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# Trajectory Generation and Control in a Special Transport Mission of a Cable-Suspended Point-Mass Load from a Quadrotor

Hassan SAYYAADI[1,*], Ali SOLTANI[2]

[1] *Sharif University of Technology, Department of Mechanical Engineering, 11155-9567, Tehran, Iran*

[2] *Sahand University of Technology, Department of Mechanical Engineering, Tabriz, Iran*

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**Abstract**

In this study, trajectory planning and control of a cable-suspended load from a quadrotor in a special transportation mission are addressed. The mission under consideration in this study includes picking up and transportation of the load toward a defined point at a specified time. Therefore, two trajectory tracking controllers are designed for the quadrotor and the load. Controllers are designed geometrically via Backstepping and configuration error functions defined on S2 and SO(3) in the literature of Lie groups. By this way, common singularities of regular quadrotor attitude controllers are avoided. Moreover, sufficient conditions for success of the proposed control laws are calculated using Lyapunov exponential stability theorem and an argument about singularly perturbed systems. All simulations are performed on an experimentally verified model of OS4 quadrotor and capabilities of the designed control laws are demonstrated.

## 1.INTRODUCTION

Unmanned aerial vehicles (UAVs) have been interested by many researchers in the recent decade. Specially, quadrotors with simple mechanical structures and easily controllable dynamics were being more attractive for the scientists. Employing UAVs in the tasks which bring risk to human life such as forest fire monitoring[1], search and surveillance operations, etc. motivates engineers to study their dynamics and to find a way to control them. In such operations, usually it is needed to transport something or somebody on a desired path from points to points. These loads can be grasped by the UAV[2] or they can be suspended by a cable from it [3]. The appended load to the body of UAV increases the moment of inertia of the system and results in slow attitude response [4]. Hence, suspending the loads through a cable from the aerial vehicle is preferred because the attitude response of the vehicle is not varied while the transportation still can be performed. However, swinging of the cable-suspend loads are troublesome in the case of dangerous or sensitive cargoes. Hence, transporting the suspended loads by regular manipulators or aerial robots with minimum swinging is studied by the researchers. For example, swing free transport of a slender payload suspended from two manipulator robots through two cables is performed by via dynamic programming in [5]. Also, problem of controlling multiple robots manipulating and transporting a payload in three dimensions via cables has been studied in [6] and robot configurations that ensure static equilibrium of the payload at a desired pose have been developed. Performances of the developed methods were examined on a team of aerial robots in simulation and experimentation. Moreover, an adaptive control approach with combined feedback and feed-forward schemes for simultaneous avoidance of swing excitation and damping of active swing has been proposed for a helicopter-slung load system in [7]. Simulations and laboratory flight tests have shown the effectiveness of the proposed combined control scheme in significant load swing reduction compared to

*Corresponding author, e-mail: sayyaadi@sharif.edu

the baseline controller. In studies [4] and [3], swing free maneuverings for safe and efficient transport by the quadrotor-suspended load systems have been achieved in simulations and experiments via optimal trajectory generation based on dynamic programming. In another study[8], a fuzzy controller has been added to the position control software of a helicopter to reduce swinging of its slung load. Distribution of the fuzzy membership functions has been optimally tuned by the method of particle swarms.

Although trajectory tracking control of the UAVs with slung load has been addressed by many researchers, trajectory tracking control of the suspended loads is rarely studied by the scientists. Vijay Kumar Lab which is one of the most active laboratories in designing and manufacturing and controlling the quadrotors has presented three conference papers about designing trajectory tracking controllers for the suspended loads from the quadrotors [9-11]. In study [9], after presenting arguments which demonstrate that hybrid system of quadrotor and its suspended load is differentially flat, proper trajectories are produced for the quadrotor to transport the suspended load on the planar paths from some restrictive initial conditions. These limited initial conditions and assumptions come from improper procedure for trajectory generation. Controller designed in [10] is three dimensional version of the proposed controller strategy in [9]. In the last study [11], a scalable algorithm is proposed to control a suspended mass point on the desired trajectories by multiple quadrotors. Although performance of the proposed control scheme are depicted experimentally in [9], only simulation is implemented to verify effectiveness of the control strategies designed in studies [10] and [11]. Moreover, in all of the researches [9-11], cables are supposed to be taut initially.

In this paper, a special mission is considered for transporting a point-mass load attached to a quadrotor through a cable. In this mission, the load is placed on the ground and the cable is not taut initially. Quadrotor is supposed to pull the cable attached to the load accordingly to manipulate it and put it in a predefined target point at a specified time. Therefore, proper trajectories are generated for the quadrotor to pull the cable accordingly. Moreover, desired trajectory for point to point transportation of the load is produced too. Furthermore, necessary control laws to track the generated trajectories are designed geometrically by implementing concepts of Lie groups via backstepping and configuration errors defined on $S^2$ and SO(3) groups. Sufficient conditions for exponential convergence of the tracking errors of the system to zero are derived by Lyapunov exponential stability theorem and an argument about singularly perturbed systems. In simulation, after generating minimum snap trajectory for the quadrotor to pull the cable and trajectory generation for transportation of the load, designed control laws are examined on a complex experimentally verified model of OS4 quadrotor [12] accordingly.

Present paper is organized as follows:
First section is about modeling dynamics of the system in slack and taut cable conditions and modeling its switching surface. Second section explains trajectory generation for the quadrotor to pull the cable and trajectory generation for the load to be placed by the quadrotor at a predefined target point at a specified time. Control laws are proposed in the third section appropriately. Fourth section describes sufficient condition for successful performance of the designed controllers. Next, simulations are organized in fifth section. Finally, associated results, discussions and future recommended studies are brought briefly in the conclusion section.

## 2. DYNAMIC MODELING FOR CONTROL

In a quadrotor and cable suspended load system, the cable may be taut or loose during flight mission. If the cable is taut then motion of the quadrotor and the load will be constrained. Otherwise, if the cable is slack, dynamics of the quadrotor and dynamics of the load are decoupled completely. In other words, quadrotor and load will move independently while tension of the cable remains zero. Consequently, dynamics of the system is hybrid and should be modeled for both of the mentioned conditions. Moreover, following assumptions are made to derive a reference dynamic model of the system for controller design:
1) Dominant forces and torques acting on the rotors of the quadrotor are thrust forces and drag torque. Moreover, thrust force and drag torque acting on i'th rotor of the quadrotor can be approximated by $F_i = c_T \omega_i^{\,2}$ and $Q_i = c_D \omega_i^2$ respectively. $c_T$ and $c_D$ are thrust and drag torque coefficients of the

propellers in hovering mode. Furthermore, gyroscopic effects of the rotors rotation are negligible. Moreover, aerodynamic forces acting on the load and the quadrotor are negligible.
2) Load is point-mass.
3) Cable is massless and does not stretch.
4) Cable is attached to center of mass of quadrotor and so cable tension does not affect rotational dynamics of quadrotor.
5) Switching of dynamics of system from slack cable condition to taut cable condition can be modeled as an inelastic collision along cable direction.
Total thrust force produced by rotors numbered in Figure 1 from one to four can be formulated as $U = c_T \sum_{i=1}^{4} \omega_i^2$ .

