the parameter estimates as a function of sample size n.

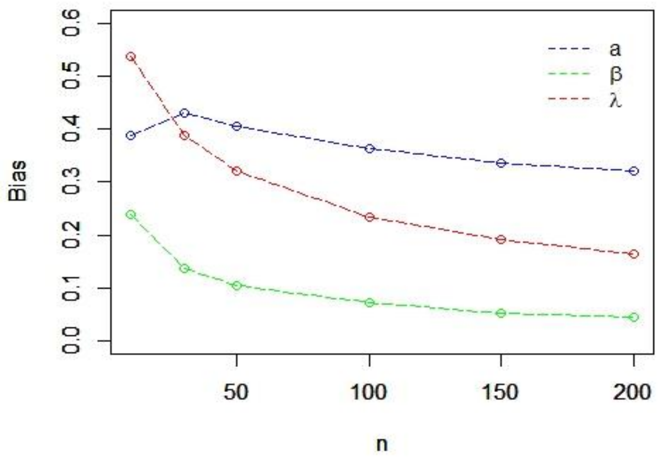

Fig 4. *Absolute bias of selected parameters* $(a = 1, \beta = 2, \lambda = 2)$ *for NWE model.*

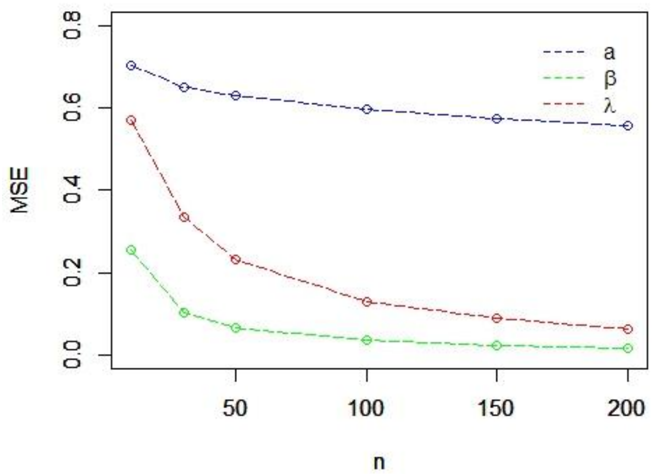

Fig. 5. *Absolute MSE's of selected parameters*$(a = 1, \beta = 2, \lambda = 2)$ *for NWE model.*

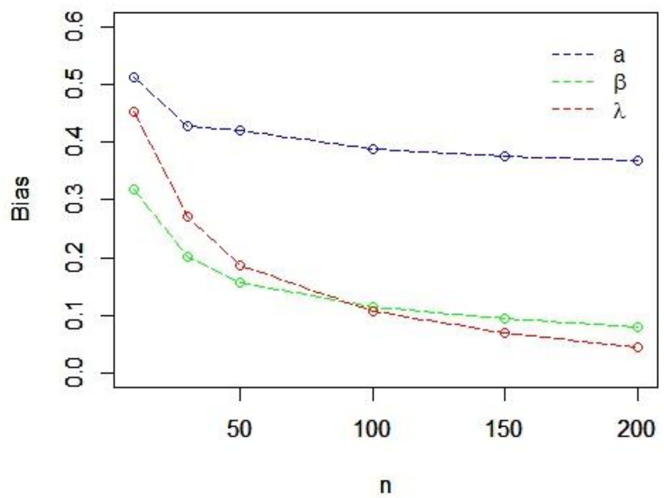

Fig. 6. *Absolute bias of selected parameters* $(a = 2, \beta = 2, \lambda = 2)$ *for NWE model.*

