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Strong Convergence of an explicit iteration method in uniformly convex Banach spaces

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Abstract

We obtain the necessary and sufficient conditions for the convergence of an explicit iterative procedure to a common fixed point of a finite family of non-self asymptotically quasi-nonexpansive type mappings in real Banach spaces. We also prove the strong convergence of this iterative method to a common fixed point of a finite family of non-self asymptotically quasi-nonexpansive in the intermediate sense mappings in uniformly convex Banach spaces. Our results mainly generalize and extend those obtained by Wang [L. Wang, Explicit iteration method for common fixed points of a finite family of nonself asymptotically nonexpansive mappings, Computers & Mathematics with applications, 53, (2007), 1012 - 1019.]

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1. Introduction

Let K be a nonempty subset of a real normed linear space E. A self-mapping $T: K \to K$ is called nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for every $x, y \in K$ and asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \geq 1$, $\| T^n x - T^n y \| \leq k_n \| x - y \|$ for all $x, y \in K$. If $F(T) = \{x \in K : Tx = x\} \neq \phi$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\| T^n x - y \| \leq k_n \| x - y \|$ for all $x \in K$, $y \in F(T)$ and every $n \geq 1$ then T is called asymptotically quasi-nonexpansive. T is called uniformly L-Lipschitzian if there exists a real number $L > 0$ such that $\| T^n x - T^n y \| \leq L \| x - y \|$ for all $x, y \in K$ and all $n \geq 1$.
The class of asymptotically nonexpansive mappings was introdued by Goebel and Kirk [5] and the class forms an important generalization of that of nonexpansive mappings. It was proved in [5] that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on K, then T has a fixed point.
Iterative methods for approximating fixed points of nonexpansive mappings have been studied by many authors (see for example [1], [2], [3], [4], [6], [8], [12], [14] and the references therein).
In Most of these papers, the well-known Mann iteration process [7],

$$x_1 \in K, \qquad x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \ \ n \geq 1, \qquad (*)$$

has been studied and the operator T has been assumed to map K into itself. The convexity of K then ensures that the sequence $\{x_n\}$ generated by (*) is well defined.
In 2001, Xu and Ori [25] introduced the following implicit iteration process for a finite family of nonexpansive self-mappings $\{T_i, i \in I\}$, where $I = \{1, 2, ..., N\}$.
For any initial point $x_0 \in K$,

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n \qquad n \geq 1,$$

where $\{\alpha_n\}$ is a real sequence in (0,1) and $T_n = T_{n(modN)}$, the mod N function takes values in I. They proved weak convergence of the above process to a common fixed point of the finite family of nonexpansive self-mappings. Later on, the implicit iteration method has been used to study the common fixed point of a finite family of strictly pseudocontractive self-mappings, asymptotically nonexpansive self-mappings or asymptotically quasi-nonexpansive self-mappings by some authors (see for example [10], [16] and [26], respectively). In 1991, Schu [15] introduced a modified iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space. More precisely, he proved the following theorem.

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Theorem 1.1. *([15]) Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ for all $n \geq 1$, $\lim k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence in [0,1] satisfying the condition $0 < a \leq \alpha_n \leq b < 1$, $n \geq 1$, for some constants a and b. Then the sequence $\{x_n\}$ generated from $x_1 \in K$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \qquad n \geq 1,$$

converges strongly to some fixed point of T.

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach space (see for example [9], [13], [12], [19]). If, however, K is a proper subset of the real Banach space E and T maps K into E (as in the case in many applications), then the sequence given by (*) may not be well-defined. one method that has been used to overcome this in the case of a single operator T is to introduce a retraction $P : E \to K$ in the recursion formula (*) as follows :

$$x_1 \in K, \qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n PTx_n, \ \ n \geq 1.$$

Recent results on approximation of fixed points of nonexpansive and asymptotically nonexpansive self and nonself single mappings can be found in ([3], [4], [6], [8], [11], [14], [17], [18], [20], [22], [24] and the references therein).
The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume et al. [4] as an important generalization of asymptotically nonexpansive self-mappings.

Definition 1.2. *[4] Let K be a nonempty subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \geq 1$,*

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq k_n \| x-y \| \qquad \text{for all } x,y \in K.$$

T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for every $n \geq 1$,

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq L \| x-y \| \qquad \text{for all } x,y \in K.$$

It is easy to see that a nonself asymptotically nonexpansive is uniformly L-Lipschitzian.
By studying the following iteration process

$$x_1 \in K, \qquad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \ \ n \geq 1$$

Chidume, Ofoedu and Zegeye [4] got some strong convergence theorems for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces.
Recently, Wang [22] proved the following strong convergence theorems for common fixed points of two nonself asymptotically nonexpansive mappings as follows ;

Theorem 1.3. *([22]) Let K a nonempty closed convex subset of a uniformly convex Banach space E. Suppose that $T_1, T_2 : K \to E$ are two nonself asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n \qquad n \geq 1, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$. If one of T_1 and T_2 is completely continuous and $F(T_1) \cap F(T_2) \neq \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Theorem 1.4. *([22]) Let K, E, T_1, T_2 and $\{x_n\}$ be as in Theorem 1.2. If one of T_1 and T_2 is demicompact then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Definition 1.5. *[11] Let K be a nonempty subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive in the intermediate sense if T is uniformly continuous and*

$$\limsup_{n\to\infty} \sup_{x,y\in K} \{\| T(PT)^{n-1}x - T(PT)^{n-1}y \| - \| x-y \|\} \quad \leq \quad 0. \tag{1.1}$$

