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Weighted Steffensen Type Inequalities Involving Convex Functions

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Abstract

The object is to obtain weighted Steffensen type inequalities for the class of convex functions using inequalities for the class of functions that are "convex at point c". Additionally, we give weaker conditions for obtained weighted Steffensen type inequalities. Moreover, by further generalizations of these inequalities we obtain refined and sharpened versions.

Keywords: *Steffensen's inequality, measure theory, generalizations, convex function*
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1. Introduction

The well-known Steffensen inequality

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt,$$

where f and g are integrable functions defined on (a,b), f is nonincreasing, $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$, has been the subject of investigation by many mathematicians. Since its appearance in [10] Steffensen's inequality has been generalized, refined and sharpened in many applications. A comprehensive survey can be found in [9].
In the sequel by $\mathscr{B}([a,b])$ we denote Borel σ-algebra on $[a,b]$. In [5] Pečarić obtained generalization of Steffensen's inequality which was extensively used in recent advances of Steffensen's inequality. The following measure theoretic version of his generalization was proved in [2].

Theorem 1.1. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g and h be measurable functions on $[a,b]$ such that h is positive, f/h is nonincreasing and $0 \leq g \leq 1$.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{1.1}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)d\mu(t). \tag{1.2}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{1.3}$$

then

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{1.4}$$

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If f/h is a nondecreasing function, then the reverse inequalities in (1.2) and (1.4) hold.

Further, the following weaker conditions for the aforementioned generalization were obtained in [2].

Theorem 1.2. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let g and h be $\mu-$integrable functions on $[a,b]$ such that h is positive.*

(a) Let λ be a positive constant such that $\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} g(t)h(t)d\mu(t)$. The inequality

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)d\mu(t)$$

holds for every nonincreasing, right-continuous function $f/h : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[a,x)} h(t)g(t)d\mu(t) \leq \int_{[a,x)} h(t)d\mu(t) \quad and \quad \int_{[x,b]} h(t)g(t)d\mu(t) \geq 0,$$

for every $x \in [a,b]$.

(b) Let λ be a positive constant such that $\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} g(t)h(t)d\mu(t)$. The inequality

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t)$$

holds for every nonincreasing, right-continuous function $f/h : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[x,b]} h(t)g(t)d\mu(t) \leq \int_{[x,b]} h(t)d\mu(t) \quad and \quad \int_{[a,x)} h(t)g(t)d\mu(t) \geq 0,$$

for every $x \in [a,b]$.

Let us define a class of functions that extends the class of convex functions. This class was introduced in [8].

Definition 1.1. Let $h : [a,b] \to \mathbb{R}$ be a positive function, $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. We say that f/h belongs to the class $\mathscr{M}_1^c[a,b]$ (resp. $\mathscr{M}_2^c[a,b]$) if there exists a constant A such that the function $\frac{F(x)}{h(x)} = \frac{f(x)}{h(x)} - Ax$ is nonincreasing (resp. nondecreasing) on $[a,c]$ and nondecreasing (resp. nonincreasing) on $[c,b]$.

Remark 1.1. A function $f/h \in \mathscr{M}_1^c[a,b]$ is said to be convex at point c.
In [8] the authors proved that a function f/h is convex (resp. concave) on $[a,b]$ if and only if it is convex (resp. concave) at every point $c \in (a,b)$. It was also shown in [8] that if $(f/h)'(c)$ exists, then $A = (f/h)'(c)$.

In the main part of this paper we obtain weighted Steffensen type inequalities for the class of functions that are "convex at point c" and as a consequence we obtain inequalities involving the class of convex functions. Additionally, motivated by Theorem 1.2 we obtain weaker conditions for these inequalities. In Section 3 we further generalize inequalities from the main part to obtain their refined and sharpened versions. We conclude the paper with applications related to linear functionals defined as the difference between the left-hand and the right-hand side of obtained inequalities.
Inequalities obtained in this paper give further insights of the well-known and classical Steffensen inequality from which the well-known Jensen-Steffensen inequality was derived. Hence, new inequalities obtained here can be used in all applications of these classical inequalities.

2. Weighted Steffensen type inequalities

Applying measure theoretic generalizations of Steffensen's inequality to a class of functions that are convex at point c we obtain the following results.

