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**Research Article**

# On codes over product of finite chain rings

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## ABSTRACT

In this paper, codes over the direct product of two finite commutative chain rings are studied. The standard form of the parity-check matrix is determined. The structure of self-dual codes is described. A distance preserving Gray map from the direct product of chain rings to a finite field is defined. Two upper bounds on minimum distance are obtained.

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## INTRODUCTION

In 1997, Rifà and Pujol introduced $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$; see [27]. The set of coordinates in $\mathbb{Z}_2\mathbb{Z}_4$-additive codes is partitioned into two parts, the first part in $\mathbb{Z}_2$ and the last part in $\mathbb{Z}_4$. Due to the appearance of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, the study of codes over mixed ring alphabets has been widely grown, for example $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$-additive codes, $\mathbb{Z}_p\mathbb{Z}_p[u]$-additive codes, $\mathbb{Z}_p[u^r, u^s]$-linear codes, etc.; see [1, 2, 3, 6, 7, 18, 4, 11, 30, 29, 25]. Very recently, Dinh et al. and Gao et al. have extensively studied the applications of mixed alphabet codes in constructing new DNA and quantum codes; see [12, 13, 22, 10].

Notice that in all the aforementioned papers, the coordinates of two parts are based on rings that are finite commutative chain rings. Recently, Borges et.al. have defined $R_1R_2$-linear codes which are $R_2$-submodules of $R_1^\alpha \times R_2^\beta$, where $R_1$ and $R_2$ are finite commutative chain rings with the same residue field $\mathbb{F}_q$. Fundamental results on $R_1R_2$-linear codes including the generator matrix, the duality concept and cyclic codes can be found in [8, 26]. Furthermore, notice that for example in $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, $\mathbb{Z}_2$ is a $\mathbb{Z}_4$-algebra and in $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes, $\mathbb{Z}_2$ is a $\mathbb{Z}_2[u]$-algebra. Based on this fact, Mahmoudi and Samei generalized all the abovementioned papers to *SR*- additive codes, where *S* is an *R*-algebra, see [24].

Motivated by all previous works done on codes over several mixed alphabets and also *SR*- additive codes, we study the structural properties of $R_1R_2$-linear codes.

One of the basic problems in coding theory is to determine the standard form of the parity-check matrix which is used in decoding algorithms efficiently. In this paper, we determine the parity-check matrix of $R_1R_2$-linear codes as well as the relation between $R_1R_2$-linear codes $C$ and $C^\perp$.

The homogeneous weight was first discovered by Constantinescu and Heise as a generalization of the

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Hamming weight on the finite fields and the Lee weight on $\mathbb{Z}_4$; see [9]. Gereferate and Schmidt used a tensor product to construct a Gray map with a non-linear image on the certain chain ring $\mathbb{Z}_3$ endowed with the homogeneous metric; see [20]. Jitman generalized the Gray map given in [20]. He presented an algebraic construction for the Gray map on chain rings equipped with the homogeneous metric that is non-linear over special chain rings $\mathbb{Z}_p m$; see [21]. In this paper, we define a weight on $R_1^{\alpha} \times R_2^{\beta}$, which is the natural generalization of the homogeneous weight over chain rings. Then, using the definition in [21], we define a distance preserving Gray map from $R_1R_2$-linear codes to codes over $\mathbb{F}_q$ equipped with the Hamming weight. Our definition is a natural generalization of the given Gray map on $\mathbb{Z}_p\mathbb{Z}_{p^k}$-linear codes in [29]. The Gray image of $R_1R_2$-linear codes presented in Examples 5.3, 5.4 and 5.5 provides optimal codes which have more simple construction than linear codes with the same parameters in Grassl table; see [19].

The study of several upper bounds on the minimum distance of a code is important in coding theory in view of the fact that codes meeting these bounds have the largest possible minimum distance. In this paper, two upper bounds for the minimum distance are obtained by the Singleton bound for the Gray image and the rank bound for codes over rings. If an $R_1R_2$-linear code meets the first bound, it is called MDS with respect to the Singleton bound (MDSS), and if it attains the rank bound, it is called MDS with respect to the rank bound (MDSR); see [5, 28]. We discuss the conditions on the $R_1R_2$-linear codes to be MDSS or MDSR.

The link between self-dual codes and many different research areas such as design theory and lattice theory makes studying self-dual codes interesting; see [14, 23]. Some sufficient conditions for constructing self-dual codes over chain rings are presented in [15, 16, 17]. Herein, we use the same two methods to build $R_1R_2$-linear self-dual codes.

This paper is organized as follows. In section 2, some basic notations and definitions about chain rings and codes over the products chain rings $R_1$ and $R_2$ are given. In section 3, the parity-check matrix of linear codes over $R_1 \times R_2$ in standard form is described and some examples are presented. In section 4, self-dual codes over $R_1 \times R_2$ are studied and self-dual codes over $R_1 \times R_2$ with larger lengths are constructed by two methods. In section 5, a weight for linear codes over $R_1 \times R_2$ is defined and a distance preserving Gray map on $R_1 \times R_2$ corresponding to the homogeneous weight over chain rings is introduced. Moreover, two upper bounds for the minimum distance of linear codes over $R_1 \times R_2$ are obtained.

## PRELIMINARIES

Throughout this paper, all rings are assumed to be finite and commutative with identity. A ring $R$ is called a chain ring if its ideals are linearly ordered by inclusion. Obviously, every chain ring has a unique maximal ideal. Consider $\gamma$ as the generator of the unique maximal ideal. Since $R$ is finite and its ideals are chain, $\gamma$ is nilpotent. Denote the nilpotency index of $\gamma$ by $e$. We have

$$R=\langle\gamma^0\rangle\supseteq\langle\gamma^1\rangle\supseteq\ldots\supseteq\langle\gamma^{e-1}\rangle\supseteq\langle\gamma^e\rangle=0.$$

Note that if $R$ is a finite field, then $\gamma = 0$ and it can be considered that $e = 1$. It is clear that all elements of $\langle\gamma\rangle$ are zero-divisors and all elements of $R\backslash\langle\gamma\rangle$ are units. The residue field $R/\langle\gamma\rangle$ is denoted by $\mathbb{F}_q$ where $q = p^m$, $p$ is a prime number and $m$ is a positive integer. Let : $R \to \mathbb{F}_q$ be the natural projection map. Let $T = \{r_0,\ldots, r_{q-1}\}$ be the Teichmüller set of representatives of $R$.

**Lemma 2.1** *[26] Assume the above notations. Let $V \subseteq R$ be a system of representatives for the equivalence classes of $R$ under congruence modulo $\gamma$. Then*

1. For all $r\in R$ there exist unique $a_0(r),a_1(r),\ldots,a_{e-1}(r) \in V$ such that $r = \sum_{i=0}^{e-1} a_i(r)\gamma^i$.
2. $|V| = q$.
3. $|\langle\gamma^j\rangle| = q^{e-j}$ for all $j\in\{0,\ldots, e-1\}$.

Clearly, $|R| = q^e$ and any elements $r \in R^n$ can be written uniquely as $r=\sum_{i=0}^{e-1} a_i(r)\gamma^i$, where $a_i(r)=(r_{i,0},r_{i,1},\ldots,r_{i,n-1}) \in V^n$ for all $i$.