In 2007, Y. X. Tian, S. S. Chang and J. L. Huang [21] introduced the following concepts for nonself mappings:

Definition 1.6. *[21] Let E be a real Banach space, C a nonempty nonexpansive retract of E and P the nonexpansive retraction from E onto C. Let $T : C \to E$ be a non-self mapping.*

(1) *T is said to be a nonself asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for every $n \geq 1$,*

$$\| T(PT)^{n-1}x - p \| \quad \leq \quad k_n \| x-p \| \qquad \text{for all } x \in K, \ p \in F(T).$$

(2) *T is said to be a nonself asymptotically nonexpansive type mapping if*

$$\limsup_{n\to\infty}\{ \sup_{x,y\in K} [\| T(PT)^{n-1}x - T(PT)^{n-1}y \| - \| x-y \|]\} \quad \leq \quad 0.$$

(3) *T is said to be a nonself asymptotically quasi-nonexpansive type mapping if $F(T) \neq \emptyset$ and*

$$\limsup_{n\to\infty}\{ \sup_{x\in K, q\in F(T)} [\| T(PT)^{n-1}x - q \| - \| x-q \|]\} \quad \leq \quad 0.$$

Remark

(i) If $T : C \to E$ is a nonself asymptotically nonexpansive mapping, then T is is a nonself asymptotically nonexpansive type mapping.
(ii) If $T : C \to E$ is a nonself asymptotically quasi-nonexpansive mapping, then T is a nonself asymptotically quasi-nonexpansive type mapping.
(iii) If $F(T) \neq \emptyset$ and $T : C \to E$ is a nonself asymptotically nonexpansive type mapping, then T is a nonself asymptotically quasi-nonexpansive type mapping.

Very recently, Lin Wang [23] constructed an explicit iteration scheme to approximate a common fixed point of a finite family of nonself asymptotically nonexpansive mappings $\{T_i : K \to E, i \in I\}$, where I denotes the set $\{1,2,...,N\}$ and proved some strong convergence theorems for such mappings in uniformly convex Banach spaces as follows; From arbitrary $x_0 \in K$,

$$\begin{aligned}
x_1 &= P((1-\alpha_1)x_0 + \alpha_1 T_1(PT_1)^{m-1}x_0), \qquad m \geq 1,\\
x_2 &= P((1-\alpha_2)x_1 + \alpha_2 T_2(PT_2)^{m-1}x_1),\\
&\;.\\
&\;.\\
&\;.\\
x_N &= P((1-\alpha_N)x_{N-1} + \alpha_N T_N(PT_N)^{m-1}x_{N-1}),\\
x_{N+1} &= P((1-\alpha_{N+1})x_N + \alpha_{N+1} T_1(PT_1)^{m-1}x_N),\\
x_{N+2} &= P((1-\alpha_{N+2})x_{N+1} + \alpha_{N+2} T_2(PT_2)^{m-1}x_{N+1}),\\
&\;.\\
&\;.\\
&\;.\\
x_{2N} &= P((1-\alpha_{2N})x_{2N-1} + \alpha_{2N} T_N(PT_N)^{m-1}x_{2N-1}),\\
x_{2N+1} &= P((1-\alpha_{2N+1})x_{2N} + \alpha_{2N+1} T_1(PT_1)^{m-1}x_{2N}),\\
&\;.\\
&\;.\\
&\;.
\end{aligned}$$

which can be rewritten in a compact form as follows

$$x_n = P((1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m-1}x_{n-1}), \qquad n \geq 1, m \geq 1, \tag{1.2}$$

where $n = (m-1)N + i$, $T_n = T_{n(modN)} = T_i$, $i \in I$, $\{\alpha_n\}$ is a real sequence in [0,1].
Motivated and inspired by the previous facts, we extend the results obtained by Lin Wang [23] to the case of nonself asymptotically quasi-nonexpansive mappings and the case of nonself asymptotically quasi-nonexpansive mappings in the intermediate sense which is slightly more general than the class nonself asymptotically nonexpansive mappings in the intermediate sense introduced by S. Plubteing and R. Wangkeeree [11] as follows ;

Definition 1.7. *Let K be a nonempty subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ with a nonempty fixed point set is called asymptotically quasi-nonexpansive in the intermediate sense if T is uniformly continuous and*

$$\limsup_{n\to\infty} \sup_{x\in K, y\in F(T)} \{\| T(PT)^{n-1}x - y \| - \| x - y \|\} \leq 0. \tag{1.3}$$

Moreover, we discuss the necessary and sufficient condition for convergence of the explicit iterative scheme (1.1) to a common fixed point (assuming existence) of a finite family of nonself asymptotically quasi-nonexpansive type mappings in real Banach spaces.