Theorem 2.1. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that*

$$\int_{[a,a+\lambda_1]} h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t) \tag{2.1}$$

and

$$\int_{(b-\lambda_2,b]} h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t). \tag{2.2}$$

If $f/h \in \mathscr{M}_1^c[a,b]$ and

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t), \tag{2.3}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda_1]} f(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)d\mu(t). \tag{2.4}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (2.3) holds, the inequality in (2.4) is reversed.

Proof. Let us prove this for $f/h \in \mathscr{M}_1^c[a,b]$. Let $F(x) = f(x) - Axh(x)$, where A is the constant from Definition 1.1. Since $F/h : [a,c] \to \mathbb{R}$ is nonincreasing, inequality (1.2) implies

$$\begin{aligned} 0 &\le \int_{[a,a+\lambda_1]} F(t)d\mu(t) - \int_{[a,c]} F(t)g(t)d\mu(t) \\ &= \int_{[a,a+\lambda_1]} f(t)d\mu(t) - \int_{[a,c]} f(t)g(t)d\mu(t) - A\left(\int_{[a,a+\lambda_1]} th(t)d\mu(t) - \int_{[a,c]} th(t)g(t)d\mu(t)\right). \end{aligned} \tag{2.5}$$

Similarly, $F/h : [c,b] \to \mathbb{R}$ is nondecreasing, so inequality (1.4) implies

$$\begin{aligned} 0 &\le \int_{(b-\lambda_2,b]} F(t)d\mu(t) - \int_{[c,b]} F(t)g(t)d\mu(t) \\ &= \int_{(b-\lambda_2,b]} f(t)d\mu(t) - \int_{[c,b]} f(t)g(t)d\mu(t) - A\left(\int_{(b-\lambda_2,b]} th(t)d\mu(t) - \int_{[c,b]} th(t)g(t)d\mu(t)\right). \end{aligned} \tag{2.6}$$

Adding up (2.5) and (2.6) we obtain

$$\begin{aligned} &\int_{[a,a+\lambda_1]} f(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\ &\ge A\left(\int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t) - \int_{[a,b]} th(t)g(t)d\mu(t)\right) = 0 \end{aligned}$$

which completes the proof.
Proof for $f/h \in \mathscr{M}_2^c[a,b]$ is similar so we omit the details. □

Remark 2.1. It is obvious from the proof that for $f/h \in \mathscr{M}_1^c[a,b]$ inequality (2.4) holds if equality (2.3) is replaced by the weaker condition

$$A\left(\int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t) - \int_{[a,b]} th(t)g(t)d\mu(t)\right) \ge 0, \tag{2.7}$$

where A is the constant from Definition 1.1.
Since $(f/h)'_-(c) \le A \le (f/h)'_+(c)$ (see [7]), if we additionally assume that $f/h \in \mathscr{M}_1^c[a,b]$ is nondecreasing, condition (2.7) can be further weakened to

$$\int_{[a,b]} th(t)g(t)d\mu(t) \le \int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t). \tag{2.8}$$

Further, if $f/h \in \mathscr{M}_1^c[a,b]$ is nonincreasing, condition (2.7) can be further weakened to (2.8) with the reverse inequality.

Theorem 2.2. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f, g and h be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that*

$$\int_{(c-\lambda_1,c]} h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t) \tag{2.9}$$

and

$$\int_{[c,c+\lambda_2]} h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t). \tag{2.10}$$

If $f/h \in \mathscr{M}_1^c[a,b]$ and

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t), \tag{2.11}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \ge \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t). \tag{2.12}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (2.11) holds, the inequality in (2.12) is reversed.