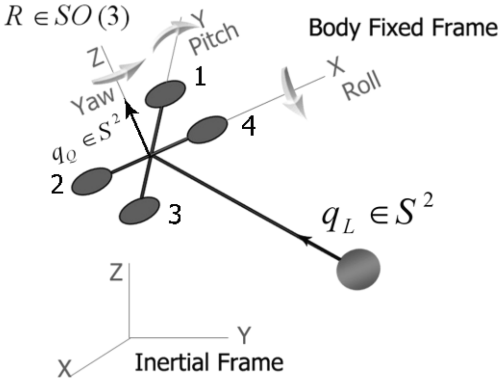


***Figure 1****. Quadrotor and Load system, Inertial frame and Body fixed frame*

Also, differences of the thrust forces provide necessary roll and pitch torques of the quadrotor which can be formulated in its body fixed frame as:

$$\begin{aligned} \tau_x &= d(F_1 - F_3) \\ \tau_y &= d(F_2 - F_4) \end{aligned} \tag{1}$$

In the recent equation, $d$ represents distances of the rotors to the center of mass of the quadrotor. Moreover, drag torque acting on each propeller result in net yaw torque. In quadrotor platform, rotors number one and three rotate in the same direction while the other pair of the rotors rotate in the opposite direction. Hence, the drag moment of these pairs will act in opposite directions. Assuming that rotors number one and three rotates in Z direction of body fixed frame then the total yaw torque can be modeled as $\tau_z = \sum_{i=1}^{4} (-1)^i Q_i$ .

Although model of the system used to design controllers is driven by making some simplifying assumptions, a complicated and experimentally verified model of OS4 quadrotor is implemented in simulations to examine performances of the controllers fairly [12].
In following, equations of motion will be presented in taut cable and slack cable condition. Next, switching surfaces of system dynamics will be modeled appropriately.

## 2.1. Equations of motion in taut cable condition

When the cable is taut, the system has eight degrees of freedom and only four of them are actuated by the four control inputs of the quadrotor (the thrust force and roll, pitch and yaw torques). Therefore, quadrotor and cable-suspended load system is an under-actuated system in this condition. Motion of the system can be described as equation (2) in this condition by using Newton-Euler method.

$$\begin{aligned} &Load \quad Dynamics: \\ &m_L \ddot{r}_L = T q_L + m_L \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \\ &Quadrotor \quad Translational \quad Dynamics: \\ &m_Q \ddot{r}_Q = U q_{z_Q} - T q_L + m_Q \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \\ &Quadrotor \quad Rotational \quad Dynamics: \\ &J_Q \dot{\Omega} = -\Omega \times (J_Q \Omega) + \tau \end{aligned} \tag{2}$$

$m_L, m_Q \in \mathbb{R}^+$ are denoting masses of the load and quadrotor respectively. In these equations $r_L, r_Q \in \mathbb{R}^3$ denote position vectors of the load and quadrotor center of mass respectively. Moreover $q_L, q_Q \in \mathbb{R}^3$ are unit vectors of the cable direction from the load to the quadrotor and thrust force direction respectively. Obviously, these unit vectors lie in the configuration space $S^2$. If cable length is denoted by $L$ then the unit vector of the cable direction is $\frac{r_Q - r_L}{L}$ clearly. $U \in \mathbb{R}^+$ is thrust force magnitude and $\tau \in \mathbb{R}^3$ is the torque vector acting on quadrotor. So, $U$ and $\tau$ are control inputs of the system. Magnitude of cable tension has been expressed in the equations by $T \in \mathbb{R}^+$. Also, $\Omega \in \mathbb{R}^3$ is angular velocity vector of the quadrotor in the body fixed frame. Tensor of moment of inertia of the quadrotor with respect to the body fixed frame has been presented by $J_Q \in \mathbb{R}^{3\times3}$ in the equations of motion. The equations of motion of the system are supported by a kinematic relation described by:

$$\dot{R} = R\hat{\Omega} \tag{3}$$

$R \in SO(3)$ rotates the inertial frame toward the body fixed frame (**Hata! Başvuru kaynağı bulunamadı.**).

*Remark.1: For two arbitrary vector $x$ and $y$, $x \times y = \hat{x}y$. In other words, $\hat{x}$ is skew symmetric matrix of vector $x$.*

Thrust force of the quadrotor is in positive direction of the z-axis of the body fixed frame. So, $q_z$ can be calculated by

$$q_z = R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

Moreover, while cable tension is not zero, distance between the quadrotor and the load is equal to the length of the cable. Hence, quadrotor and load motions are constrained when the cable is taut. Cable length constraint and its derivatives with respect to the time can be explained by:

$$\begin{aligned}
&Cable\ \ Length\ Constraint:\\
&\left\| r_Q - r_L \right\| = L\\
&Relative\, Velocity\ \ Constraint:\\
&(\dot{r}_Q - \dot{r}_L)^T q_L = 0\\
&Relative\ \ Acceleration\ Constraint:\\
&(\ddot{r}_Q - \ddot{r}_L)^T q_L = -\frac{\left\| \dot{r}_Q - \dot{r}_L \right\|^2}{L}
\end{aligned} \tag{5}$$

Importing acceleration constraint to the equations of motion of the system results in:

$$(m_L + m_q)(\ddot{r}_L + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}) = U\,(q_L \cdot q_{z_Q})q_L + m_Q \frac{\left\| \dot{r}_Q - \dot{r}_L \right\|^2}{L} \tag{6}$$

Also, dynamics of $q_L$ can be explained by following equations.

$$\begin{aligned}
&\dot{q}_L = \omega_L \times q_L\\
&m_Q L \dot{\omega}_L = U q_L \times q_{z_Q}
\end{aligned} \tag{7}$$

$\omega_L$ is angular velocity vector of the unit vector $q_L$ .