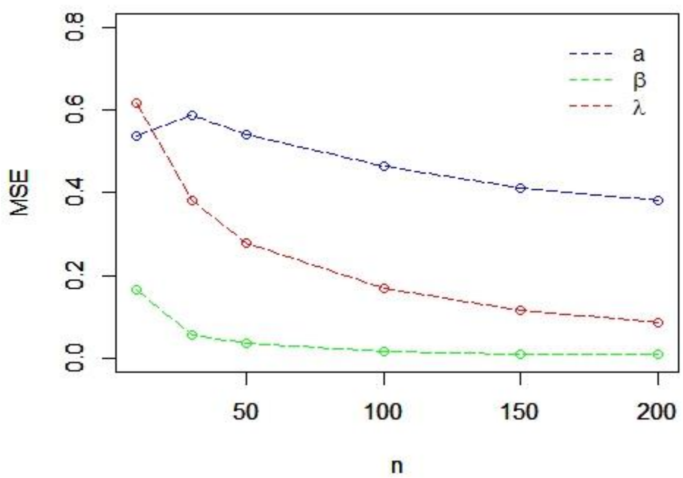

Fig. 7. *Absolute MSE's of selected parameters*$(a = 2, \beta = 2, \lambda = 2)$ *for NWE model.*

This simulation method seems to work well, giving estimates close to the true values of parameters. Clearly, the bias and MSE of three parameters converge to zero when n increases.

7. DATA ANALYSIS AND APPLICATIONS

In this section, we illustrate the usefulness of the NWE distribution. We fit proposed distribution to real data sets in complete and censored cases by ML method and compare the results with the gamma, Weibull, generalized exponential (GE), weighted exponential (WE), generalized Weibull (GW), two parameter weighted exponential (TWE) and log-logistic (LL) with respective densities

$$f_{gamma}(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}, \qquad x \geq 0,$$

$$f_{Weibull}(x) = \frac{\beta}{\lambda^{\beta}} x^{\beta-1} e^{-(\frac{x}{\lambda})^{\beta}}, \qquad x \geq 0,$$

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}, \qquad x \geq 0\,,$$

$$f_{WE}(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1 - e^{-\alpha\lambda x}\right), \qquad x \geq 0\,,$$

$$f_{GW}(x) = \beta\alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}}(1 - e^{-\lambda x^{\alpha}})^{\beta}, \qquad x \geq 0\,,$$

$$f_{TWE}(x) = \frac{\alpha}{B(1/\alpha\,,3)}\lambda e^{-\lambda x}(1 - e^{-\lambda\alpha x})^{2}, \qquad x \geq 0\,,$$

$$f_{LL}(x) = \frac{\alpha\left(\frac{x}{\lambda}\right)^{\alpha-1}}{\lambda\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{2}}, \qquad x \geq 0\,.$$

Censored Data Set: the survival times in months of 100 patients who have been infected by HIV were provided by Hosmer and Lemeshow (1999), where the plus sign in the data indicates a right-censored time.

5 6+ 8 3 22 1+ 7 9 3 12 2+ 12 1 15 34 1 4 19+ 3+ 2 2+ 6 60+ 7+ 60+ 11 2+ 5 4+ 1+ 13 3+ 2+ 1+ 30 7+ 4+ 8+ 5+ 10 2+ 9+ 36 3+ 9+ 3+ 35 8+ 1+ 5+ 11 56+ 2+ 3+ 15 1+ 10 1+ 7+ 3+ 3+ 2+ 32 3+ 10+ 11 3+ 7+ 5+ 31 5+ 58 1+ 2+ 1 3+ 43 1+ 6+ 53 14 4+ 54 1+ 1+ 8+ 5+ 1+ 1+ 2+ 7+ 1+ 10 24+ 7+ 12+ 4+ 57 1+ 12+.

For this data set, there are 37 uncensored time and 63 right censored time. We estimate the parameters of our proposed model by using the likelihood method

Complete Data Set: Bjerkedal (1960) provides a data set consists of survival times of 72 guinea pigs injected with different amount of tubercle. This species of Guinea pigs are known to have high susceptibility of human tuberculosis, which is one of the reasons for choosing. We consider only the study in which animals in a single cage are under the same regimen. The data represents the survival times of Guinea pigs in days. The data are given below:

12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376.