Proof. Let $f/h \in \mathscr{M}_1^c[a,b]$ and let $F(x) = f(x) - Axh(x)$, where A is the constant from Definition 1.1. Since $F/h : [a,c] \to \mathbb{R}$ is nonincreasing, inequality (1.4) implies

$$\begin{aligned} 0 &\le \int_{[a,c]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c]} f(t)d\mu(t) \\ &\quad - A\left(\int_{[a,c]} th(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c]} th(t)d\mu(t)\right). \end{aligned} \tag{2.13}$$

Similarly, $F/h : [c,b] \to \mathbb{R}$ is nondecreasing, so inequality (1.2) implies

$$\begin{aligned} 0 &\le \int_{[c,b]} f(t)g(t)d\mu(t) - \int_{[c,c+\lambda_2]} f(t)d\mu(t) \\ &\quad - A\left(\int_{[c,b]} th(t)g(t)d\mu(t) - \int_{[c,c+\lambda_2]} th(t)d\mu(t)\right). \end{aligned} \tag{2.14}$$

Adding up (2.13) and (2.14) we obtain

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t)$$
$$\geq A\left(\int_{[a,b]} th(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t)\right) = 0$$

which completes the proof for $f/h \in \mathscr{M}_1^c[a,b]$. Similarly for $f/h \in \mathscr{M}_2^c[a,b]$. □

Remark 2.2. Similarly as in Remark 2.1, it is obvious from the proof that for $f/h \in \mathscr{M}_1^c[a,b]$ inequality (2.12) holds if equality (2.11) is replaced by the weaker condition

$$A\left(\int_{[a,b]} th(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t)\right) \geq 0, \tag{2.15}$$

where A is the constant from Definition 1.1.
If we additionally assume that $f/h \in \mathscr{M}_1^c[a,b]$ is nondecreasing, condition (2.15) can be further weakened to

$$\int_{[a,b]} th(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t). \tag{2.16}$$

Further, if $f/h \in \mathscr{M}_1^c[a,b]$ is nonincreasing, condition (2.15) can be further weakened to (2.16) with the reverse inequality.
As a consequence of Theorems 2.1 and 2.2 we obtain the following weighted Steffensen type inequalities that involve convex functions.

Corollary 2.1. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1,\lambda_2 \in \mathbb{R}_+$ be such that (2.1) and (2.2) hold. If $f/h : [a,b] \to \mathbb{R}$ is convex and (2.3) holds, then the inequality (2.4) holds.*
If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (2.4) is reversed.

Proof. Since f/h is convex, from Remark 1.1, we have that $f/h \in \mathscr{M}_1^c[a,b]$ for every $c \in (a,b)$. Hence, we can apply Theorem 2.1. □

Corollary 2.2. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1,\lambda_2 \in \mathbb{R}_+$ be such that (2.9) and (2.10) hold. If $f/h : [a,b] \to \mathbb{R}$ is convex and (2.11) holds, then the inequality (2.12) holds.*
If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (2.12) is reversed.

Proof. Similar to the proof of Corollary 2.1 applying Theorem 2.2. □

Motivated by Theorem 1.2 we obtain the following weaker conditions for weighted Steffensen type inequalities for convex functions at a point.

Theorem 2.3. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let g and h be $\mu-$integrable functions on $[a,b]$ such that h is positive and*

$$\int_{[a,x)} h(t)g(t)d\mu(t) \leq \int_{[a,x)} h(t)d\mu(t) \quad \text{and} \quad \int_{[x,c]} h(t)g(t)d\mu(t) \geq 0,$$

for every $x \in [a,c]$ and

$$\int_{[x,b]} h(t)g(t)d\mu(t) \leq \int_{[x,b]} h(t)d\mu(t) \quad \text{and} \quad \int_{[c,x)} h(t)g(t)d\mu(t) \geq 0,$$

for every $x \in [c,b]$.
Let f/h be a right-continuous function on $[a,b]$ and let $\lambda_1,\lambda_2 \in \mathbb{R}_+$ be such that (2.1) and (2.2) hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and (2.3) holds, the inequality (2.4) holds. If $f/h \in \mathscr{M}_2^c[a,b]$ and (2.3) holds, the inequality in (2.4) is reversed.