## LINEAR CODES OVER CHAIN RINGS

A linear code of length n over $R$ is an $R$-submodule of $R^n$. In [20], the homogeneous weight of each element $r \in R$ in the sense of [9], denoted by $w_{\text{hom}}(r)$, is defined as follows:

$$w_{\text{hom}}(r)=\begin{cases} q^{e-1} & r\in\gamma^{e-1}R\setminus\{0\},\\ q^{(e-2)}(q-1) & r\in R\setminus\gamma^{e-1}R,\\ 0 & \text{o.w.}\end{cases}$$

Naturally, the homogeneous weight can be extended to $R^n$ componentwise. Then, the homogeneous weight of $r = (r_1,\ldots,r_n) \in R^n$ becomes $w_{\text{hom}}(r) = \sum_{i=1}^{n} w_{\text{hom}}(r_i)$. The homogeneous distance $d_{\text{hom}}(r,s)$ between vectors $r,s$ in $R^n$ is defined to be $w_{\text{hom}}(r-s)$.

In [21], the Gray map from $R^n$ to $\mathbb{F}_q^{q^{e-1}}$ is defined as follows.

Let

$$\varepsilon=\xi_0(\varepsilon)+\xi_1(\varepsilon)p+\cdots+\xi_{m-1}(\varepsilon)p^{m-1}$$

be the $p$-adic representation of $\varepsilon\in\mathbb{Z}_{p^m}$ where $\xi_i(\varepsilon) \in \{0,1,\cdots,p-1\}$ for all $i \in \{0,\cdots,m-1\}$. Let $\alpha$ be a fixed primitive element of $\mathbb{F}_{q^m}$. Corresponding to every $\varepsilon$, consider $\alpha_\varepsilon$ as

$$\alpha_\varepsilon=\xi_0(\varepsilon)+\xi_1(\varepsilon)\alpha+\cdots+\xi_{m-1}(\varepsilon)\alpha^{m-1}.$$

Moreover, let

$$w = \tilde{\xi}_0(w) + \tilde{\xi}_1(w)p^m + \cdots + \tilde{\xi}_{e-2}(w)p^{m(e-2)}$$

be the $p$-adic representation of $w \in \mathbb{Z}_{p^{m(e-1)}}$, where $\tilde{\xi}_i(w) \in \{0,1,\cdots,p^m-1\}$ for all $i \in \{0,\cdots,e-2\}$. Now, the Gray map $\phi: R^n \to \mathbb{F}_q^{q^{e-1}n}$ is defined as

$$\phi(r) = \left(b_0, b_1, \ldots, b_{q^{e-1}-1}\right)$$

for all $r = a_0(r) + a_1(r)\gamma + \cdots + a_{e-1}(r)\gamma^{e-1} \in R^n$, where

$$b_{wp^m+\varepsilon} = \alpha_\varepsilon \overline{a_0(r)} + \sum_{l=1}^{e-2} \alpha_{\tilde{\xi}_{l-1}(w)} \overline{a_l(r)} + \overline{a_{e-1}(r)}$$

for all $w \in \{0,\cdots,p^{m(e-2)}-1\}$ and $\varepsilon \in \{0,\cdots,p^m-1\}$.

**Example 2.2**

1. if $R = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$, where $u^3 = 0$, the Gray map $\phi: R \to \mathbb{F}_2^4$ is

$$\phi\left(a_0 + a_1u + a_2u^2\right) = \left(a_2, a_0 + a_2, a_1 + a_2, a_0 + a_1 + a_2\right).$$

2. For $R = \mathbb{F}_5 + u\mathbb{F}_5$ where $u^2 = 0$, the Gray map $\phi: R \to \mathbb{F}_5^5$ is

$$\phi\left(a_0 + a_1u\right) = \left(a_1, a_0 + a_1, 2a_0 + a_1, 3a_0 + a_1, 4a_0 + a_1\right).$$

**Proposition 2.3** *[21] The Gray map $\phi$ is an isometry from $(R^n, d_{hom})$ to $(\mathbb{F}_q^{q^{e-1}n}, d_H)$, where $d_H$ denotes the Hamming distance on $\mathbb{F}_q^{q^{e-1}n}$.*

It is well known that Singleton bound for every code $C$ over an alphabet of size $q$ is given by

$$d_H(C) \le n - \log_q|C| + 1. \tag{2.1}$$

Furthermore, by Theorem 3.7 in [28], if $C$ is a code of length $n$ over chain ring $R$ equipped with the homogeneous distance $d_{\mathrm{hom}}$ then the rank bound for $C$ is established as follows:

$$\left\lfloor \frac{d_{\mathrm{hom}}(C) - 1}{q^{e-1}} \right\rfloor \le n - rank(C), \tag{2.2}$$

where *rank* $(C)$ is the minimum cardinality of the generator set of $C$.

## LINEAR CODES OVER $R_1 \times R_2$

From now on, assume that $R_1 = R_{\gamma_1,e_1,q}$ and $R_2 = R_{\gamma_2,e_2,q}$ denote two finite chain rings where $\gamma_1$ and $\gamma_2$ are the generators of their maximal ideals with nilpotency indices $e_1$ and $e_2$ respectively. Besides, assume that $R_1$ and $R_2$ have the same residue field $\mathbb{F}_q$ and $e_1 \le e_2$. Moreover, suppose that $T_1 = \{r_0,\cdots,r'_{q-1}\}$ and $T_2 = \{r'_0,\cdots,r'_{q-1}\}$ are the Teichmüller sets of representatives of $R_1$ and $R_2$, respectively.

Define the surjective ring homomorphism $\pi$ from $R_2$ to $R_1$ such that $\pi(\gamma_2) = \gamma_1$ and $\pi(r'_j) = r_j$. It is obvious that $\pi(\gamma_2^{\,i}) = 0$ for all $i \ge e_1$. Consider $a \in R_2$ and $u = (u|u') = (u_1,\cdots,u_\alpha|u'_1,\cdots,u'_\beta) \in R_1^\alpha \times R_2^\beta$. In [8], it is asserted that $R_1^\alpha \times R_2^\beta$ is an $R_2$-module with the following scalar multiplication

$$a * u = \left(\pi(a)u_1, \cdots, \pi(a)u_\alpha \mid au'_1, \cdots, au'_\beta\right).$$

**Definition 2.4** *[8] A subset $C \subseteq R_1^\alpha \times R_2^\beta$ is called an $R_1R_2$-linear code if $C$ is a submodule of $R_1^\alpha \times R_2^\beta$.*

**Proposition 2.5** *[8] Let $C$ be an $R_1R_2$-linear code, then $C$ is permutation equivalent to a code with a standard generator matrix of the form*