2. Preliminaries

Let E be a real normed linear space. *The modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2} \| x+y \| : \| x \| = \| y \| = 1, \| x-y \| = \varepsilon\}.$$

E is *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$.
A subset K of E is said to be a *retract* of E if there exists a continuous map $P : E \to K$ such that $Px = x$, $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a *retraction* if $P^2 = P$. It follows that if P is a retraction then $Py = y$ for all $y \in R(P)$, the range of P.
A mapping $T : K \to K$ is said to be *semicompact* if, for any bounded sequence $\{x_n\}$ in K such that $\| x_n - Tx_n \| \to 0$ as $n \to 0$, there exists a subsequence $\{x_{n_j}\}$, say, of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K. T is said to be *completely continuous* if, for any bounded sequence $\{x_n\}$, there exists a subsequence $\{Tx_{n_j}\}$, say, of $\{Tx_n\}$ such that $\{Tx_{n_j}\}$ converges strongly to some element of the range of the range of T.
In what follows we shall use the following results.

Lemma 2.1. *[19] Let $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \leq \lambda_n + \mu_n$, $n \geq 1$ and $\sum_{n=1}^{\infty} \mu_n < \infty$ then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$ then $\lambda_n \to 0$ as $n \to \infty$.*

Lemma 2.2. *[15] Let E be a real uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all positive integers $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that*

$$\limsup_{n\to\infty} \| x_n \| \leq r, \qquad \limsup_{n\to\infty} \| y_n \| \leq r \quad \text{and} \qquad \limsup_{n\to\infty} \| t_n x_n + (1-t_n) y_n \| = r$$

hold for some $r \geq 0$, then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.

Lemma 2.3. *[4] Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E and let $T : K \to E$ be asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $k_n \to 1$ as $n \to \infty$, then $I - T$ is demiclosed at zero.*

3. Main Results

3.1. Asymptotically quasi-nonexpansive type mappings

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction $P : E \to K$. Suppose that $T_i : K \to E$, $i \in I$ be N nonself asymptotically quasi-nonexpansive type mappings with a nonempty closed common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$. Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined iteratively by (1.2) with the sequence $\{\alpha_n\}_{n=1}^{\infty}$ satisfying that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i, $i \in I$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F)$ is the distance from x_n to the set F.*

Proof. Necessity of the condition is obvious. Since if $x_n \to q$ as $n \to \infty$, $q \in F$, then $\lim_{n\to\infty} d(x_n,F) = d(\lim_{n\to\infty} x_n, F) = d(q,F) = 0$. Hence, $\liminf_{n\to\infty} d(x_n,F) = 0$.
Next, we prove sufficiency. Since T_i, $i \in I$ are N nonself asymptotically quasi-nonexpansive type mappings, that is, for each $i \in I$, $F(T_i) \neq \emptyset$ and

$$\limsup_{n\to\infty} \{ \sup_{x\in K, q\in F(T_i)} [\| T_i(PT_i)^{n-1}x - q \| - \| x - q \|]\} \quad \leq \quad 0.$$

Then given any $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$,

$$\sup_{x\in K, q\in F(T_i)} [\| T_i(PT_i)^{n-1}x - q \| - \| x - q \|]\} \quad < \quad \varepsilon, \qquad i \in I.$$

Since $\{x_n\} \subset K$, then for any $m \geq n_0$ we have

$$\| T_i(PT_i)^{m-1}x_n - q \| - \| x_n - q \| \quad < \quad \varepsilon, \qquad i \in I,\ n \geq 1. \tag{3.1}$$

Hence for every $x^* \in F$ and for any $m \geq n_0$, $n \geq 1$, it follows from (1.2) and (3.1) that

$$\begin{aligned}
\| x_n - x^* \| &= \| P((1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m-1}x_{n-1}) - x^* \| \\
&\leq \| (1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m-1}x_{n-1} - x^* \| \\
&\leq (1-\alpha_n) \| x_{n-1} - x^* \| + \alpha_n \| T_n(PT_n)^{m-1}x_{n-1} - x^* \| \\
&\leq (1-\alpha_n) \| x_{n-1} - x^* \| + \alpha_n(\| T_n(PT_n)^{m-1}x_{n-1} - x^* \| - \| x_{n-1} - x^* \|) + \\
&\quad \alpha_n \| x_{n-1} - x^* \| \\
&\leq \| x_{n-1} - x^* \| + \alpha_n \varepsilon.
\end{aligned}$$

That is, we have

$$\| x_{n+1} - x^* \| \quad \leq \quad \| x_n - x^* \| + \alpha_{n+1}\varepsilon.$$

By arbitrariness of $x^* \in F$, we get, upon taking infimum over $x^* \in F$,

$$\inf_{x^*\in F} \| x_{n+1} - x^* \| \quad \leq \quad \inf_{x^*\in F} \| x_n - x^* \| + \alpha_{n+1}\varepsilon,$$

so that

$$d(x_{n+1},F) \quad \leq \quad d(x_n,F) + \alpha_{n+1}\varepsilon,$$

i. e, $\lambda_{n+1} \leq \lambda_n + \mu_n$, $n \geq 1$, where $\lambda_n = d(x_n,F)$ and $\mu_n = \alpha_{n+1}\varepsilon$, $n \geq 1$. Clearly, $\sum_{n=1}^{\infty} \mu_n < \infty$ by our assumption. Then $\lim_{n\to\infty} d(x_n,F)$ exists, by Lemma 2.1. But $\liminf_{n\to\infty} d(x_n,F) = 0$, then $\lim_{n\to\infty} d(x_n,F) = 0$.
Now, for any $x^* \in F$,