### 2.2. Equations of Motion in Slack Cable Condition

Quadrotor and the suspended load move independently, when the cable is slack. In this condition, the point-mass load has three degrees of freedom while the quadrotor has six degrees of freedom and only four of them are actuated. Equations of motion in this situation can be expressed as

$$\begin{aligned}
&Load\ \ Dynamics:\\
&m_L \ddot{r}_L = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}\\
&Quadrotor\ Translational\ \ Dynamics:\\
&m_Q \ddot{r}_Q = U q_{z_Q} + m_Q \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}\\
&Quadrotor\ \ Rotational\ \ Dynamics:\\
&J_Q \dot{\Omega} = -\Omega \times (J_Q \Omega) + \tau
\end{aligned} \tag{8}$$

It is obvious from (8) that there is not any actuation on the load while the cable is not taut. However, quadrotor is still controllable in this condition.

### 2.3. Modeling Switching Surface of System Dynamics

When the cable goes from taut to slack, there is not any change in the states of the system. But, velocity of the load and the quadrotor changes when tension of the cable reaches from zero to a non-zero value because it is assumed that the cable does not stretch. To model this phenomena, one should consider relative velocity constraint expressed by equation (5). By this constraint and linear momentum conservation along the cable direction, velocity change of the quadrotor and load can be modeled as an inelastic collision as:

$$v' = \frac{(m_Q v_Q + m_L v_L)^T q_L}{m_Q + m_L} \tag{9}$$

In the recent equation, $v'$ is velocity of the quadrotor and load along the cable due to the inelastic collision. Therefore, velocities of quadrotor and load after collision can be formulated as equation (10) by imposing equation (9) to linear momentum conservation rule:

$$\begin{aligned} v'_Q &= v_Q - (v_Q^T q_L) q_L + v' q_L \\ v'_L &= v_L - (v_L^T q_L) q_L + v' q_L \end{aligned} \tag{10}$$

In equation (10), $v'_Q$ and $v'_L$ denote velocities of quadrotor and load after the collision. This model of velocity change will be implemented as a discrete velocity change in the simulations.

## 3. TRAJECTORY GENERATION

In this section, considered mission is introduced at first. So, trajectory generation procedure to perform the mission is described in detail.

### 3.1. Mission Definition

Load is assumed to be on the ground initially and initial position of the quadrotor is hovering at a point such that cable is not taut. To pick up the load, quadrotor should move to a point directly above the load and stop there at a specified time. Distance between this point and the load is equal to the cable length. By this way, the cable will be taut. Then, quadrotor should locate the load on a specified point at a desired time.

### 3.2. Trajectory Generation for Quadrotor to Pull Cable

When the cable is slack, quadrotor is free and it can be shown that input torques of the quadrotor are functions of snap of quadrotor's position and second derivative of yaw angle[13]. Consequently, minimum snap trajectories are usually preferred for position of quadrotor. Therefore, desired trajectory of quadrotor $r_Q^d$ from start point to target point of quadrotor in time domain $[t_0, t_f]$ should be planned such that following integral is minimized.

$$\int_{t_0}^{t_f} \left\| \frac{d^4 r_Q^d}{dt^4} \right\|^2 .dt \tag{11}$$

This integral is minimum if $\frac{d^8 r_Q^d}{dt^8} = 0$ [14]. Therefore, one can choose a 7th order polynomial as desired trajectory such as:

$$r_Q^d = \sum_{i=0}^{7} C_i t^i, C_i = \begin{bmatrix} C_{ix} & C_{iy} & C_{iz} \end{bmatrix} \tag{12}$$

Coefficients of this polynomial can be calculated from initial and desired final conditions. Therefore, initial and final conditions should be chosen as following in the considered mission.

$$\begin{aligned} & r_Q^d(t_0) = r_Q(t_0), r_Q^d(t_f) = r_{Q_f} \\ & \frac{d^i r_Q^d}{dt^i}(t_0) = \frac{d^i r_Q^d}{dt^i}(t_f) = 0, i = 1, \ldots, 3 \end{aligned} \tag{13}$$

Furthermore, desired yaw angle of quadrotor is considered equal to zero in this study.

### 3.3. Trajectory Generation for Load in Taut Cable Condition

When the cable is taut, dynamics of the system switches as described previously. It can be proven that torque inputs of the quadrotor are functions of sixth derivative of the load position [10]. Therefore, desired trajectory of the load position $r_L^d$ should be generated such that below integral is minimized.

$$\int_{t_0}^{t_f} \left\| \frac{d^6 r_L^d}{dt^6} \right\|^2 .dt \tag{14}$$

Same as previous optimizing problem, this function is minimum if $\frac{d^{12} r_L^d}{dt^{12}} = 0$ . So, desired trajectory of the load from the start point to the target point can be adopted as a 11th order polynomial such as:

$$r_L^d = \sum_{i=0}^{11} D_i t^i, D_i = \begin{bmatrix} D_{ix} & D_{iy} & D_{iz} \end{bmatrix} \tag{15}$$

Similarly, coefficients of the polynomial can be calculated from initial and final conditions. These conditions in the considered mission are as follows accordingly.

$$\begin{aligned} & r_L^d(t_0) = r_L(t_0), r_L^d(t_f) = r_{L_f} \\ & \frac{d^i r_L^d}{dt^i}(t_0) = \frac{d^i r_L^d}{dt^i}(t_f) = 0, i = 1, \ldots, 5 \end{aligned} \tag{16}$$

So, all 12 coefficient vectors can be found directly.

## 4. CONTROLLER DESIGN

Trajectory tracking controller is designed for system considering that the cable may be taut or slack. In slack cable condition, a controller calculates necessary thrust force vector such that the cable be taut in proper direction in a finite time. Moreover, in taut cable condition, another controller determines trajectory for thrust vector such that the hung load follows its predefined path. The proper direction of the thrust force vector calculated by any of these controllers is tracked by an attitude controller designed geometrically via backstepping and a configuration error defined on SO (3) groups in Lie groups literature. On the other hand, magnitude of desired thrust force vector and input torques determined by attitude controller are achieved by varying angular velocity of rotors appropriately.