$$G = \left[\begin{array}{c|c} B & T \\ S & A \end{array}\right], \tag{2.3}$$

where

$$B = \begin{bmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} & \cdots & B_{0,e_1-1} & B_{0,e_1} \\ 0 & \gamma_1 I_{k_1} & \gamma_1 B_{1,2} & \gamma_1 B_{1,3} & \cdots & \gamma_1 B_{1,e_1-1} & \gamma_1 B_{1,e_1} \\ 0 & 0 & \gamma_1^2 I_{k_2} & \gamma_1^2 B_{2,3} & \cdots & \gamma_1^2 B_{2,e_1-1} & \gamma_1^2 B_{2,e_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_1^{e_1-1} I_{k_{e_1-1}} & \gamma_1^{e_1-1} B_{e_1-1,e_1} \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & \cdots & \gamma_2^{e_2-e_1} T_{0,1} & \gamma_2^{e_2-e_1} T_{0,2} & \cdots & \gamma_2^{e_2-e_1} T_{0,e_1} \\ 0 & \cdots & 0 & \gamma_2^{e_2-e_1+1} T_{1,2} & \cdots & \gamma_2^{e_2-e_1+1} T_{1,e_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \gamma_2^{e_2-1} T_{e_1-1,e_1} \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & S_{0,1} & S_{0,2} & \cdots & S_{0,e_1-1} & S_{0,e_1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,e_1-1} & S_{1,e_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & S_{e_2-e_1-1,1} & S_{e_2-e_1-1,2} & \cdots & S_{e_2-e_1-1,e_1-1} & S_{e_2-e_1-1,e_1} \\ 0 & 0 & \gamma_1 S_{e_2-e_1,2} & \cdots & \gamma_1 S_{e_2-e_1,e_1-1} & \gamma_1 S_{e_2-e_1,e_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_1^{e_1-2} S_{e_2-3,e_1-1} & \gamma_1^{e_1-2} S_{e_2-3,e_1} \\ 0 & 0 & 0 & \cdots & 0 & \gamma_1^{e_1-1} S_{e_2-2,e_1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} I_{l_0} & A_{0,1} & A_{0,2} & A_{0,3} & \cdots & A_{0,e_2-1} & A_{0,e_2} \\ 0 & \gamma_2 I_{l_1} & \gamma_2 A_{1,2} & \gamma_2 A_{1,3} & \cdots & \gamma_2 A_{1,e_2-1} & \gamma_2 A_{1,e_2} \\ 0 & 0 & \gamma_2^2 I_{l_2} & \gamma_2^2 A_{2,3} & \cdots & \gamma_2^2 A_{2,e_2-1} & \gamma_2^2 A_{2,e_2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_2^{e_2-1} I_{l_{e_2-1}} & \gamma_2^{e_2-1} A_{e_2-1,e_2} \end{bmatrix},$$

such that the entries in $\gamma_1^i B_{i,j}$ and $\gamma_1^i S_{i,j}$ are in $\langle \gamma_1^i \rangle$ and the entries in $\gamma_2^t A_{t,j}$ and $\gamma_2^t T_{t,j}$ are in $\langle \gamma_1^t \rangle$.

It is said that $C$ is of type $(\alpha,\beta; k_0,\cdots,l_0,\cdots,l_{e_2-1})$. According to Proposition 2.5, it can be concluded that $rank(C)=\sum_{i=0}^{e_1-1}k_i+\sum_{j=0}^{e_2-1}l_i$ and

$$\begin{aligned}|C|&=\left|\langle\gamma_1^0\rangle\right|^{k_0}\times\left|\langle\gamma_1^1\rangle\right|^{k_1}\times\cdots\times\left|\langle\gamma_1^{e_1-1}\rangle\right|^{k_{e_1-1}}\times\left|\langle\gamma_2^0\rangle\right|^{l_0}\\&\quad\times\left|\langle\gamma_2^1\rangle\right|^{l_1}\times\cdots\times\left|\langle\gamma_2^{e_2-1}\rangle\right|^{l_{e_2-1}}\\&=q^{\sum_{i=0}^{e_1-1}(e_1-i)k_i+\sum_{j=0}^{e_2-1}(e_2-j)l_j}.\end{aligned}$$

Consider injective map $\iota:R_1\to R_2$ by definition $\iota(\gamma_1)=\gamma_2$ and $\iota(r_j)=r'_j$. It is obvious that $\pi\iota=Id$.

**Definition 2.6** *[8] The inner product of vectors $u=(u,u')$ and $v=(v,v')$ in $R_1^\alpha\times R_2^\beta$ is defined by*

$$<u,v>=\gamma_2^{e_2-e_1}\iota(u.v)+u'.v'\in R_2,$$

where $u.v$ and $u'.v'$ are standard inner product.

The $R_1R_2$-dual code of an $R_1R_2$-linear code $C$ is defined in the standard way by

$$C^\perp=\left\{v\in R_1^\alpha\times R_2^\beta:<u,v>=0,\quad for\,all\,u\in C\right\},$$

which is an $R_2$-submodule of $R_1^\alpha\times R_2^\beta$. We say that an $R_1R_2$-linear code $C$ is self-orthogonal if $C\subseteq C^\perp$ and is self-dual if $C\subseteq C^\perp$.

Let $C_X$ be the punctured $R_1R_2$-linear code of $C$ by deleting the first $\alpha$ coordinates and $C_Y$ be that of by deleting the last $\beta$ coordinates. The code $C$ is called separable if $C=C_X\times C_Y$. If $C$ is separable, then its generator matrix is in the form

$$G=\left[\begin{array}{c|c}B&0\\0&A\end{array}\right],\tag{2.4}$$

where $A$ and $B$ are matrices in Proposition 2.5. The dual-code of every separable code $C$ is separable and $C^\perp=C_X^\perp\times C_Y^\perp$.

## PARITY-CHECK MATRICES OF $R_1R_2$-LINEAR CODES

The next theorem generalizes the structure of the parity-check matrices presented in [1, 3] to the case of $R_1R_2$-linear codes.

Define $k(B)=k_0+k_1+\ldots+k_{e_1-1}$ and $l(A)=l_0+l_1+\ldots+l_{e_2-1}$.

**Theorem 3.1** *Let $C$ be an $R_1R_2$-linear code with the generator matrix $G$ given in Proposition 2.5. Then the standard form for the generator matrix of $C^\perp$ is*

$$H=\left[\begin{array}{c|c}\tilde{B}+F&U\\V&\tilde{A}+E\end{array}\right],\tag{3.1}$$

where

$$\tilde{B}=\begin{bmatrix}\tilde{B}_{0,e_1}&\tilde{B}_{0,e_1-1}&\tilde{B}_{0,e_1-2}&\cdots&\tilde{B}_{0,3}&\tilde{B}_{0,2}&\tilde{B}_{0,1}&I_{\alpha-k(B)}\\\gamma_1\tilde{B}_{1,e_1}&\gamma_1\tilde{B}_{1,e_1-1}&\gamma_1\tilde{B}_{1,e_1-2}&\cdots&\gamma_1\tilde{B}_{1,3}&\gamma_1\tilde{B}_{1,2}&\gamma_1 I_{k_{e_1-1}}&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\\gamma_1^{e_1-2}\tilde{B}_{e_1-2,e_1}&\gamma_1^{e_1-2}\tilde{B}_{e_1-2,e_1-1}&\gamma_1^{e_1-2}I_{k_2}&\cdots&\cdots&0&0&0\\\gamma_1^{e_1-1}\tilde{B}_{e_1-1,e_1}&\gamma_1^{e_1-2}I_{k_1}&0&\cdots&\cdots&0&0&0\end{bmatrix}$$

$$F=\begin{bmatrix}F_{0,e_1-2}&F_{0,e_1-3}&\cdots&F_{0,2}&F_{0,1}&0&0&0\\\gamma_1F_{1,e_1-2}&\gamma_1F_{1,e_1-3}&\cdots&\gamma_1F_{1,2}&0&0&0&0\\\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots&\vdots\\\gamma_1^{e_1-4}F_{e_1-4,e_1-2}&\gamma_1^{e_1-4}F_{e_1-4,e_1-3}&\cdots&0&0&0&0&0\\\gamma_1^{e_1-3}F_{e_1-3,e_1-2}&0&\cdots&0&0&0&0&0\\0&0&\cdots&0&0&0&0&0\\0&0&\cdots&0&0&0&0&0\end{bmatrix}$$

$$U=\begin{bmatrix}U_{0,e_2-1}&U_{0,e_2-2}&\cdots&U_{0,3}&U_{0,2}&U_{0,1}&0&0\\\gamma_2U_{1,e_2-1}&\gamma_2U_{1,e_2-2}&\cdots&\gamma_2U_{1,3}&\gamma_2U_{1,2}&0&0&0\\\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots&\vdots\\\gamma_2^{e_1-2}U_{e_1-2,e_1-1}&\gamma_2^{e_1-2}U_{e_1-2,e_2-2}&\cdots&0&0&0&0&0\\\gamma_2^{e_1-1}U_{e_1-1,e_2-1}&\gamma_2^{e_1-1}U_{e_1-1,e_2-2}&\cdots&0&0&0&0&0\end{bmatrix}$$