$$\| x_{n+l} - x_n \| \quad \leq \quad \| x_{n+l} - x^* \| + \| x_n - x^* \|,$$

taking infimum on both sides over $x^* \in F$, we obtain

$$\| x_{n+l} - x_n \| \quad \leq \quad d(x_{n+l},F) + d(x_n,F),$$

letting $n \to \infty$ on both sides of the above inequality yields that $\lim_{n\to\infty} \| x_{n+l} - x_n \| = 0$, which shows that $\{x_n\}$ is a Cauchy sequence. Since K is a closed subset of the real Banach space E, then K is also complete. Hence there exists $p \in K$ such that $x_n \to p$ as $n \to \infty$. Finally, we prove that $p \in F$. Since $\lim_{n\to\infty} d(x_n,F) = d(\lim_{n\to\infty} x_n, F) = d(p,F) = 0$. Then $p \in \overline{F}$, but F is closed, then $p \in F$ and the proof is complete. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a real Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction $P : E \to K$. Suppose that $T_i : K \to E$, $i \in I$ be N continuous nonself asymptotically quasi-nonexpansive type mappings with a nonempty common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$. Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined iteratively by (1.2) with the sequence $\{\alpha_n\}_{n=1}^{\infty}$ satisfying that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i, $i \in I$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F)$ is the distance from x_n to the set F.*

We only need to show that F is closed so that the conclusion of Theorem 3.2 follow from the conclusion of Theorem 3.1 immediately. Let $\{p_n\}$ be a sequence of elements of F, i. e, $T_i p_n = p_n$, $n \geq 1$, $i \in I$. Assume that $p_n \to p^*$ as $n \to \infty$. We claim that $p^* \in F$. Indeed, since for each $i \in I$, we have

$$\begin{aligned} \| T_i p^* - p^* \| &\leq \| T_i p^* - p_n \| + \| p_n - p^* \| \\ &= \| T_i p^* - T_i p_n \| + \| p_n - p^* \| . \end{aligned} \tag{3.2}$$

Since T_i is continuous, $i \in I$, then letting $n \to \infty$ on both sides of (3.2) yields that

$$\lim_{n\longrightarrow\infty} \| T_i p^* - p^* \| = 0,$$

which implies that $T_i p^* = p^*$, $i \in I$ and hence $p^* \in F$.

3.2. Asymptotically quasi-nonexpansive mappings

Lemma 3.3. *Let K be a nonempty closed convex subset of a normed linear space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let $\{T_i : i \in I\}$ be N nonself asymptotically quasi-nonexpansive mappings from K to E with sequences $\{k_n^{(i)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\lim_{n\to\infty} k_n^{(i)} = 1$ for all $i \in I$, respectively. Let $\{\alpha_n\}$ be a real sequence in $[0,1)$ and let $\{x_n\}$ be the sequence defined by (1.2). If $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, then $\lim_{n\longrightarrow\infty} \| x_n - x^* \|$ exists for each $x^* \in F$.*

Proof. For each positive integer n, put $k_n = \max_{i\in I} k_n^{(i)} = 1 + u_n$.
Thus, $1 \leq k_n \leq \sum_{n=1}^{N} k_n^{(i)} - (N-1)$. Since for each $i \in I$, $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ then $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, consequently $\sum_{n=1}^{\infty} u_n < \infty$. For any $x^* \in F$, $n = (m(n)-1)N + i(n)$, $i(n) \in I$, it follows from (1.2) that

$$\begin{aligned} \| x_n - x^* \| &= \| P[(1-\alpha_n)x_{n-1} + \alpha_n T_n (PT_n)^{m(n)-1} x_{n-1}] - x^* \| \\ &\leq \| (1-\alpha_n)x_{n-1} + \alpha_n T_n (PT_n)^{m(n)-1} x_{n-1} - x^* \| \\ &\leq (1-\alpha_n) \| x_{n-1} - x^* \| + \alpha_n \| T_n (PT_n)^{m(n)-1} x_{n-1} - x^* \| \\ &\leq (1-\alpha_n) \| x_{n-1} - x^* \| + \alpha_n (1+u_m) \| x_{n-1} - x^* \| \\ &\leq (1+u_m) \| x_{n-1} - x^* \|, \end{aligned}$$

that is,

$$\| x_n - x^* \| \leq \| x_{n-1} - x^* \| + u_m \| x_{n-1} - x^* \| . \tag{3.3}$$

Furthermore, we have

$$\begin{aligned} \| x_n - x^* \| &= \| x_{(m(n)-1)N+i(n)} - x^* \| \\ &= \| P[(1-\alpha_{(m(n)-1)N+i(n)})x_{(m(n)-1)N+i(n)-1} + \\ &\quad \alpha_{(m(n)-1)N+i(n)} T_{(m(n)-1)N+i(n)} (PT_{(m(n)-1)N+i(n)})^{m(n)-1} x_{(m(n)-1)N+i(n)-1}] - x^* \| \\ &\leq \| (1-\alpha_{(m(n)-1)N+i(n)})x_{(m(n)-1)N+i(n)-1} + \\ &\quad \alpha_{(m(n)-1)N+i(n)} T_{(m(n)-1)N+i(n)} (PT_{(m(n)-1)N+i(n)})^{m(n)-1} x_{(m(n)-1)N+i(n)-1} - x^* \| \\ &\leq (1-\alpha_{(m(n)-1)N+i(n)}) \| x_{(m(n)-1)N+i(n)-1} - x^* \| + \\ &\quad \alpha_{(m(n)-1)N+i(n)} \| T_{(m(n)-1)N+i(n)} (PT_{(m(n)-1)N+i(n)})^{m(n)-1} x_{(m(n)-1)N+i(n)-1} - x^* \| \\ &\leq (1-\alpha_{(m(n)-1)N+i(n)}) \| x_{(m(n)-1)N+i(n)-1} - x^* \| + \\ &\quad \alpha_{(m(n)-1)N+i(n)} (1+u_m) \| x_{(m(n)-1)N+i(n)-1} - x^* \| \\ &\leq (1+u_m) \| x_{(m(n)-1)N+i(n)-1} - x^* \| \\ &\leq (1+u_m)^2 \| x_{(m(n)-1)N+i(n)-2} - x^* \| \\ &\leq \ldots \leq (1+u_m)^{i(n)} \| x_{(m(n)-1)N} - x^* \| . \end{aligned} \tag{3.4}$$