### 4.1. Load Trajectory Tracking Controller in Taut Cable Condition

To design the control law in this condition, equation (6) is used. If dynamics of the trajectory tracking error of the load is defined by:

$$\ddot{e}_r = -k_d^r \dot{e}_r - k_p^r e_r \tag{17}$$

with positive constants $k_d^r, k_p^r$ and $e_r = r_L - r_L^d$ then its equilibrium point($\dot{e}_r = 0, e_r = 0$) will be exponentially stable. As a result, magnitude and direction of thrust force and direction of cable should satisfy:

$$U(q_L \cdot q_{z_Q})q_L = (m_L + m_q)(\ddot{r}_s + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}) - m_Q \frac{\left\|\dot{r}_Q - \dot{r}_L\right\|^2}{L} \tag{18}$$

$\ddot{r}_s$ in equation (18) is defined by equation (17) as

$$\ddot{r}_s = \ddot{r}_d - k_d^r \dot{e}_r - k_p^r e_r \tag{19}$$

So, desired cable direction $q_L^d$ is

$$q_L^d = \frac{F}{\|F\|} \tag{20}$$

where $F$ denotes the right hand side of equation (18). Moreover, desired direction and magnitude of thrust force can be chosen as

$$U^d(q_L \cdot q_{z_Q}^d) = F \cdot q_L \tag{21}$$

Furthermore, thrust force vector should be set such that desired direction of cable is achieved. To do this, a configuration error function defined on $S^2$ groups is implemented as:

$$\psi_L = 1 - q_L^d . q_L \tag{22}$$

Derivative of this error function with respect to the time is:

$$\dot{\psi}_L = e_{q_L} \cdot e_{\omega_L} \tag{23}$$

In the recent equation, parameters $e_{q_L}$ and $e_{\omega_L}$ are

$$\begin{aligned} e_{q_L} &= q_L \times (q_L \times q_L^d) \\ e_{\omega_L} &= (\omega_L - \omega_L^d) \times q_L \\ \omega_L^d &= q_L^d \times \dot{q}_L^d \end{aligned} \tag{24}$$

If $\psi_L(0) \neq 2$ and $e_{\omega_L} = -\dfrac{k_{\psi_L}}{2-\psi_L} e_{q_L}$ then $\dot{\psi}_L = -k_{\psi_L} \psi_L$ which causes that $\psi_L$ tends to zero exponentially for any positive constant value $k_{\psi_L}$ and desired direction of the cable is achieved accordingly. Therefore, an error vector $E_{\psi_L} = e_{\omega_L} + \dfrac{k_{\psi_L}}{2-\psi_L} e_{q_L}$ is defined. A Lyapunov function is candidate for Backstepping as

$$V_{\psi_L} = C_{\psi_L} \psi_L + \frac{\left\| E_{\psi_L} \right\|^2}{2} \tag{25}$$

It can be shown that if

$$\begin{gathered} \dot{\omega} \times q_L = \dot{\omega}_L^d \times q_L + (\omega_L^d . q_L) \omega_L^d - \frac{k_{\psi_L}}{2-\psi_L} ((q_L . q_L^d) \dot{q}_L + q_L \times (q_L \times \dot{q}_d^L)) \\ - \frac{k_{\psi_L}}{(2-\psi_L)^2} (e_{q_L} . e_{\omega_L}) e_{q_L} - C_{\psi_L} e_{q_L} - k_{E_{\psi_L}} E_{\psi_L} \end{gathered} \tag{26}$$

then

$$\dot{V}_{\psi_L} = -k_{\psi_L} C_{\psi_L} \psi_L - k_{E_{\psi_L}} \left\| E_{\psi_L} \right\|^2 \tag{27}$$

To ensure that $\psi_L$ will not be equal to 2 during the control procedure, it is sufficient that:

$$C_{\psi_L} > \frac{\frac{1}{2} \left\| E_{\psi_L}(0) \right\|^2}{2 - \psi_L(0)} \tag{28}$$

Considering (26) and(7), one can conclude that

$$U^d q_L \times (q_{z_Q}^d \times q_L) = m_Q L A \tag{29}$$

where $A$ denotes the right hand side of(26). By equation (21), it can be concluded that:

$$U^d q_{z_Q}^d = m_Q L A + B q_L \tag{30}$$

In the recent equation, $B$ denotes the right hand side of (21) accordingly. Consequently,

$$U^d = \left\| m_Q LA + Bq_L \right\|$$
$$q_{z_Q}^d = \frac{m_Q LA + Bq_L}{\left\| m_Q LA + Bq_L \right\|} \tag{31}$$

Therefore, desired trajectory of thrust force vector are calculated by the load position controller appropriately.

### 4.2. Trajectory Tracking Controller of Quadrotor in Slack Cable Condition

When cable is slack, load is not controllable but quadrotor can be controlled. If desired trajectory of the quadrotor is denoted by $r_Q^d$ then desired magnitude and direction of thrust force can be adopted as the next equation to control the quadrotor on its desired trajectory.

$$U = m_Q \left\| \ddot{r}_Q^s + ge_3 \right\|, q_{z_Q}^d = \frac{\ddot{r}_Q^s + ge_3}{\left\| \ddot{r}_Q^s + ge_3 \right\|} \tag{32}$$

In this equation, $\ddot{r}_Q^s$ is defined as

$$\ddot{r}_Q^s = \ddot{r}_Q^d - k_d^Q \dot{e}_{r_Q} - k_p^Q e_{r_Q}, e_{r_Q} = r_Q - r_Q^d \tag{33}$$

where $k_d^Q$ and $k_p^Q$ are positive constant gains.

### 4.3. Attitude Controller of Quadrotor

Although desired magnitude of the thrust force can be achieved directly, thrust force should be oriented in the desired direction by an attitude controller. To obtain desired rotation matrix, a unit vector as $q_c = \begin{bmatrix} -sin(\psi) & cos(\psi) & 0 \end{bmatrix}^T$ are defined for desired yaw angle $\psi^d$. It can be demonstrated that unit vector of x-axis of body fixed frame in the desired attitude can be calculated as

$$q_{x_Q} = \frac{q_c \times q_{z_Q}^d}{\left\| q_c \times q_{z_Q}^d \right\|} \tag{34}$$

Consequently, the desired rotation matrix is defined as

$$R_d = \begin{bmatrix} q_x^{Q^d} & q_z^{Q^d} \times q_x^{Q^d} & q_z^{Q^d} \end{bmatrix} \tag{35}$$