Before analyzing this data set, we use the scaled-TTT plot to verify our model validity, see Aarset (1987). It allows to identify the shape of hazard function graphically. We provide the empirical scaled-TTT plot of above data set. Fig. 8. Shows the scaled-TTT plot is concave. It indicates that the hazard function is increasing; therefore it verifies our model validity.

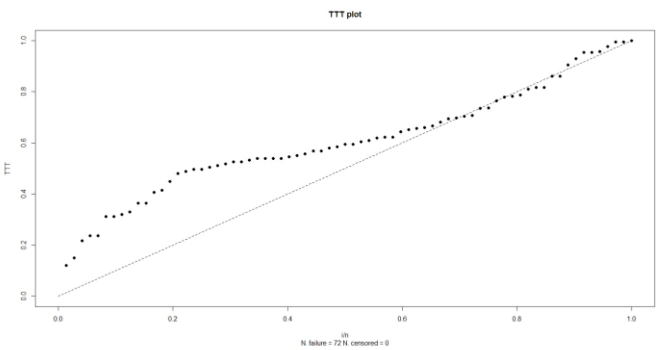

Fig.8. *Scaled-TTT plot of the Guinea pigs data set.*

7.1. Analysis results for censored data set

Here, we fit the NWE distribution to the censored data set and compare it with the gamma, generalized exponential, weighted exponential, Weibull, generalized Weibull, two parameter weighted exponential, weighted exponential and log-logistic densities. Table 1 shows the MLEs of parameters, log-likelihood, Akaike information criterion (AIC) for the censored data set. The NWE distribution provides the best fit for the data set as it shows the lowest AIC than other considered models.

Table 1. *The MLEs of parameters for HIV data.*

Model	MLEs of parameters	Log-likelihood	AIC
NWE	$\hat{\boldsymbol{\alpha}}$ =3.577, $\hat{\boldsymbol{\beta}}$ =17.362, $\hat{\boldsymbol{\lambda}} = 0.051$	-159.641	325.282
gamma	$\hat{\alpha}$ =1.306, $\hat{\lambda} = 0.047$	-162.438	328.877
Weibull	$\hat{\beta}$ =1.182, $\hat{\lambda} = 29.514$	-162.632	329.263
GE	$\hat{\alpha}$ =1.325, $\hat{\lambda}$ =0.043	-162.416	328.832
GW	$\hat{\alpha}$ =2.414, β =0.700, $\hat{\lambda}$ =0.170	-162.232	330.464
TWE	$\hat{\alpha}$ =46.11, $\hat{\lambda}$ =0.035	-162.155	328.31
WE	$\hat{\alpha}$ =15.793, $\hat{\lambda}$ =0.037	-162.160	328.32
log-logistic	$\hat{\alpha}$ =0.010, $\hat{\lambda}$ =1.559	-162.808	329.616

7.2. Analysis results for complete data set

Here, we fit the NWE distribution to the complete data set and compare it with the gamma, generalized exponential, weighted exponential and Weibull densities. Table 2 shows the MLEs of parameters, log-likelihood, Akaike information criterion (AIC), (K-S) distance and related P-value for the complete data set. Analysis of Table 2 shows that the model NWE provides the best fit among other models all those used here to fit dataset. The relative histograms, fitted NWE, gamma, generalized exponential and Weibull PDFs for complete data are plotted in Fig. 9(a). The plots of empirical and fitted survival functions for the NWE and other fitted distributions are displayed in Fig. 9 (b). These plots also support the results in Table 2.