In addition, since $m = 1$, while $1 \leq n \leq N$, then

$$\begin{aligned} \| x_1 - x^* \| &\leq \| (1-\alpha_1)x_0 + \alpha_1 T_1 (PT_1)^{m(n)-1} x_0 - x^* \| \\ &\leq (1-\alpha_1) \| x_0 - x^* \| + \alpha_1 \| T_1 (PT_1)^{m(n)-1} x_0 - x^* \| \\ &\leq (1-\alpha_1) \| x_0 - x^* \| + \alpha_1 (1+u_1) \| x_0 - x^* \| \\ &\leq (1+u_1) \| x_0 - x^* \|, \end{aligned}$$

$$\begin{aligned} \| x_2 - x^* \| &\leq \| (1-\alpha_2)x_1 + \alpha_2 T_2(PT_2)^{m(n)-1}x_1 - x^* \| \\ &\leq (1-\alpha_2) \| x_1 - x^* \| + \alpha_2 \| T_2(PT_2)^{m(n)-1}x_1 - x^* \| \\ &\leq (1-\alpha_2) \| x_1 - x^* \| + \alpha_2(1+u_1) \| x_1 - x^* \| \\ &\leq (1+u_1) \| x_1 - x^* \| \leq (1+u_1)^2 \| x_0 - x^* \|, \end{aligned}$$

hence,

$$\| x_N - x^* \| \leq (1+u_1)^N \| x_0 - x^* \| .$$

Similarly, we have

$$\begin{aligned} \| x_{2N} - x^* \| &\leq \| (1-\alpha_{2N})x_{2N-1} + \alpha_{2N} T_{2N}(PT_{2N})^{m(n)-1}x_{2N-1} - x^* \| \\ &\leq (1-\alpha_{2N}) \| x_{2N-1} - x^* \| + \alpha_{2N} \| T_{2N}(PT_{2N})^{m(n)-1}x_{2N-1} - x^* \| \\ &\leq (1-\alpha_{2N}) \| x_{2N-1} - x^* \| + \alpha_{2N}(1+u_2) \| x_{2N-1} - x^* \| \\ &\leq (1+u_2) \| x_{2N-1} - x^* \| \leq (1+u_2)^N \| x_N - x^* \| \\ &\leq (1+u_2)^N(1+u_1)^N \| x_0 - x^* \| . \end{aligned}$$

Therefore,

$$\| x_{(m(n)-1)N} - x^* \| \leq (1+u_1)^N(1+u_2)^N ... (1+u_{m(n)-1})^N \| x_0 - x^* \| . \tag{3.5}$$

Finally (3.4) together with (3.5) imply that

$$\begin{aligned} \| x_n - x^* \| &\leq (1+u_m)^{i(n)} \| x_{(m(n)-1)N} - x^* \| \\ &\leq (1+u_1)^N(1+u_2)^N ... (1+u_{m(n)-1})^N(1+u_m)^{i(n)} \| x_0 - x^* \|, \end{aligned}$$

$i(n) \in I$. Thus

$$\| x_n - x^* \| \leq (1+u_1)^N(1+u_2)^N ... (1+u_{m(n)-1})^N(1+u_m)^N \| x_0 - x^* \| . \tag{3.6}$$

Since $1+x \leq e^x$, $x \geq 0$, then

$$\| x_n - x^* \| \leq e^{Nu_1}e^{Nu_2}...e^{Nu_m} \| x_0 - x^* \| = e^{N\sum_{j=1}^{m} u_j} \| x_0 - x^* \| \leq e^{N\sum_{j=1}^{\infty} u_j} \| x_0 - x^* \| .$$

But $\sum_{j=1}^{\infty} u_j < \infty$, then $\{x_n\}$ is a bounded sequence and there exists a constant $M > 0$ such that $\leq \| x_n - x^* \| M$, $n \geq 0$. It follows, from (3.3), that

$$\| x_n - x^* \| \leq \| x_{n-1} - x^* \| + u_m M.$$

Since $n \to \infty$ is equivalent to $m \to \infty$, it follows from Lemma 2.1 that $\lim_{n\to\infty} \| x_n - x^* \|$ exists for any $x^* \in F$. The proof is complete. □

Lemma 3.4. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let T_i, $i \in I$ be N nonself asymptotically quasi-nonexpansive mappings from K to E with sequences $\{k_n^{(i)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1) < \infty$ and $\lim_{n\to\infty} k_n^{(i)} = 1$ for all $i \in I$, respectively and suppose that T_i are uniformly L_i-Lipschitzian with the uniform Lipschitz constants $L_i > 0$, $i \in I$, respectively. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, then $\lim_{n\longrightarrow\infty} \| x_n - T_i x_n \| = 0$ for each $i \in I$.*