To design attitude controller, a configuration error function $\psi_R = \frac{1}{2} tr(I - R_d{}^T R)$ defined on SO(3) groups is implemented for Backstepping. It should be noted that $0 \le \psi_R \le 2$ similar to $\psi_L$. Derivative of this error function with respect to the time is $\dot{\psi}_R = e_R . e_\Omega$ in which $e_R$ and $e_\Omega$ are

$$\begin{aligned} e_R &= \frac{1}{2}(R_d{}^T R - R^T R_d)^{\vee} \\ e_\Omega &= \Omega - R^T R_d \Omega_d \\ \Omega_d &= (R_d{}^T \dot{R}_d)^{\vee} \end{aligned} \tag{36}$$

"$\vee$" is invert of "$\wedge$" operator mentioned formerly. If $\psi_R(0) \neq 2$ and $e_\Omega = -\frac{k_{\psi_R}}{2-\psi_R} e_R$ then $\dot{\psi}_R = -k_{\psi_R}\psi_R$ . It means that $\psi_R$ exponentially will converge to zero and rotation matrix will track its desired value appropriately. Therefore, a Lyapunov function is candidate for backstepping as

$$V_{\psi_R} = C_{\psi_R}\psi_R + \frac{1}{2}\left\| E_{\psi_R} \right\|^2 \tag{37}$$

in which $E_{\psi_R} = e_\Omega + \frac{k_{\psi_R}}{2-\psi_R} e_R$ . It can be shown that if

$$\begin{aligned} A_{\psi_R} &= -\frac{k_{\psi_R}}{\psi_R - 2}\dot{e}_R - \left(\frac{k_{\psi_R}}{(\psi_R - 2)^2}(e_R . e_\Omega) + C_{\psi_R}\right)e_R - k_{E_{\psi_R}} E_{\psi_R} \\ B_{\psi_R} &= \dot{R}^T R_d \Omega_d + R^T R_d \dot{\Omega}_d \end{aligned} \tag{38}$$

and

$$\dot{\Omega} = A_{\psi_R} + B_{\psi_R} \tag{39}$$

then

$$\dot{V}_{\psi_R} = -k_{\psi_R} C_{\psi_R}\psi_R - k_{E_{\psi_R}} \left\| E_{\psi_R} \right\|^2 \tag{40}$$

and $V_{\psi_R}$ exponentially converges to zero. If following inequality is satisfied then it can be guaranteed that $\psi_R$ will not be equal to 2 during tracking.

$$C_{\psi_R} > \frac{\frac{1}{2}\left\| E_{\psi_R}(0) \right\|^2}{2 - \psi_R(0)} \tag{41}$$

Consequently, torque input vector can be adopted as next equation appropriately via rotational dynamics of quadrotor and equation(39).

$$\tau = J_Q^{-1}(A_{\psi_R} + B_{\psi_R} + \Omega \times J_Q \Omega) \tag{42}$$

In this study, $k_{\psi_R}$ and $k_{E_{\psi_R}}$ are equal to $\frac{K_{\psi_R}}{\epsilon}$ and $\frac{K_{E_{\psi_R}}}{\epsilon}$ respectively for any positive constants $K_{\psi_R}$ , $K_{E_{\psi_R}}$ and $0<\epsilon<1$ . In other words, $\epsilon$ is singular perturbation parameter which guarantees rapid convergence of the quadrotor attitude to its desired configuration.

### 4.4. Slow Model of System

For full model of system, $\dot{V}_{\psi_R}$ is calculated from

$$\epsilon \dot{V}_{\psi_R} = -K_{\psi_R} C_{\psi_R} \psi_R - K_{E_{\psi_R}} \left\| E_{\psi_R} \right\|^2 \tag{43}$$

By setting $\epsilon = 0$ slow model of system is obtained in which $\psi_R = 0$. Consequently in slow model of system $R = R_d$ . In following, exponential stability of tracking errors in both of taut and slack cable conditions is examined for slow model. So, an argument of singularly perturbed systems is implemented to ensure that the load or quadrotor would track their desired trajectories by the designed control laws in full system.

### 4.5. Exponential Stability of Tracking Error of Quadrotor in Slack Cable Condition

For slow model of system in slack cable condition, one can write

$$\ddot{r}_Q = \ddot{r}_s^Q \rightarrow \ddot{e}_{r_Q} + k_d^Q \dot{e}_{r_Q} + k_p^Q e_{r_Q} \tag{44}$$

So, tracking errors will exponentially converges to zero. Since full model of system in slack cable condition satisfies all conditions of theorem 9.3 of [15], there is $\epsilon^*$ such that for all $\epsilon$ in $(0,\epsilon^*)$ interval, tracking errors of full model of system converges to zero exponentially.

### 4.6. Exponential Stability of Tracking Error of Load in Taut Cable Condition

In the slow model, $V_{\psi_L}$ converges exponentially to zero. Therefore, desired direction of cable will be achieved exponentially. Slow model of tracking error of load position in taut cable condition is

$$\ddot{e}_r = -k_d^r \dot{e}_r - k_p^r e_r + \frac{\|F\|}{m_Q + m_L} e_{q_L} \tag{45}$$

To find sufficient condition of successful trajectory tracking by load, a Lyapunov function is candidate as below:

$$V_{r_L} = \frac{k_p^r}{2} \|e_r\|^2 + \frac{1}{2} \|\dot{e}_r\|^2 + C_{r_L} e_r . \dot{e}_r \tag{46}$$

with $0 < C_{r_L} < \sqrt{k_p^r}$ . So,

$$\dot{V}_{r_L} = -k_p^r C_{r_L} \|e_r\|^2 - C_{r_L} k_d^r e_r . \dot{e}_r + (C_{r_L} - k_d^r) \|\dot{e}_r\|^2 + \frac{\|F\|}{m_Q + m_L} e_{q_L} . \dot{e}_r + \frac{C_{r_L} \|F\|}{m_Q + m_L} e_{q_L} . e_r \tag{47}$$

If $\psi_L < 1$ then

$$\dot{V}_{r_L} \le -k_p^r C_{r_L} \|e_r\|^2 + C_{r_L} k_d^r \|e_r\| \|\dot{e}_r\| + (C_{r_L} - k_d^r) \|\dot{e}_r\|^2 + \frac{\|F\|}{m_Q + m_L} \psi_L \|\dot{e}_r\| + \frac{C_{r_L} \|F\|}{m_Q + m_L} \psi_L \|e_r\| \quad (48)$$

Therefore, one can write next inequality easily.

$$\dot{V}_{r_L} + \dot{V}_{\psi_L} \le -z_{r_L}^T M_{r_L} z_{r_L} - z_{\psi_L}^T M_{\psi_L} z_{\psi_L} + z_{r_L}^T M_{r_L,\psi_L} z_{\psi_L} \quad (49)$$