$$V=\begin{bmatrix}V_{0,e_1}&V_{0,e_1-1}&V_{0,e_1-2}&\cdots&V_{0,3}&V_{0,2}&V_{0,1}&0\\\gamma_1V_{1,e_1}&\gamma_1V_{1,e_1-1}&\gamma_1V_{1,e_1-2}&\cdots&\gamma_1V_{1,3}&\gamma_1V_{1,2}&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\\gamma_1^{e_1-1}V_{e_1-1,e_1}&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&0&0&0&0\end{bmatrix}$$

$$\tilde{A}=\begin{bmatrix}\tilde{A}_{0,e_2} & \tilde{A}_{0,e_2-1} & \tilde{A}_{0,e_2-2} & \cdots & \tilde{A}_{0,3} & \tilde{A}_{0,2} & \tilde{A}_{0,1} & I_{\beta-l(A)}\\ \gamma_2\tilde{A}_{1,e_2} & \gamma_2\tilde{A}_{1,e_2-1} & \gamma_2\tilde{A}_{1,e_2-2} & \cdots & \gamma_2\tilde{A}_{1,3} & \gamma_2\tilde{A}_{1,2} & \gamma_2 I_{l_{e_2}-1} & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \gamma_2^{e_2-2}\tilde{A}_{e_2-2,e_2} & \gamma_2^{e_2-2}\tilde{A}_{e_2-2,e_2-1} & \gamma_2^{e_2-2}I_{l_2} & \cdots & \cdots & 0 & 0 & 0\\ \gamma_2^{e_2-1}\tilde{A}_{e_2-1,e_2} & \gamma_2^{e_2-1}I_{l_1} & 0 & \cdots & \cdots & 0 & 0 & 0\end{bmatrix}$$

$$E=\begin{bmatrix}E_{0,e_2-2} & E_{0,e_2-3} & E_{0,e_2-4} & \cdots & E_{0,2} & E_{0,1} & & & \\ \gamma_2E_{1,e_2-2} & \gamma_2E_{1,e_2-3} & \gamma_2E_{1,e_2-4} & \cdots & \gamma_2E_{1,2} & & & & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \gamma_2^{e_1-2}E_{e_1-2,e_2-2} & \gamma_2^{e_1-2}E_{e_1-2,e_2-3} & \gamma_2^{e_1-2}E_{e_1-2,e_2-4} & \cdots & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

such that

$$\tilde{B}_{i,j}=-\sum_{k=i+1}^{j-1}\tilde{B}_{i,k}\tilde{B}^t_{e_1-j,e_1-k}-B^t_{e_1-j,e_1-i},\quad 0\le i\le j\le e_1,$$

$$\tilde{A}_{i,j}=-\sum_{k=i+1}^{j-1}\tilde{A}_{i,k}A^t_{e_2-j,e_2-k}-A^t_{e_2-j,e_2-i},\quad 0\le i\le j\le e_2,$$

$$U_{i,j}=-\sum_{k=i+1}^{j-1}U_{i,k}A^t_{e_2-j,e_2-k-1}-\gamma_2^{j-e_1+1-i}\left(\iota\left(\gamma_1^a\left(\sum_{l=i+1}^{j-1}\tilde{B}_{i,l}S^t_{e_2-j-1,e_1-l}+\sum_{m=i+1}^{j-3}F_{i,m}S^t_{e_2-j-1,e_1-m-2}+S^t_{e_2-j-1,e_1-i}\right)\right)\right),$$

$$a=\begin{cases} i & j\ge e_1,\\ e_1-j+i & j\ge e_1,\end{cases}\quad m\le e_1-3,\quad l\le e_1-1,$$

$$E_{i,j}=-\sum_{k=i+1}^{j-1}E_{i,k}A^t_{e_2-j-2,e_2-k-2}-\gamma_2^{j+2-i-e_1}\left(\iota\left(\gamma_1^a\sum_{l=i+1}^{j}V_{i,l}S^t_{e_2-j-2,e_1-l}\right)\right),$$

$$a=\begin{cases} i & j\ge e_1-1,\\ e_1-j-1+i & j\ge e_1-1,\end{cases}\quad l\le e_1-1,$$

$$\iota\left(\gamma_1^{e_1-2-j+i}\left(F_{i,j}\right)\right)=-\iota\left(\gamma_1^{e_1-2-j+i}\sum_{k=i+1}^{j-1}F_{i,j}B^t_{e_1-j-2,e_1-k-2}\right)-\gamma_2^{e_1-2-j+i}\left(\sum_{k=i+1}^{j}U_{i,k}T^t_{e_1-j-2,e_1-k-1}\right),$$

$$\iota\left(\gamma_1^{e_1-j+i}V_{i,j}\right)=-\iota\left(\gamma_1^{e_1-j+i}\left(\sum_{k=i+1}^{j-1}V_{i,k}B^t_{e_1-j,e_1-k}\right)\right)-\gamma_2^{e_1-j+i}\left(\sum_{k=i+1}^{j-1}\tilde{A}_{i,k}T^t_{e_1-j,e_1-k}+\sum_{k=i+1}^{j-3}E_{i,i+j-k-2}.T^t_{e_1-j,e_1+k-i-j}+T^t_{e_1-j,e_1-i}\right).$$

***Proof.*** It is time-consuming but easy to check that $HG_t = 0$. Hence, we conclude that the rows of $H$ are orthogonal to the rows of $G$, i.e. $C'\subset C^{\perp}$, where $C'$ is the code generated by $H$. Moreover,

$$|C'|=|R_1|^{\alpha-k(B)}\times|\gamma_1R_1|^{k_{e_1-1}}\times\cdots\times|\gamma_1^{e_1-1}R_1|^{k_1}\times|R_2|^{\beta-l(A)}\times|\gamma_2R_2|^{l_{e_2-1}}\times\cdots\times|\gamma_2^{e_2-1}R_2|^{l_1},$$

which implies

$$|C||C'|=q^{e_1\alpha+e_1\beta}=|R_1^{\alpha}\times R_2^{\beta}|.$$

Therefore, $H$ generates the dual space of $C$.

**Corollary 3.2** *For every $R_1R_2$-linear code C we have*

1. $|C||C^{\perp}|=|R_1^{\alpha}\times R_2^{\beta}|.$
2. $(C^{\perp})^{\perp}=C.$

***Proof.*** The statement was shown throughout the proof of Theorem 3.1. To prove we note that $C\subset(C^{\perp})^{\perp}$. Since $C^{\perp}$ is of type

$$(\alpha,\beta;\alpha-k(B),k_{e_1-1},\ldots,k_1;\beta-1(A),l_{e_2-1},\ldots,l_1),$$

then $(C^{\perp})^{\perp}$ is of type $(\alpha,\beta;k_0,\ldots,k_{e_1-1};l_0,\ldots,l_{e_2-1})$ and hence $C$ and $(C^{\perp})^{\perp}$ have the same size, completing (2).