Table 2. *The MLEs of parameters for Guinea pigs data.*

Model	MLEs of parameters		Log-likelihood	AIC	K-S test	P-value
NWE	$\hat{\boldsymbol{\alpha}}$ =3.966, $\hat{\boldsymbol{\beta}}$ =0.001, $\hat{\boldsymbol{\lambda}} = 0.014$		-391.367	788.734	0.113	0.321
gamma	$\hat{\alpha}$ =2.081, $\hat{\lambda} = 0.020$	_	-394.248	792.495	0.139	0.112
Weibull	$\hat{\beta}$ =1.393, $\hat{\lambda} = 110.530$	_	-397.148	798.295	0.149	0.082
GE	$\hat{\alpha}$ =2.473, $\hat{\lambda}$ =0.017	_	-393.110	790.221	0.135	0.135
WE	$\hat{\alpha}$ =1.623, $\hat{\lambda}$ =0.014	_	-393.570	791.138	0.117	0.275

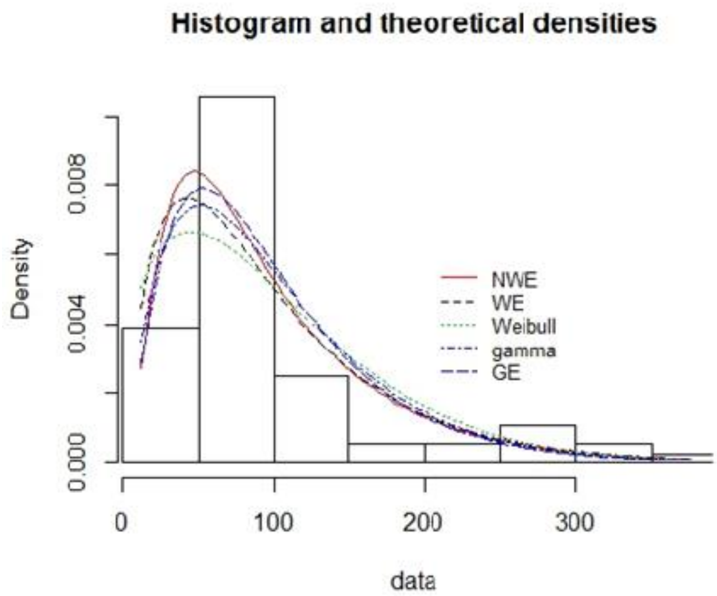

Fig 9 (a). *The fitted PDFs and the relative histogram for the Guinea pigs data histogram for the Guinea pigs data.*

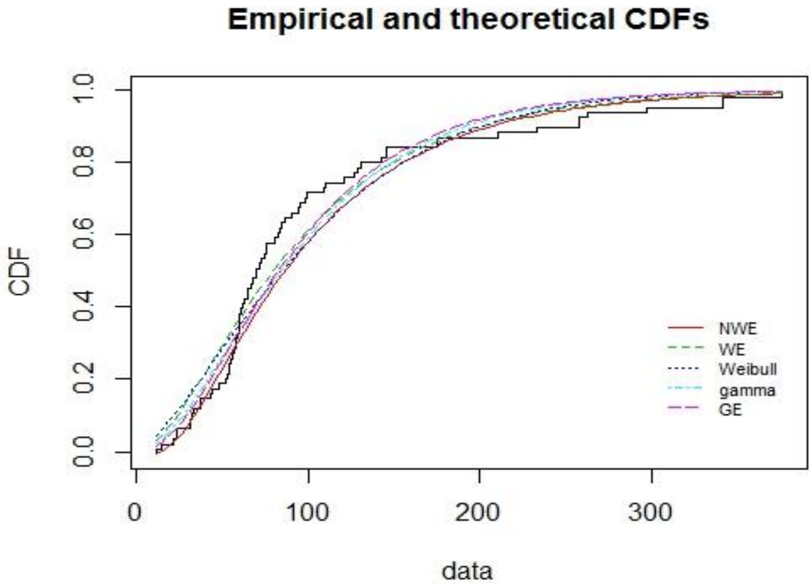

Fig 9(b). *Empirical and fittrd survival functions for Guinea pigs data*

8. CONCLUSIONS

In this paper, we have proposed the new weigthed exponential distribution denoted by NWE. The proposed distribtion generalizes the WE distributions and contains this distribution as its sub-model. It is investigated that the new model has increasing hazard function. Two applications of the NWE distribution to the real data sets are provided to illustrate that this distribution provides a better fit than its sub-models and other common statistical distributions.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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