In the recent inequality,

$$z_{r_L} = \begin{bmatrix} \|e_r\| \\ \|\dot{e}_r\| \end{bmatrix}, z_{\psi_L} = \begin{bmatrix} \psi_L \\ \|E_{\psi_L}\| \end{bmatrix}$$

$$M_{r_L} = \begin{bmatrix} k_p^r C_{r_L} & -\frac{C_{r_L} k_d^r}{2} \\ -\frac{C_{r_L} k_d^r}{2} & k_d^r - C_{r_L} \end{bmatrix}, M_{\psi_L} = \begin{bmatrix} k_{\psi_L} C_{\psi_L} & 0 \\ 0 & k_{E_{\psi_L}} \end{bmatrix} \quad (50)$$

$$M_{r_L,\psi_L} = \begin{bmatrix} C_{r_L}\alpha & 0 \\ 0 & \alpha \end{bmatrix}, \alpha = \max(\|F\|)$$

So, If

$$C_{r_L} < \min\left\{ \sqrt{k_p^r}, k_d^r, \frac{4k_p^r k_d^r}{(k_d^r)^2 + k_p^r} \right\}$$

$$\lambda_{\min}(M_{\psi_L}) > \frac{\|M_{r_L,\psi_L}\|^2}{4\lambda_{\min}(M_{r_L})} \quad (51)$$

then

$$\dot{V}_{r_L} + \dot{V}_{\psi_L} \le -\beta(V_{r_L} + V_{\psi_L}) \quad (52)$$

for $\beta < \lambda_{min}(M)$ and

$$M = \begin{bmatrix} M_{r_L} & \frac{1}{2} M_{r_L,\psi_L} \\ \frac{1}{2} M_{r_L,\psi_L} & M_{\psi_L} \end{bmatrix} \quad (53)$$

Consequently, Lyapunov function of slow model $V_{r_L} + V_{\psi_L}$ will converge to zero exponentially.

*Remark.2: If* $\psi_L(0) > 1$ *then* $\psi_L$ *will be less than 1 in a finite time* $t^*$ *as proven formerly. Hence, for successful trajectory tracking,* $z_{r_L}$ *should remain bounded in time interval* $[0, t^*]$.

Since full model of system satisfies all conditions of theorem 9.3 of [15], there is $\epsilon^*$ such that for all $\epsilon$ in $(0,\epsilon^*)$ interval, tracking errors of full model of system converges to zero exponentially.

## 5. SIMULATION

Simulations are performed in Simulink on an experimentally verified and complicated model of OS4 quadrotor which has been presented in [12]. In this model, all of aerodynamic forces and torques acting on the rotors such as hub force, rolling moments are included. Moreover, aerodynamic friction forces acting on the body of the quadrotor has been modeled. Also, gyroscopic effects of rotors rotation has been included in the model. Furthermore, dynamics of the motors considered in the model causes that the desired angular speeds of the rotors are achieved with a delay. More detailed information about this model can be found in study [12] .In the simulation, the quadrotor is hovering at the origin of the inertial of frame initially. Also, initial yaw angle of the quadrotor is zero. Therefore, initial rotation matrix is $I_{3\times3}$ . Moreover, load is supposed to be located on a surface at coordinate $\begin{bmatrix}-0.5 & -0.2 & 0\end{bmatrix}^T (m)$ . Furthermore mass of the load is 0.1 kg mass and it is attached to the quadrotor through a cable with one meter length.

### 5.1. Trajectory Generation for Cable Pulling Mission

The quadrotor is supposed to reach the point just one meter above the load at 8 seconds and stabilized there. Therefore, a minimum snap trajectory is generated for quadrotor as described formerly and is feed to controller. Desired trajectory of the quadrotor are depicted in ***Figure 2***.

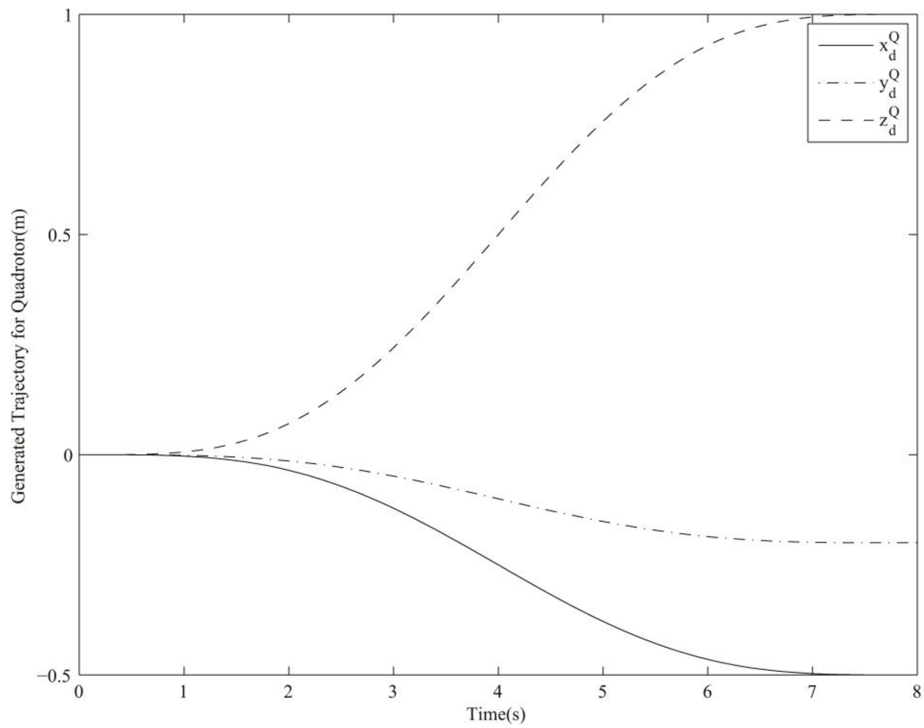


***Figure 2.*** *Desired trajectory for quadrotor to pull cable*

### 5.2. Trajectory Generation for Transport Mission

After that quadrotor stabilized at its final desired position, load should be transported to point $\begin{bmatrix}1 & -2 & 3\end{bmatrix}^T (m)$ and stabilized there 20 seconds after that transport mission starts. Generated trajectory for the load in transport mission is plotted in ***Figure 3***.