**Example 3.3** *Let $R_1 = R_{\gamma_1,2,2}$, $R_2 = R_{\gamma_2,3,2}$ and C be an $R_1R_2$-linear code of type* (3,4; 1,1; 1,1,1) *generated by*

$$G=\left[\begin{array}{ccc|cccc}1 & 1 & 1 & 0 & 0 & \gamma_2 & \gamma_2\\ 0 & \gamma_1 & \gamma_1 & 0 & 0 & 0 & \gamma_2^2\\ 0 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & \gamma_1 & 0 & \gamma_2 & \gamma_2 & \gamma_2\\ 0 & 0 & 0 & 0 & 0 & \gamma_2^2 & \gamma_2^2\end{array}\right].$$

We calculate

$$\begin{array}{lll} 3V_{0,1}=1, & V_{0,2}=1, & \gamma_1 V_{1,2}=\gamma_1, \\ U_{0,1}=\gamma_2, & U_{0,2}=\gamma_2, & \gamma_2 U_{1,2}=\gamma_2^2, \\ \tilde{B}_{0,1}=1, & \tilde{B}_{0,2}=0, & \gamma_1 \tilde{B}_{1,2}=\gamma_1, \\ \tilde{A}_{0,1}=1, & \tilde{A}_{0,2}=\tilde{A}_{0,3}=0, & \gamma_2 \tilde{A}_{1,2}=\gamma_2, \\ \gamma_2 \tilde{A}_{1,3}=0, & \gamma_2^2 \tilde{A}_{2,3}=\gamma_2^2, & E_{0,1}=\gamma_2. \end{array}$$

Therefore, the parity-check matrix is in the form

$$H=\left[\begin{array}{ccc|cccc} 0 & 1 & 1 & \gamma_2 & \gamma_2 & 0 & 0 \\ \gamma_1 & \gamma_1 & 0 & \gamma_2^2 & 0 & 0 & 0 \\ 1 & 1 & 0 & \gamma_2 & 0 & 1 & 1 \\ \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_2 & 0 \\ 0 & 0 & 0 & \gamma_2^2 & \gamma_2^2 & 0 & 0 \end{array}\right].$$

**Example 3.4** *[2] Suppose C is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code generated by*

$$G=\left[\begin{array}{cc|ccc} 1 & 1 & u & u & u \\ 0 & 1 & 1 & 0 & 1+u \\ 1 & 0 & 1 & u & u \end{array}\right].$$

$G$ is permutation equivalent with

$$G'=\left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & u \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right].$$

Consider the natural injective map $\imath: \mathbb{Z}_2 \to \mathbb{Z}_2[u]$ where $\imath(0) = 0$ and $\imath(1) = 1$. We have

$$F=E=0,\ \ V_{0,1}=1,\ \ \tilde{B}_{0,1}=1,\ \ \tilde{A}_{0,2}=(0\ 0),\ \ U_{0,1}=(u\ 0).$$

Thus, the parity-check matrix is

$$H=\left[\begin{array}{cc|ccc} 1 & 1 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

**Example 3.5** *[8] Let $R_1=\mathbb{Z}_2$, $R_2=R_{u,3,2}$ and C be an $R_1R_2$-linear code generated by*

$$G=\left[\begin{array}{ccc|cccc} 1 & 1 & 0 & u & u+u^2 & 1+u & 1+u \\ 0 & 1 & 0 & 1 & u & u^2 & 0 \\ 0 & 1 & 1 & 0 & u^2 & 0 & u^2 \\ 1 & 1 & 1 & u^2 & u & u+u^2 & 0 \end{array}\right]$$

It is easy to show that $G$ is permutation equivalent with

$$G'=\left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & u & u+u^2 \end{array}\right].$$

Hence, $C$ is of type (3,4; 1; 2,1,0) and

$$3E=F=0,\qquad V_{0,1}=0,$$

$$U_{0,1}=\begin{bmatrix} u \\ u \end{bmatrix},\qquad U_{0,2}=\begin{bmatrix} 0 & u^2 \\ u^2 & 0 \end{bmatrix},\qquad \tilde{B}_{0,1}=\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\tilde{A}_{0,2}=1+u,\qquad \tilde{A}_{0,3}=\begin{bmatrix} u & 1 \end{bmatrix},\qquad \tilde{A}_{2,3}=\begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Thererfore,

$$H=\left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 0 & u^2 & u & 0 \\ 1 & 0 & 1 & u^2 & 0 & u & 0 \\ 0 & 0 & 0 & u & 1 & 1+u & 1 \\ 0 & 0 & 0 & 0 & 0 & u^2 & 0 \end{array}\right].$$

## $R_1R_2$-LINEAR SELF-DUAL CODES

**Definition 4.1** *Let $u = (u|u') \in R_1^{\alpha} \times R_2^{\beta}$. The weight function $wt^*(u)$ is defined by*

$$wt^*(u)=w_{\mathrm{hom}_{R1}}(u)+w_{\mathrm{hom}_{R2}}(u'),$$

where $w_{\mathrm{hom}_{R1}}$ and $w_{\mathrm{hom}_{R2}}$ are the homogeneous weights over $R_1^{\alpha}$ and $R_2^{\beta}$, respectively.

The distance between two elements $u,\ v \in R_1^{\alpha} \times R_2^{\beta}$, denoted by $d^*(u,v)$, is $wt^*(u-v)$. The minimum distance of an $R_1R_2$-linear code $C$, denoted by $d^*(C)$ or $d^*$, is the minimum value of $d^*(u,v)$ for all $u,v \in C$ such that $u \neq v$. Consider two notations $k_{e_1} = \alpha - k(B)$ and $l_{e_2} = \beta - l(A)$.

**Lemma 4.2** *Let C be an $R_1R_2$-linear code of type ($\alpha$,$\beta$; $k_0,\cdots, k_{e_1-1}$; $l_0,\cdots, l_{e_2-1}$) If C is self-dual, then $k_i = k_{e_1-i}$ and $l_j = l_{e_2-j}$ for all $i \in \{0,\cdots, e_1\}$ and $j \in \{0,\cdots, e_2\}$.*

***Proof.*** By Theorem 3.1, $C^{\perp}$ is of type ($\alpha$,$\beta$; $k_{e_1}$, $k_{e_1-1},\cdots, k_1$; $l_{e_2}$, $l_{e_2-1}\cdots, l_0$). Since $C = C^{\perp}$, their types are equal and the result follows.

**Theorem 4.3** *An $R_1R_2$-linear self-orthogonal code C is self-dual if and only if $|C|=q^{\frac{e_1\alpha+e_2\beta}{2}}$.*

***Proof.*** By Corollary 3.2, we have $|C||C^{\perp}|=q^{e_1\alpha+e_2\beta}$ which gives the result.

**Example 4.4** *Assume $R_1 = R_{\gamma_1,2,5}$, $R_2 = R_{\gamma_2,3,5}$ and C is an $R_1R_2$-linear code with the generator matrix*

$$G=\left[\begin{array}{c|cc} \gamma_1 I_3 & T_1 & T_2 \\ 0 & 2\gamma_1 I_4 & A \\ 0 & 0 & \gamma_2^2 I_4 \end{array}\right]$$

where $T_1$ and $T_2$ are arbitrary matrices over $\gamma_2^2 R_2$, $I_3$ and $I_4$ are identity matrices and

$$A=\begin{bmatrix}3\gamma_2 & 3\gamma_2 & 2\gamma_2 & 3\gamma_2\\ 3\gamma_2 & 3\gamma_2 & 3\gamma_2 & 2\gamma_2\\ 3\gamma_2 & 2\gamma_2 & 3\gamma_2 & 3\gamma_2\\ 2\gamma_2 & 3\gamma_2 & 3\gamma_2 & 3\gamma_2\end{bmatrix}.$$

It can be easily seen that $C$ is self-orthogonal and $|C| = q_{15}$ and therefore $C$ is self-dual.