Proof. Similar to the proof of Theorem 2.1 using weaker conditions from Theorem 1.2. □

Theorem 2.4. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let g and h be $\mu-$integrable functions on $[a,b]$ such that h is positive and*

$$\int_{[x,c]} h(t)g(t)d\mu(t) \leq \int_{[x,c]} h(t)d\mu(t) \quad \text{and} \quad \int_{[a,x)} h(t)g(t)d\mu(t) \geq 0,$$

for every $x \in [a,c]$ and

$$\int_{[c,x)} h(t)g(t)d\mu(t) \leq \int_{[c,x)} h(t)d\mu(t) \quad \text{and} \quad \int_{[x,b]} h(t)g(t)d\mu(t) \geq 0,$$

for every $x \in [c,b]$.
Let f/h be a right-continuous function on $[a,b]$ and let $\lambda_1,\lambda_2 \in \mathbb{R}_+$ be such that (2.9) and (2.10) hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and (2.11) holds, the inequality (2.12) holds. If $f/h \in \mathscr{M}_2^c[a,b]$ and (2.11) holds, the inequality in (2.12) is reversed.

Proof. Similar to the proof of Theorem 2.2 using weaker conditions from Theorem 1.2. □

Remark 2.3. Condition (2.3) (resp. (2.11)) in Corollary 2.1 and Theorem 2.3 (resp. Corollary 2.2 and Theorem 2.4) can be replaced by weaker conditions given in Remark 2.1 (resp. Remark 2.2).
Similar as in Corollaries 2.1 and 2.2 we have that Theorems 2.3 and 2.4 still hold if f/h is a convex function.

3. Further generalizations of weighted Steffensen type inequalities

In [4] Mercer proved a generalization of Steffensen's inequality for which Liu, Wu and Srivastava proved to be incorrect as stated (see [3] and [11]). As noted in [6] corrected version of Mercer's result follows from generalization obtained in [5]. Making substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorems 2.1 and 2.2 we obtain the following weighted Steffensen type inequalities for convex functions at a point related to corrected version of Mercer's result.

Theorem 3.1. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le h$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{[a,a+\lambda_1]} h(t)d\mu(t) = \int_{[a,c]} g(t)d\mu(t)$ and $\int_{(b-\lambda_2,b]} h(t)d\mu(t) = \int_{[c,b]} g(t)d\mu(t)$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} tg(t)d\mu(t) = \int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t), \tag{3.1}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \int_{[a,a+\lambda_1]} f(t)h(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)h(t)d\mu(t). \tag{3.2}$$

If $f \in \mathscr{M}_2^c[a,b]$ and (3.1) holds, the inequality in (3.2) is reversed.

Theorem 3.2. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le h$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{(c-\lambda_1,c]} h(t)d\mu(t) = \int_{[a,c]} g(t)d\mu(t)$ and $\int_{[c,c+\lambda_2]} h(t)d\mu(t) = \int_{[c,b]} g(t)d\mu(t)$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} tg(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t), \tag{3.3}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \ge \int_{(c-\lambda_1,c+\lambda_2]} f(t)h(t)d\mu(t). \tag{3.4}$$

If $f \in \mathscr{M}_2^c[a,b]$ and (3.3) holds, the inequality in (3.4) is reversed.

In [4] Mercer also proved a generalization of Steffensen's inequality which is, as proved in [6], equivalent to generalization obtained by Pečarić in [5]. Weighted version of this result was proved in [2]. Using substitutions $h \mapsto kh$, $g \mapsto g/k$ and $f \mapsto fk$ in Theorems 2.1 and 2.2 we obtain related Steffensen type inequalities given in the following theorems.

Theorem 3.3. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g,h and k be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le k$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{[a,a+\lambda_1]} k(t)h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t)$ and $\int_{(b-\lambda_2,b]} k(t)h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t)$. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{[a,a+\lambda_1]} tk(t)h(t)d\mu(t) + \int_{(b-\lambda_2,b]} tk(t)h(t)d\mu(t), \tag{3.5}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \int_{[a,a+\lambda_1]} f(t)k(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)k(t)d\mu(t). \tag{3.6}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (3.5) holds, the inequality in (3.6) is reversed.

Theorem 3.4. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g,h and k be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le k$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{(c-\lambda_1,c]} k(t)h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t)$ and $\int_{[c,c+\lambda_2]} k(t)h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t)$. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} tk(t)h(t)d\mu(t), \tag{3.7}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \ge \int_{(c-\lambda_1,c+\lambda_2]} f(t)k(t)d\mu(t). \tag{3.8}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (3.7) holds, the inequality in (3.8) is reversed.