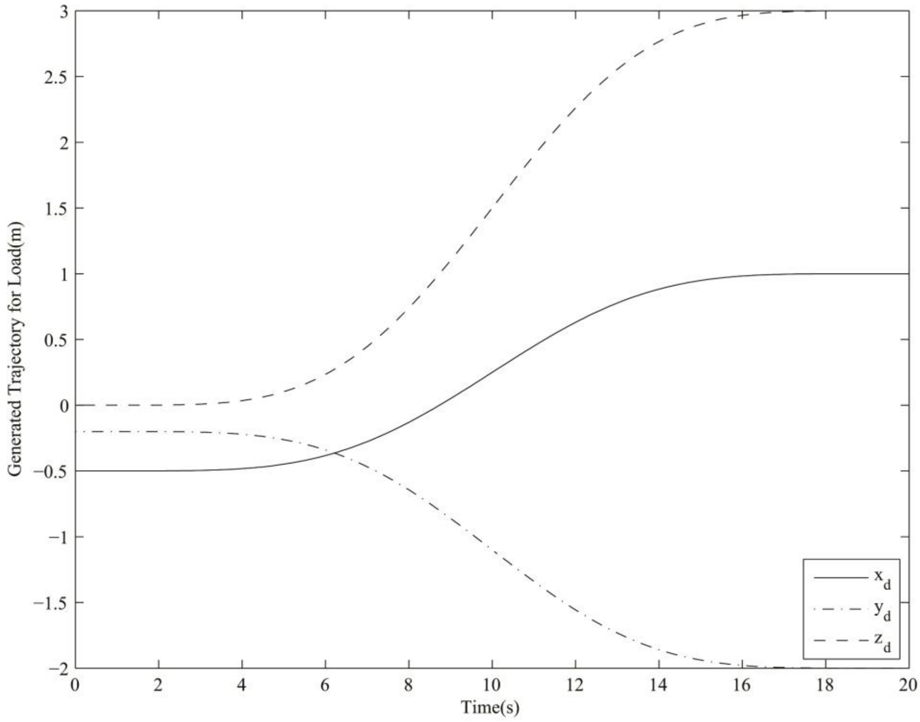


***Figure 3.*** *Desired trajectory for load in transport mission*

### 5.3. Pulling Cable Mission

Parameters of the controller in this part of mission are tuned by response optimization facility of Simulink such that tracking error is less than 4cm. Controller parameters are listed in ***Table 1***.

***Table 1.*** *Parameters of controller in cable pulling mission*

| | | |
|---|---|---|
| $k_p^Q$ | $k_d^Q$ | $k_{\psi_R}$ |
| 40 | 4 | 1.7 |
| $k_{E_{\psi_R}}$ | $C_{\psi_R}$ | ò |
| 2.05 | 2.6 | 0.5 |

Simulation is stopped about one second before t=8s because cable is taught and distance between the quadrotor and the load becomes equal to cable length due to tracking error. Trajectory tracking error of quadrotor versus time are plotted in ***Figure 4***.

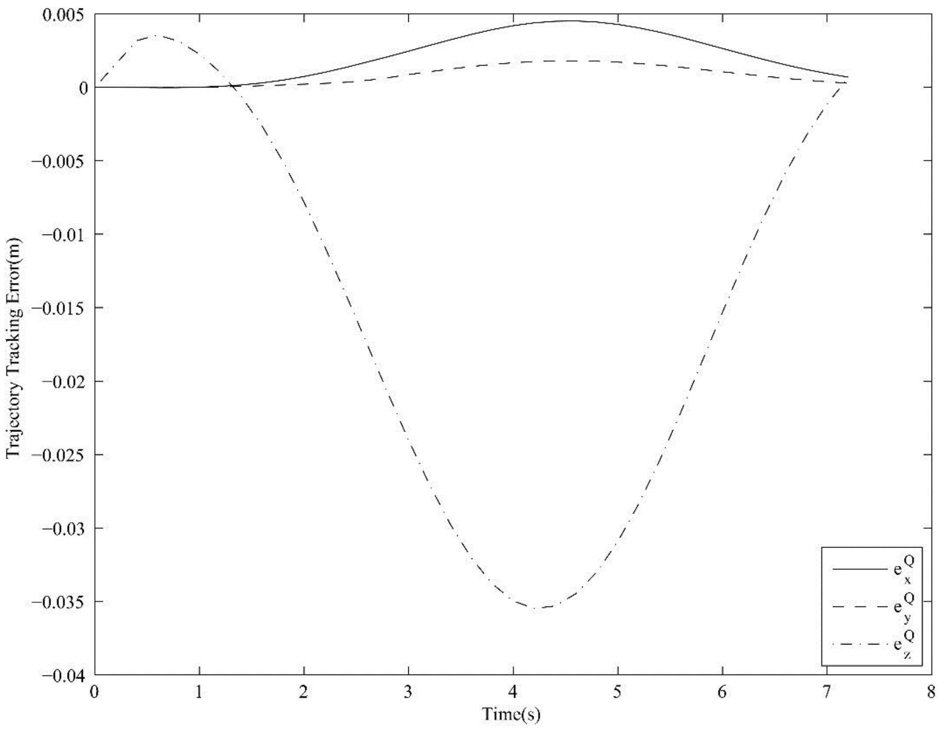


***Figure 4**. Trajectory tracking error for quadrotor in pulling cable mission*

Also, traveled path by quadrotor is compared with its desired path in ***Figure 5***.

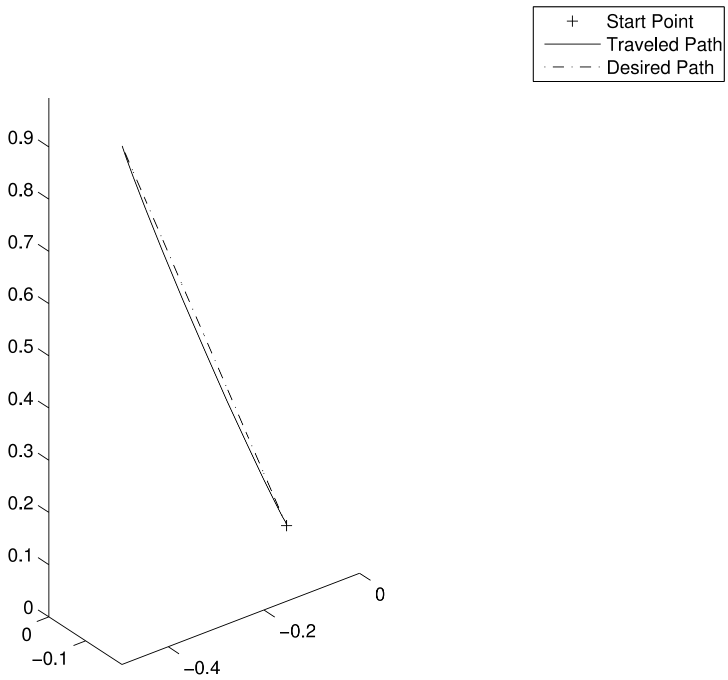


***Figure 5**. Traveled path by quadrotor versus desired path*

Speed of rotors during cable pulling mission are depicted in ***Figure 6*** appropriately. As depicted in this figure, speeds of rotors are only a bit higher than hovering speeds because the generated trajectory is minimum snap.