**Theorem 4.5** *Let $q = 2$ and $C$ be an $R_1R_2$-linear self-dual code. Then $d^*(C^\perp) \le q^{e_2-1}\beta$.*

***Proof.*** For any arbitrary element $u = (u_1,\ldots,u_\alpha|u'_1,\ldots,u'_\beta) \in$ C, we have

$$\gamma_2^{e_2-e_1}\sum_{i=1}^{\alpha}\iota\left(u_i^2\right)+\sum_{j=1}^{\beta}\left(u'_j\right)^2=0.$$

Consider the natural homomorphism $\rho : R_2 \to \frac{R_2}{\langle\gamma_2\rangle} = \mathbb{F}_2$. Thus

$$\rho\left(\gamma_2^{e_2-e_1}\sum_{i=1}^{\alpha}\iota\left(u_i^2\right)\right)+\rho\left(\sum_{j=1}^{\beta}\left(u'_j\right)^2\right)=0.$$

So $\Sigma_{j=1}^{\beta}(\rho(u'_j))^2 = 0$ and since $\mathbb{F}^2$ has characteristic 2, $\Sigma_{j=1}^{\beta}\rho(u'_j) = 0$, which implies $\Sigma_{j=1}^{\beta} u'_j \in <\gamma_2>$. Take $v = (0,\ldots,0|\gamma_2^{e_2-1},\ldots,\gamma_2^{e_2-1})$. We have $<u,v> = \gamma_2^{e_2-1}\,\Sigma_{j=1}^{\beta}\, u'_j = 0$. As a result, $C^\perp$ contains the element $v$ and hence $d^*(C^\perp) \le q^{e_2-1}\beta$.

If $C_X$ and $C_Y$ are self-dual codes over chain rings $R_1$ and $R_2$, respectively, then the separable code $C = C_X \times C_Y$ is a separable self-dual code over $R_1 \times R_2$. Some sufficient conditions for the existence of self-dual codes over chain rings are presented in [15, 16, 17]. In the following, we present some conditions for a self-dual code to be non-separable.

**Theorem 4.6** *Let $C$ and $C'$ be self-dual linear codes in $R_1^\alpha \times R_2^\beta$ and $R_1^{\alpha'} \times R_2^{\beta'}$ with generator matrices $G=(G_1|G_2)$ and $G' = (G'_1|G'_2)$, respectively. Then*

$$\mathcal{G}=\left[\begin{array}{cc|cc}G_1 & 0 & G_2 & 0\\ 0 & G'_1 & 0 & G'_2\end{array}\right]$$

generates the self-dual linear code $D$ in $R_1^{\alpha+\alpha'} \times R_2^{\beta+\beta'}$. Moreover, if either $G'_1$ or $G_2$ is non-zero, then $D$ is a non-separable self-dual code

***Proof.*** Since the inner product of any two rows of $G$ is zero, $D$ is self-orthogonal. In addition, we have

$$|\mathcal{D}|=|\mathcal{C}||\mathcal{C}'|=q^{\frac{e_1(\alpha+\alpha')+e_2(\beta+\beta')}{2}},$$

which implies $D$ is self-dual.

The following theorem describes a technique for constructing $R_1R_2$-linear self-dual codes with larger lengths, which is a generalization of the presented technique in [17].

**Theorem 4.7** *Let $B_i$, $T_i$, $S_j$ and $A_j$ be the rows of B, T, S and A in Matrix (2.3), respectively. Assume that there are $c_1 \in R_1$ and $c_2 \in R_2$ such that $c_1^2 = -1$ and $c_2^2 = -1$. Let C be an $R_1R_2$-linear self-dual code of length n generated by the matrix (2.3). Consider $b = (b_1,\ldots, b_\alpha) \in R_1^\alpha$ and $a=(a_1,\ldots, a_\beta) \in R_2^\beta$ satisfying $b.b = -1$ and $a.a = -1$. Put $u_i = b.B_i^t$, $v_i = a.T_i^t$, $z_j = b.S_j^t$ and $w_j = a.A_j^t$. Then*

$$g=\left[\begin{array}{cccc|cccc}1 & 0 & b_1\ \cdots & b_\alpha & 0 & 0 & 0\ \cdots & 0\\ -u_1 & c_1u_1 & & & -v_1 & c_2v_1 & & \\ \vdots & \vdots & B & & \vdots & \vdots & T & \\ -u_{k(B)} & c_1u_{k(B)} & & & -v_{k(B)} & c_2u_{k(B)} & & \\ & & & & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & a_1\ \cdots & a_\beta\\ -z_1 & c_1z_1 & & & -w_1 & c_2w_1 & & \\ \vdots & \vdots & S & & \vdots & \vdots & A & \\ -z_{l(A)} & c_1z_{l(A)} & & & -w_{l(A)} & c_2w_{l(A)} & & \end{array}\right]$$

generates the $R_1R_2$-linear self-dual code $D$ of length $n + 4$.

***Proof.*** To simplify, we display the first $k(B)$ rows of $G$ by $g_i$, $i \in \{1,\ldots, k(B)\}$, and the last $l(A)$ rows of $G$ by $h_j$, $j \in \{1,\ldots, l(A)\}$. We have $<g_1,g_1> = 1+b.b = 0$ and $<h_1,h_1> = 1 + a.a = 0$. Besides, since $C$ is self-dual, for all $i \ne 1$ and $j \ne 1$, we obtain

$$\begin{aligned}\langle g_i,g_i\rangle &= \gamma_2^{e_2-e_1}\iota\left(u_i^2+c_1^2u_i^2+B_i.B_i\right)\\ &\quad+\left(v_i^2+c_2^2v_i^2+T_i.T_i\right)\\ &= \gamma_2^{e_2-e_1}\iota\left(B_i.B_i\right)+T_i.T_i=0,\\ \langle h_j,h_j\rangle &= \gamma_2^{e_2-e_1}\iota\left(z_j^2+c_1^2z_j^2+S_j.S_j\right)\\ &\quad+\left(w_j^2+c_2^2w_j^2+A_j.A_j\right)\\ &= \gamma_2^{e_2-e_1}\iota\left(S_j.S_j\right)+A_j.A_j=0.\end{aligned}$$

Thus, the rows of matrix $G$ are orthogonal to themselves. Moreover, we have

$$\begin{aligned}&3\langle g_1,h_1\rangle=0,\\ &\langle g_1,h_j\rangle=\gamma_2^{e_2-e_1}\iota\left(-z_j+b.S_j\right)+0=0 \text{ for all } j\ne 1,\\ &\langle g_i,h_1\rangle=\gamma_2^{e_2-e_1}\iota(0)+\left(-v_i+a.T_i\right)=0 \text{ for all } i\ne 1.\end{aligned}$$

Furthermore, since $C$ is self-dual, for $i \ne 1$ and $j \ne 1$ we get

$$\begin{aligned}\langle g_i,h_j\rangle &= \gamma_2^{e_2-e_1}\iota\left(u_iz_j+c_1^2u_iz_j+B_i.S_j\right)\\ &\quad+\left(v_iw_j+c_2^2v_iw_j+T_i.A_j\right)\\ &= \gamma_2^{e_2-e_1}\iota\left(B_i.S_j\right)+\left(T_i.A_j\right)=0.\end{aligned}$$