In [11] Wu and Srivastava gave not only corrected but also a refined version of Mercer's result. Their refinement with weaker conditions on the parameter λ was obtained in [6]. Further, the following refinement in measure theory settings was proved in [2].

Theorem 3.5. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g and h be measurable functions on $[a,b]$ such that h is positive f/h is nonincreasing and $0 \le g \le 1$.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that (1.1) holds, then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\le \int_{[a,a+\lambda]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda)}{h(a+\lambda)}\right] h(t)[1-g(t)] \right) d\mu(t) \\ &\le \int_{[a,a+\lambda]} f(t)d\mu(t).\end{aligned} \tag{3.9}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ such that (1.3) holds, then

$$\begin{aligned}\int_{(b-\lambda,b]} f(t)d\mu(t) &\leq \int_{(b-\lambda,b]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda)}{h(b-\lambda)}\right] h(t)[1-g(t)]\right) d\mu(t) \\ &\leq \int_{[a,b]} f(t)g(t)d\mu(t).\end{aligned} \tag{3.10}$$

Using this idea we obtain the following refined versions of results given in Theorems 2.1 and 2.2.

Theorem 3.6. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f, g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that (2.1) and (2.2) hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) &= \int_{[a,a+\lambda_1]} (th(t) - [t-a-\lambda_1]h(t)[1-g(t)])\, d\mu(t) \\ &+ \int_{(b-\lambda_2,b]} (th(t) - [t-b+\lambda_2]h(t)[1-g(t)])\, d\mu(t),\end{aligned} \tag{3.11}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \leq &\int_{[a,a+\lambda_1]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right] h(t)[1-g(t)]\right) d\mu(t) \\ &+ \int_{(b-\lambda_2,b]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right] h(t)[1-g(t)]\right) d\mu(t).\end{aligned} \tag{3.12}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (3.11) holds, the inequality in (3.12) is reversed.

Proof. Similar to the proof of Theorem 2.1 applying Theorem 3.5(a) for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Theorem 3.5(b) for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Theorem 3.7. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f, g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that (2.9) and (2.10) hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) &= \int_{(c-\lambda_1,c]} (th(t) - [t-c+\lambda_1]h(t)[1-g(t)])\, d\mu(t) \\ &+ \int_{[c,c+\lambda_2]} (th(t) - [t-c-\lambda_2]h(t)[1-g(t)])\, d\mu(t),\end{aligned} \tag{3.13}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \geq &\int_{(c-\lambda_1,c]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right] h(t)[1-g(t)]\right) d\mu(t) \\ &+ \int_{[c,c+\lambda_2]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right] h(t)[1-g(t)]\right) d\mu(t).\end{aligned} \tag{3.14}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (3.13) holds, the inequality in (3.14) is reversed.

Proof. Similar to the proof of Theorem 2.2 applying Theorem 3.5(b) for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Theorem 3.5(a) for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Motivated by sharpened and generalized version of Theorem 1.1 obtained by Jakšetić, Pečarić and Smoljak in [2] we obtain the following results.

Theorem 3.8. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f, g, h and ψ be measurable functions on $[a,b]$ such that h is positive and $0 \leq \psi \leq g \leq 1-\psi$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that (2.1) and (2.2) hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) &= \int_{[a,a+\lambda_1]} th(t)d\mu(t) - \int_{[a,c]} |t-a-\lambda_1| h(t)\psi(t)d\mu(t) \\ &+ \int_{(b-\lambda_2,b]} th(t)d\mu(t) + \int_{[c,b]} |t-b+\lambda_2| h(t)\psi(t)d\mu(t),\end{aligned} \tag{3.15}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \leq &\int_{[a,a+\lambda_1]} f(t)d\mu(t) - \int_{[a,c]} \left|\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right| h(t)\psi(t)d\mu(t) \\ &+ \int_{(b-\lambda_2,b]} f(t)d\mu(t) + \int_{[c,b]} \left|\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right| h(t)\psi(t)d\mu(t).\end{aligned} \tag{3.16}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (3.15) holds, the inequality in (3.16) is reversed.