Proof. Lemma 3.3 asserts that $\lim_{n\longrightarrow\infty} \| x_n - x^* \|$ exists for each $x^* \in F$. We may assume that, for some $x^* \in F$, $\lim_{n\longrightarrow\infty} \| x_n - x^* \| = c$ for some $c \geq 0$. If $c = 0$, we are done. So let $c > 0$. Set $n = (m(n)-1)N + i(n)$, $i(n) \in I$. Since

$$\begin{aligned} \| x_{n+1} - x^* \| &= \| P[(1-\alpha_{n+1})x_n + \alpha_{n+1}T_{n+1}(PT_{n+1})^{m(n)-1}x_n] - x^* \| \\ &\leq \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \end{aligned} \tag{3.7}$$

Taking lim inf on both sides of (3.7), we obtain

$$\liminf_{n\longrightarrow\infty} \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \geq c. \tag{3.8}$$

Also,

$$\| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \leq (1+u_m) \| x_n - x^* \|,$$

which on taking lim sup on both sides yields that

$$\begin{aligned} &\limsup_{n\longrightarrow\infty} \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \\ &\qquad \leq \limsup_{m\longrightarrow\infty}(1+u_m) \| x_n - x^* \| = c. \end{aligned} \tag{3.9}$$

Inequalities (3.8) and (3.9) imply

$$\lim_{n\longrightarrow\infty} \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| = c. \tag{3.10}$$

Since $\lim_{n\longrightarrow\infty} \| x_n - x^* \| = c$ and $\limsup_{n\longrightarrow\infty} \| T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^* \| \leq c$, it follows from Lemma 2.2 that

$$\lim_{n\longrightarrow\infty} \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \| = 0. \tag{3.11}$$

Since

$$\| x_{n+1} - x_n \| \leq \alpha_{n+1} \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \|$$

then, by (3.11), we have

$$\lim_{n\longrightarrow\infty} \| x_{n+1} - x_n \| = 0.$$

By induction, we have

$$\lim_{n\longrightarrow\infty} \| x_{n+r} - x_n \| = 0 \tag{3.12}$$

for any positive integer r.
Let $L = \max_{i\in I}\{L_i\}$. When $n > N$ $(m \geq 2)$, we have

$$\begin{aligned}
\| x_n - T_{n+1}x_n \| &\leq \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \| + \| T_{n+1}(PT_{n+1})^{m(n)-1}x_n - T_{n+1}x_n \| \\
&\leq \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \| + L \| P[T_{n+1}(PT_{n+1})^{m(n)-2}]x_n - x_n \| \\
&\leq \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \| + L \| T_{n+1}(PT_{n+1})^{m(n)-2}x_n - x_n \| \\
&\leq \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \| + L\{\| x_n - x_{n-N} \| + \\
&\quad \| x_{n-N} - T_{n+1-N}(PT_{n+1-N})^{m(n)-2}x_{n-N} \| + \\
&\quad \| T_{n+1-N}(PT_{n+1-N})^{m(n)-2}x_{n-N} - T_{n+1}(PT_{n+1})^{m(n)-2}x_n \|\}
\end{aligned}$$

Hence

$$\begin{aligned}
\| x_n - T_{n+1}x_n \| &\leq \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \| + L\{(1+L) \| x_n - x_{n-N} \| + \\
&\quad \| x_{n-N} - T_{n+1-N}(PT_{n+1-N})^{m(n)-2}x_{n-N} \|\}
\end{aligned} \tag{3.13}$$

Noticing that $n = (m(n)-1)N + i(n)$, $i(n) \in I$, we have $n - N = (m(n)-1)N + i(n) - N = (m(n)-2)N + i(n) = (m(n-N)-1)N + i(n-N)$, thus $m(n-N) = m(n)-1$ and $i(n-N) = i(n)$, $n \geq 1$. Hence

$$\| x_{n-N} - T_{n+1-N}(PT_{n+1-N})^{m(n)-2}x_{n-N} \| = \| x_{n-N} - T_{n+1-N}(PT_{n+1-N})^{m(n-N)-1}x_{n-N} \| .$$

Using (3.11), we get

$$\lim_{n\to\infty} \| x_{n-N} - T_{n+1-N}(PT_{n+1-N})^{m(n)-2}x_{n-N} \| = 0. \tag{3.14}$$

Using (3.12) and (3.14), it follows from (3.13) that

$$\lim_{n\to\infty} \| x_n - T_{n+1}x_n \| = 0. \tag{3.15}$$

Furthermore, for each $i \in I$

$$\begin{aligned}
\| x_n - T_{n+i}x_n \| &\leq \| x_n - x_{n+i-1} \| + \| x_{n+i-1} - T_{n+i}x_{n+i-1} \| + \| T_{n+i}x_{n+i-1} - T_{n+i}x_n \| \\
&\leq (1+L) \| x_n - x_{n+i-1} \| + \| x_{n+i-1} - T_{n+i}x_{n+i-1} \| .
\end{aligned}$$