Besides, for $i_1, i_2 \in \{1,\ldots,k(B)\}$ and $j_1, j_2 \in \{1,\ldots,l(A)\}$ such that $i_1 \neq i_2$ and $j_1 \neq j_2$ we have

$$\begin{aligned}
\left\langle g_{i_1}, g_{i_2} \right\rangle &= \gamma_2^{e_2-e_1}\iota\left(u_{i_1}u_{i_2} + c_1^2 u_{i_1}u_{i_2} + B_{i_1}.B_{i_2}\right) \\
&\quad + (v_{i_1}v_{i_2} + c_2^2 v_{i_1}v_{i_2} + T_{i_1}.T_{i_2}) \\
&= \gamma_2^{e_2-e_1}\iota\left(B_{i_1}.B_{i_2}\right) + T_{i_1}.T_{i_2} = 0, \\
\left\langle h_{j_1}, h_{j_2} \right\rangle &= \gamma_2^{e_2-e_1}\iota\left(z_{j_1}z_{j_2} + c_1^2 z_{j_1}z_{j_2} + S_{j_1}.S_{j_2}\right) + \\
&\quad \left(w_{j_1}w_{j_2} + c_2^2 w_{j_1}w_{j_2} + A_{j_1}.A_{j_2}\right) \\
&= \gamma_2^{e_2-e_1}\iota\left(S_{j_1}.S_{j_2}\right) + A_{j_1}.A_{j_2} = 0.
\end{aligned}$$

Thus, any two distinct rows of $G$ are orthogonal and therefore $D$ is self-orthogonal. To complete the proof, we need the size of $D$. By the elementary row and column operations, we conclude that $G$ is of type

$$(\alpha+2, \beta+2; k_0+1, k_1,\ldots k_{e_1-1}; l_0+1, l_1,\ldots, l_{e_2-1}).$$

So

$$\begin{aligned}
|D| &= q^{e_1(k_0+1)+\sum_{i=1}^{e_1-1}(e_1-i)k_i + e_2(l_0+1)+\sum_{j=1}^{e_2-1}(e_2-j)l_i} \\
&= q^{e_1+e_2}|C| = q^{\frac{e_1(\alpha+2)+e_2(\beta+2)}{2}}.
\end{aligned}$$

Consequently, $D$ is self-dual.

**Example 4.8** *Let* $R_1 = \mathbb{F}_5 + u\mathbb{F}_5$, *where* $u^2 = 0$, *and* $R_2 = \mathbb{F}_5 + u\mathbb{F}_5 + u^2\mathbb{F}_5$, *where* $u^3 = 0$. *Take* $c_1 = c_2 = 3$, $b = (3, 2+u, 4+2u, 3u) \in R_1^4$ *and* $a = (1+4u+3u^2, 0, 2+3u+u^2, 3) \in R_2^4$. *Then* $R_1R_2$*-linear code* $C$ *with the generator matrix* $\mathcal{G} = [\mathcal{G}_1 | \mathcal{G}_2]$, *where*

$$\mathcal{G}_2 = \begin{bmatrix} 1 & 0 & 3 & 2+u & 3u & 0 \\ 1+2u & 2+4u & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1+3u & 2+3u & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{G}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2+3u=u^2 & 4+u=2u^2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1+4u=3u^2 & 0 & 2+3u+u^2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1+u+2u^2 & 2+2u+4u^2 & 0 & 0 & 3 & 1 \end{bmatrix},$$

*is a non-separable self-dual code.*

## BOUNDS ON MINIMUM DISTANCE

In this section, a Gray map is introduced, which is a generalization of the given Gray map in [29].

**Definition 5.1** *Define the Gray map* $\Phi$ *from* $R_1^\alpha \times R_2^\beta$ *to* $F_q^{q^{e_1-1}\alpha+q^{e_2-1}\beta}$ *as* $\Phi(u|u') = (\phi_1(u)|\phi_2(u'))$, *where* $\phi_1$ *and* $\phi_2$ *are the Gray maps over* $R_1$ *and* $R_2$, *respectively.*

The following theorem can be easily verified according to Definition 5.1 and Proposition 2.3.

**Theorem 5.2** *The Gray map* $\Phi$ *is an isometry from* $(R_1^\alpha \times R_2^\beta, d^*)$ *to* $F_q^{q^{e_1-1}\alpha+q^{e_2-1}\beta}$, $d_H$, *where* $d_H$ *denotes the Hamming distance on* $F_q^{q^{e_1-1}\alpha+q^{e_2-1}\beta}$.

The following examples provide optimal codes which are obtained directly in spite of the indirect construction presented in [19].

**Example 5.3** *Let* $R_1 = R_{\gamma_1,2,2}$, $R_2 = R_{\gamma_2,3,2}$ *and* $C$ *be a linear code over* $R_1 \times R_2$ *generated by*

$$G = \left[\begin{array}{cccc|c} 1 & 0 & 1+\gamma_1 & \gamma_1 & 0 \\ 0 & 1 & \gamma_1 & 1+\gamma_1 & 0 \\ 0 & \gamma_1 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2^2 \end{array}\right]$$

Then $\Phi(C)$ is a binary linear code with parameters [12,5,4].

**Example 5.4** *Let* $R_1 = R_{\gamma_1,2,2}$, $R_2 = R_{\gamma_2,3,2}$ *and* $C$ *be a linear code over generated by*

$$\begin{aligned}
G = \Bigg[\begin{array}{c|cccccccc} \gamma_1 & 0 & \gamma_2^2 & \gamma_2^2 & 0 & \gamma_2^2 & 0 & \gamma_2^2 & 0 \\ 0 & 1 & 0 & 1 & \gamma_2 & \gamma_2 & 1+\gamma_2 & 1+\gamma_2 & \gamma_2^2 \end{array} \\
\begin{array}{ccccc} \gamma_2^2 & 0 & \gamma_2^2 & 0 & \gamma_2^2 \\ \gamma_2^2 & 1+\gamma_2^2 & 1+\gamma_2^2 & \gamma_2+\gamma_2^2 & \gamma_2+\gamma_2^2 \end{array} \\
\begin{array}{cc} 0 & \gamma_2^2 \\ 1+\gamma_2+\gamma_2^2 & 1+\gamma_2+\gamma_2^2 \end{array}\Bigg].
\end{aligned}$$

Then $\Phi(C)$ is a binary linear code with parameters [62,4,32].

**Example 5.5** *Let* $R_1 = R_{\gamma_1,2,2}$, $R_2 = R_{\gamma_2,3,2}$ *and* $C$ *be a linear code over* $R_1 \times R_2$ *generated by*

$$\begin{aligned}
G = \Bigg[\begin{array}{c|cccccccc} 1 & 1 & 0 & 2 & \gamma_2 & 1+\gamma_2 & 1 & 2 & 1+\gamma_2 \\ 0 & 0 & \gamma_2 & \gamma_2 & 0 & 0 & \gamma_2 & 2\gamma_2 & \gamma_2 \end{array} \\
\begin{array}{ccccc} 2 & \gamma_2 & 2+\gamma_2 & 1+2\gamma_2 & 2\gamma_2 \\ \gamma_2 & \gamma_2 & \gamma_2 & 2\gamma_2 & \gamma_2 \end{array}\Bigg].
\end{aligned}$$

Then $\Phi(C)$ is a ternary linear code with parameters [40,3,27].

The next theorem presents two bounds on the minimum distance of $R_1R_2$-linear codes.