Proof. We use the following inequalities, which hold for f/h nonincreasing, proved in [2]:

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} \left|\frac{f(t)}{h(t)} - \frac{f(a+\lambda)}{h(a+\lambda)}\right| h(t)\psi(t)d\mu(t) \tag{3.17}$$

and

$$\int_{(b-\lambda,b]} f(t)d\mu(t) + \int_{[a,b]} \left|\frac{f(t)}{h(t)} - \frac{f(b-\lambda)}{h(b-\lambda)}\right| h(t)\psi(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{3.18}$$

For f/h nondecreasing inequalities in (3.17) and (3.18) are reversed.
The proof is similar to that of Theorem 2.1 applying (3.17) for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and (3.18) for $F/h : [c,b] \to \mathbb{R}$ nondecreasing, where $F(x) = f(x) - Axh(x)$. □

Theorem 3.9. *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g,h and ψ be measurable functions on $[a,b]$ such that h is positive and $0 \leq \psi \leq g \leq 1-\psi$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that (2.9) and (2.10) hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) = &\int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t) - \int_{[a,c]} |t-c+\lambda_1| h(t)\psi(t)d\mu(t)\\ &+ \int_{[c,b]} |t-c-\lambda_2| h(t)\psi(t)d\mu(t),\end{aligned} \tag{3.19}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \geq &\int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t) + \int_{[a,c]} \left|\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right| h(t)\psi(t)d\mu(t)\\ &- \int_{[c,b]} \left|\frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right| h(t)\psi(t)d\mu(t).\end{aligned} \tag{3.20}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (3.19) holds, the inequality in (3.20) is reversed.

Proof. Similar to the proof of Theorem 3.8. □

Remark 3.1. Steffensen type inequalities obtained in this section also hold if the function f/h is convex (resp. concave). This follows from Remark 1.1.

4. Concluding remarks and some applications

Motivated by weighted Steffensen type inequalities (2.4), (2.12), (3.6), (3.8), (3.12) and (3.14), which are linear in f, we can define the following linear functionals:

$$L_1(f) = \int_{[a,a+\lambda_1]} f(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \tag{4.1}$$

$$L_2(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t) \tag{4.2}$$

$$L_3(f) = \int_{[a,a+\lambda_1]} f(t)k(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)k(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \tag{4.3}$$

$$L_4(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} f(t)k(t)d\mu(t) \tag{4.4}$$

$$\begin{aligned}L_5(f) = &\int_{[a,a+\lambda_1]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right] h(t)[1-g(t)]\right) d\mu(t)\\ &+ \int_{(b-\lambda_2,b]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right] h(t)[1-g(t)]\right) d\mu(t)\\ &- \int_{[a,b]} f(t)g(t)d\mu(t)\end{aligned} \tag{4.5}$$

$$\begin{aligned}L_6(f) = &\int_{[a,b]} f(t)g(t)d\mu(t)\\ &- \int_{(c-\lambda_1,c]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right] h(t)[1-g(t)]\right) d\mu(t)\\ &- \int_{[c,c+\lambda_2]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right] h(t)[1-g(t)]\right) d\mu(t).\end{aligned} \tag{4.6}$$

Remark 4.1. Under assumptions of Theorems 2.1, 2.2, 3.3, 3.4, 3.6, 3.7 we have that $L_i(f) \geq 0$, $i = 1, \ldots, 6$ for all functions $f/h \in \mathscr{M}_1^c[a,b]$. Further, from Remark 1.1 $L_i(f) \geq 0$, $i = 1, \ldots, 6$ for all convex functions f/h.

In [1] Bernstein defined exponentially convex functions on an open interval I in the following way.

Definition 4.1. A function $\psi : I \to \mathbb{R}$ is *exponentially convex* on I if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(x_i + x_j\right) \geq 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $x_i + x_j \in I$, $1 \leq i, j \leq n$.

Remark 4.2. Using similar construction as in [7] we could produce exponentially convex functions and obtain new Cauchy means related to functionals L_i, $i = 1, \ldots, 6$ but here we omit the details.
Hence, linear functionals L_i, $i = 1, \ldots, 6$ are important in obtaining results which further improve well-known inequalities, and also in obtaining conversions of these inequalities.

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