Using (3.12) and (3.15), we obtain

$$\lim_{n\to\infty} \| x_n - T_{n+i}x_n \| = 0, \qquad i \in I.$$

Thus

$$\lim_{n\to\infty} \| x_n - T_ix_n \| = 0, \qquad i \in I.$$

This completes the proof. □

3.3. Asymptotically quasi-nonexpansive in the intermediate sense mappings

Lemma 3.5. *Let K be a nonempty closed convex subset of a normed linear space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let $\{T_i : i \in I\}$ be N nonself asymptotically quasi-nonexpansive in the intermediate sense mappings from K to E with a nonempty common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$. For each $i \in I$, put $G_m^{(i)} = \max\{\sup_{x\in K, x^* \in F}(\| T_i(PT_i)^{m-1}x - x^* \| - \| x - x^* \|), 0\}$ so that $\sum_{m=1}^{\infty} G_m^{(i)} < \infty$, $i \in I$. If $\{x_n\}$ is the sequence defined by (1.2), then $\lim_{n\longrightarrow\infty} \| x_n - x^* \|$ exists for each $x^* \in F$.*

Proof. For any $x^* \in F$, we have

$$\begin{aligned}
\| x_n - x^* \| &= \| P[(1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m(n)-1}x_{n-1}] - x^* \| \\
&\leq \| (1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m(n)-1}x_{n-1} - x^* \| \\
&\leq (1-\alpha_n) \| x_{n-1} - x^* \| + \alpha_n \| T_n(PT_n)^{m(n)-1}x_{n-1} - x^* \| \\
&\leq (1-\alpha_n) \| x_{n-1} - x^* \| + \alpha_n (G_m^{(n)} + \| x_{n-1} - x^* \|).
\end{aligned}$$

Thus

$$\| x_n - x^* \| \leq \| x_{n-1} - x^* \| + G_m^{(n)}.$$

Since $\sum_{m=1}^{\infty} G_m^{(n)} < \infty$, $n \geq 1$ and $n \longrightarrow \infty$ is equivalent to $m \longrightarrow \infty$, then applying Lemma 2.1 implies that $\lim_{n\longrightarrow\infty} \| x_n - x^* \|$ exists for each $x^* \in F$. The proof is complete. ☐

Lemma 3.6. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let $\{T_i : i \in I\}$ be N nonself asymptotically quasi-nonexpansive in the intermediate sense mappings from K to E with a nonempty common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$. For each $i \in I$, put $G_m^{(i)} = \max\{\sup_{x\in K, x^* \in F}(\| T_i(PT_i)^{m-1}x - x^* \| - \| x - x^* \|), 0\}$ so that $\sum_{m=1}^{\infty} G_m^{(i)} < \infty$, $i \in I$. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then $\lim_{n\longrightarrow\infty} \| x_n - T_i x_n \| = 0$ for each $i \in I$.*

Proof. It follows from Lemma 3.5 that $\lim_{n\longrightarrow\infty} \| x_n - x^* \|$ exists for each $x^* \in F$. Assume that $\lim_{n\longrightarrow\infty} \| x_n - x^* \| = c$, $x^* \in F$ for some $c \geq 0$. If $c = 0$, we are done. So let $c > 0$. Set $n = (m(n)-1)N + i(n)$, $i(n) \in I$. Since

$$\begin{aligned}
\| x_{n+1} - x^* \| &= \| P[(1-\alpha_{n+1})x_n + \alpha_{n+1} T_{n+1}(PT_{n+1})^{m(n)-1}x_n] - x^* \| \\
&\leq \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \qquad (3.16)
\end{aligned}$$

Taking lim inf on both sides of (3.16), we obtain

$$\liminf_{n\longrightarrow\infty} \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \geq c. \qquad (3.17)$$

In addition,

$$\begin{aligned}
\| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \leq\ & (1-\alpha_{n+1}) \| x_n - x^* \| \\
& + \alpha_{n+1}(G_{n+1}^{m(n)} + \| x_n - x^* \|).
\end{aligned}$$

Hence

$$\| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \leq \| x_n - x^* \| + G_{n+1}^{m(n)},$$

which on taking lim sup on both sides yields that

$$\begin{aligned}
&\limsup_{n\longrightarrow\infty} \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| \\
&\qquad \leq \limsup_{n\longrightarrow\infty} \| x_n - x^* \| + \limsup_{m\longrightarrow\infty} G_{n+1}^{m(n)} = c. \qquad (3.18)
\end{aligned}$$

Inequalities (3.17) and (3.18) imply

$$\lim_{n\longrightarrow\infty} \| (1-\alpha_{n+1})(x_n - x^*) + \alpha_{n+1}(T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^*) \| = c. \qquad (3.19)$$

Since $\lim_{n\longrightarrow\infty} \| x_n - x^* \| = c$ and $\limsup_{n\longrightarrow\infty} \| T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x^* \| \leq c$, it follows from Lemma 2.2 that

$$\lim_{n\longrightarrow\infty} \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \| = 0. \qquad (3.20)$$

Since

$$\| x_{n+1} - x_n \| \leq \alpha_{n+1} \| x_n - T_{n+1}(PT_{n+1})^{m(n)-1}x_n \|$$

then, by (3.20), we have

$$\lim_{n\longrightarrow\infty} \| x_{n+1} - x_n \| = 0.$$

By induction, we have

$$\lim_{n\longrightarrow\infty} \| x_{n+r} - x_n \| \quad = \quad 0 \tag{3.21}$$

for any positive integer r.
Now, we have

$$\begin{aligned} \| x_n - T_{n+1}x_n \| \quad \leq \quad & \| x_n - x_{n+N} \| + \| x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} \| + \\ & \| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n \| + \\ & \| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n - T_{n+1}x_n \| . \end{aligned}$$