**Theorem 5.6** *Let* $C$ *be an* $R_1R_2$*-linear code of type* $(\alpha,\beta; k_0,\ldots, k_{e_1-1}; l_0,\ldots, l_{e_2-1}$. *Then*

$$3\frac{d^*(C)-1}{q^{e_2-1}} \leq \frac{\alpha}{q^{e_2-e_1}} + \beta - \frac{\sum_{i=0}^{e_1-1}(e_1-i)k_i + \sum_{j=0}^{e_1-1}(e_2-j)l_j}{q^{e_2-1}}, \tag{5.1}$$

$$\left\lfloor \frac{d^*(C)-1}{q^{e_2-1}} \right\rfloor \leq \alpha+\beta-(\textstyle\sum_{i=0}^{e_1-1}k_i + \sum_{j=0}^{e_2-1}l_j). \tag{5.2}$$

***Proof.*** Note that $\Phi(C)$ is a code over $F_q^{q^{e_1-1}\alpha+q^{e_1-1}\beta}$ with size $|C|$. Applying Inequality (2.1) on $\Phi(C)$, we obtain

$$d_H(\Phi(C)) \leq q^{e_1-1}\alpha + q^{e_1-1}\beta - (\textstyle\sum_{i=0}^{e_1-1}(e_1-i)k_i + \sum_{j=0}^{e_1-1}(e_2-j)l_j) + 1,$$

which implies Inequality (5.1).

Next, let $\chi: R_1 \to R_2$ be a map such that $\chi(x) = \gamma_2^{e_2-e_1}\iota(x)$. Extend $\chi$ to the map $(\chi, Id)$ from $R_1^\alpha \times R_2^\beta$ to $R_2^\alpha \times R_2^\beta$, where $Id$ is the identity map over $R_2^\beta$. Obviously $d^*(C) \leq d_{hom_{R_2}}((\chi, Id)(C))$. In addition, it is clear that rank$((\chi, Id)(C))$ = rank$(C)$. Applying Inequality (2.2) on $(\chi, Id)(C)$, we obtain

$$\left\lfloor \frac{d^*(C)-1}{q^{e_2-1}} \right\rfloor \leq \alpha+\beta-\text{rank}(C) = \alpha+\beta-(\textstyle\sum_{i=0}^{e_1-1}k_i + \sum_{j=0}^{e_2-1}l_j).$$

We say that an $R_1R_2$-linear code is a maximum distance separable (MDS) code if $d^*(C)$ meets the bound given in Inequality (5.1) or (5.2). In the first case, we say that $C$ is MDS with respect to the Singleton bound (MDSS). in the second case, $C$ is MDS with respect to the rank bound (MDSR); see [5].

**Lemma 5.7** *Let $C$ be an $R_1R_2$-linear code of type $(\alpha,\beta; k_0,\ldots, k_{e_1-1}; l_0,\ldots, l_{e_2-1})$.*

1. If $\alpha + \beta$ = rank$(C)$, then $C$ is MDSR and $1 \leq d^*(C) \leq q^{e_2-1}$.
2. If $k_0 + l_0 = 0$, then $C^\perp$ is MDSR and $1 \leq d^*(C^\perp) \leq q^{e_2-1}$.

***Proof.***

1. We know that rank$(C) = \sum_{i=0}^{e_1-1}k_i + \sum_{j=0}^{e_2-1}l_j$. Now use the second inequality in Theorem 5.6.
2. Using Theorem 3.1, $C^\perp$ is of type $(\alpha,\beta; k_{e_1}, k_{e_1-1}, k_1; l_{e_2}, l_{e_2-1}, \ldots, l_1)$, where $k_{e_1} = \alpha - k(B)$ and $l_{e_2} = \beta - l(A)$. So rank $C^\perp = \alpha - k_0 + \beta - l_0$. Now the proof is similar to the first part.

**Example 5.8** *Every $R_1R_2$-linear code of type $(\alpha,\beta; \alpha, 0,\ldots, 0; \beta, 0,\ldots,0)$ is MDSR. Furthermore, the dual code of every $R_1R_2$-linear code of type $(\alpha,\beta; 0, k_1,\ldots, k_{e_1-1}; 0, l_1,\ldots, l_{e_2-1}$ is MDSR. Moreover, if $C$ is an $R_1R_2$-linear code of type $(\alpha,\beta; 0, \alpha, 0,\ldots, 0; 0, \beta, 0,\ldots,0)$, then $C$ and $C^\perp$ are MDSR.*

**Example 5.9** *The code $C$ given in Example 4.4 is an MDSR self-dual code.*

**Example 5.10** *Let $C$ be an $R_1R_2$-linear code of type $(\alpha,\beta; 0,\ldots, 0, 1; 0,\ldots,0)$ generated by*

$$G = <\left(\gamma_1^{e_1-1},\cdots,\gamma_1^{e_1-1} \middle| \gamma_2^{e_2-1},\cdots,\gamma_2^{e_2-1}\right)>.$$

Clearly, $d^*(C) = \alpha q^{e_1-1} + \beta q^{e_2-1}$ and so $C$ is an MDSS code. Choose $\alpha$ such that $\alpha \leq q^{e_2-e_1}$. We have $\left\lfloor \frac{d^*(C)-1}{q^{e_2-1}} \right\rfloor = \beta$. Now if $\alpha = 1$, then $C$ is MDSR and if $\alpha > 1$, then $C$ is not MDSR.

**Example 5.11** *Assume $R_1=R_{\gamma_{1,2,5}}$ and $R_2=R_{\gamma_{2,3,5}}$. The $R_1R_2$-linear code $C$ with the generator matrix*

$$G = \left[\begin{array}{c|cccc} \gamma_1 & 0 & \gamma_2^2 & 3\gamma_2^2 \\ 0 & \gamma_2^2 & 3\gamma_2^2 & 2\gamma_2^2 \end{array}\right]$$

is of type (1,3; 0,1; 0,0,1) with the minimum distance $d^*(C)$ = 54. Applying the bound (5.1) and (5.2), we obtain that $C$ is MDSR and is not MDSS.

**Example 5.12** *Let $R_1=Z_{49}$ and $R_2=R_{\gamma_2,3,7}$. The $R_1R_2$-linear code $C$ with the generator matrix*

$$\left[\begin{array}{c|cccccc} 7 & 0 & 0 & \gamma_2^2 & 4\gamma_2^2 & \gamma_2^2 \\ 0 & \gamma_2^2 & \gamma_2^2 & 0 & 4\gamma_2^2 & 0 \\ 0 & 0 & \gamma_2^2 & \gamma_2^2 & 0 & 4\gamma_2^2 \\ 0 & 0 & 0 & \gamma_2^2 & 5\gamma_2^2 & 6\gamma_2^2 \end{array}\right]$$

is of type (1,5; 0,1; 0,0,1) with the minimum distance $d^*(C)$ = 105. Clearly, $C$ is MDSR and is not MDSS.

## CONCLUSION

In this paper, we study $R_1R_2$-linear codes of length $n = \alpha + \beta$. We first determine the parity-check matrix of $R_1R_2$-linear codes as well as the relation between $R_1R_2$-linear codes $C$ and $C^\perp$. Also, we provide some examples to show that our results on duality and parity-check matrix recover that of on several mixed alphabet codes. As an application of the results on dual codes, we construct some separable and non-separable self-dual $R_1R_2$-linear codes. After that, we define a weight function on $R_1^\alpha \times R_2^\beta$ which is the natural generalization of the homogeneous weight over chain rings. Then, we define a distance preserving Gray map from $R_1R_2$-linear codes to codes over $\mathbb{F}_q$ equipped with the Hamming weight. The Gray image of $R_1R_2$-linear codes presented in Examples 5.3, 5.4 and 5.5 provide optimal codes which have more simple construction than linear codes with the same parameters in Grassl table. Moreover, two upper bounds for the minimum distance are obtained by the Singleton bound for the Gray image and the rank bound for codes over rings. Finally, we discuss the conditions on the $R_1R_2$-linear codes to be MDSS or MDSR.

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## AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

## DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

## CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

## ETHICS

There are no ethical issues with the publication of this manuscript.

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