Since $n = (m(n)-1)N + i(n)$, $i(n) \in I$, then $n+N = (m(n)-1)N + i(n) + N = m(n)N + i(n) = (m(n+N)-1)N + i(n+N)$, thus $m(n+N) = m(n)+1$, $i(n+N) = i(n)$ and $T_{n+1} = T_{n+N+1} = T_{i(n+1)}$, $n \geq 1$. Hence

$$\begin{aligned} \| x_n - T_{n+1}x_n \| \quad \leq \quad & \| x_n - x_{n+N} \| + \| x_{n+N} - T_{n+N+1}(PT_{n+N+1})^{m(n+N)-1}x_{n+N} \| + \\ & \| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n \| + \\ & \| T_{n+1}(PT_{n+1})^{m(n)}x_n - T_{n+1}x_n \| . \end{aligned} \tag{3.22}$$

But (3.20) implies that

$$\| PT_{n+1}(PT_{n+1})^{m(n)-1}x_n - x_n \| \quad \leq \quad \| T_{n+1}(PT_{n+1})^{m(n)-1}x_n - x_n \| \to 0, \qquad n \to \infty,$$

since T_{n+1} are uniformly continuous, then

$$\| T_{n+1}(PT_{n+1})^{m(n)}x_n - T_{n+1}x_n \| = \| T_{n+1}PT_{n+1}(PT_{n+1})^{m(n)-1}x_n - T_{n+1}x_n \| \to 0, \quad n \to \infty \; . \tag{3.23}$$

Also, uniform continuity of T_{n+1} and (3.21) yield

$$\| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n \| \longrightarrow 0, \quad n \to \infty. \tag{3.24}$$

Finally, using (3.20), (3.21), (3.23) and (3.24), it follows from (3.22) that

$$\lim_{n\to\infty} \| x_n - T_{n+1}x_n \| \quad = \quad 0. \tag{3.25}$$

Furthermore, for each $i \in I$

$$\| x_n - T_{n+i}x_n \| \quad \leq \quad \| x_n - x_{n+i-1} \| + \| x_{n+i-1} - T_{n+i}x_{n+i-1} \| + \| T_{n+i}x_{n+i-1} - T_{n+i}x_n \|,$$

using (3.21), (3.25) and uniform continuity of T_{n+i}, we get

$$\lim_{n\to\infty} \| x_n - T_{n+i}x_n \| \quad = \quad 0, \qquad i \in I.$$

Thus

$$\lim_{n\to\infty} \| x_n - T_i x_n \| \quad = \quad 0, \qquad i \in I.$$

The proof is complete. □

Now, we are in a position to state our main theorems

Theorem 3.7. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let T_i, $i \in I$ be N nonself asymptotically quasi-nonexpansive mappings from K to E with sequences $\{k_n^{(i)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\lim_{n\to\infty} k_n^{(i)} = 1$ for all $i \in I$, respectively. Suppose that T_i are uniformly L_i-Lipschitzian with the uniform Lipschitz constants $L_i > 0$, $i \in I$, respectively. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and if one of the mappings T_i, $i \in I$ is completely continuous, then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_i, $i \in I$.*

Theorem 3.8. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let $\{T_i : i \in I\}$ be N nonself asymptotically quasi-nonexpansive in the intermediate sense mappings from K to E with a nonempty common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$. For each $i \in I$, put $G_m^{(i)} = \max\{\sup_{x\in K, x^*\in F}(\| T_i(PT_i)^{m-1}x - x^* \| - \| x - x^* \|), 0\}$ so that $\sum_{m=1}^{\infty} G_m^{(i)} < \infty$, $i \in I$. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If one of the mappings T_i, $i \in I$ is completely continuous Then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_i, $i \in I$.*

Proof. The proof of theorems 3.7 and 3.8 follows from the proof of Theorem 3.4 in [23]. □

Theorem 3.9. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let T_i, $i \in I$ be N nonself asymptotically quasi-nonexpansive mappings from K to E with sequences $\{k_n^{(i)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\lim_{n\to\infty} k_n^{(i)} = 1$ for all $i \in I$, respectively. Suppose that T_i are uniformly L_i-Lipschitzian with the uniform Lipschitz constants $L_i > 0$, $i \in I$, respectively. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and one of the mappings T_i, $i \in I$ is demicompact then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_i, $i \in I$.*

Theorem 3.10. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E with a nonexpansive retraction P. Let $\{T_i : i \in I\}$ be N nonself asymptotically quasi-nonexpansive in the intermediate sense mappings from K to E with a nonempty common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$. For each $i \in I$, put $G_m^{(i)} = \max\{\sup_{x\in K, x^*\in F}(\| T_i(PT_i)^{m-1}x - x^* \| - \| x - x^* \|), 0\}$ so that $\sum_{m=1}^{\infty} G_m^{(i)} < \infty$, $i \in I$. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If one of the mappings T_i, $i \in I$ is demicompact then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_i, $i \in I$.*

Proof. The proof of theorems 3.9 and 3.10 follows from the proof of Theorem 3.5 in [23]. □

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