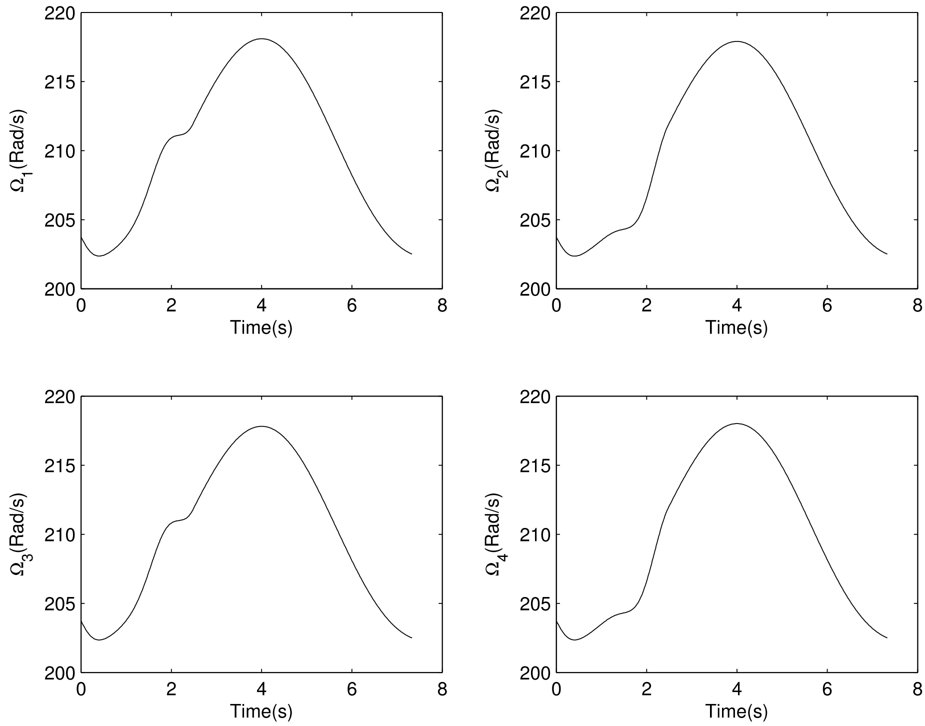


***Figure 6.*** *Speed of rotors in pulling cable mission*

### 5.4. Transport Mission

States of the system are changed when the cable is taught according to the model proposed for switching surface of system dynamics. The generated desired trajectory for the load is fed to the controller accordingly and transport mission starts. Also, simulation time is reset to zero. Controller parameters of transport mission are tuned by response optimization facility of Simulink too such that tracking error is less than 25cm during mission. These parameters are listed in ***Table 2***.

***Table 2.*** *Parameters of controller in transport mission*

| $k_p^r$ | $k_d^r$ | $k_{\psi_L}$ | $k_{\psi_R}$ | $k_{E_{\psi_L}}$ | $k_{E_{\psi_R}}$ | $C_{\psi_L}$ | $C_{\psi_R}$ | ò |
|---|---|---|---|---|---|---|---|---|
| 10 | 6 | 0.75 | 28.1 | 3.6 | 48.1 | 1 | 0.07 | 0.1 |

Plot of trajectory tracking error for the load shows that load tracked its desired trajectory well by designed control laws (***Figure 7***).

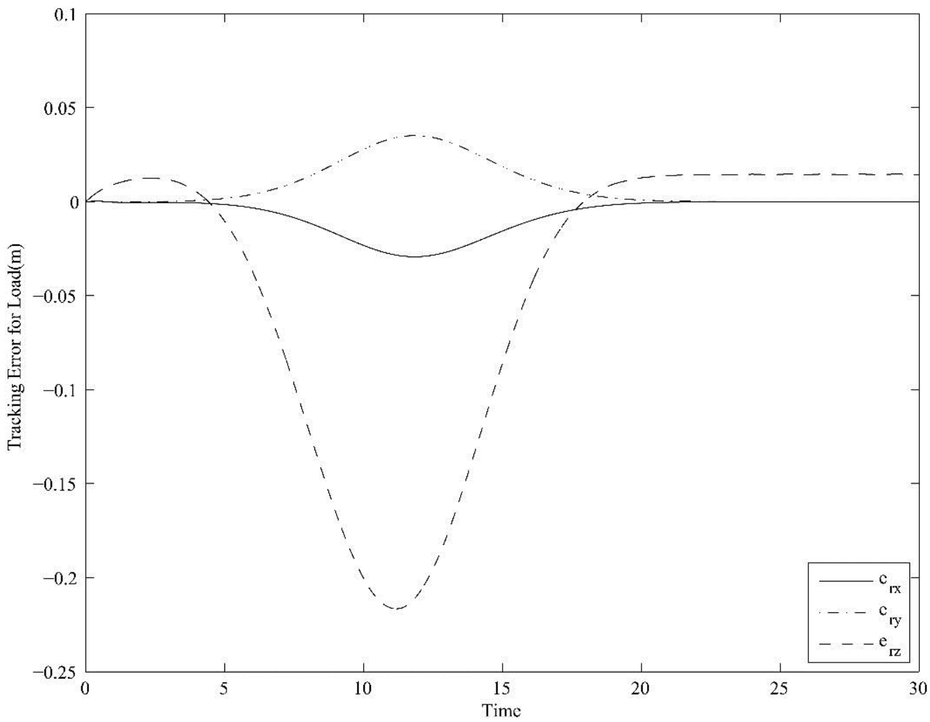


***Figure 7.*** *Trajectory tracking error for load during transport mission*

As depicted in Figure7, final steady state error in z-direction is not zero which can be improved by adding an integrator of tracking error to controller probably. Cable direction tracking error $\psi_L$ and quadrotor attitude tracking error $\psi_R$ are close to zero during the mission as depicted in ***Figure 8*** and ***Figure 9***.

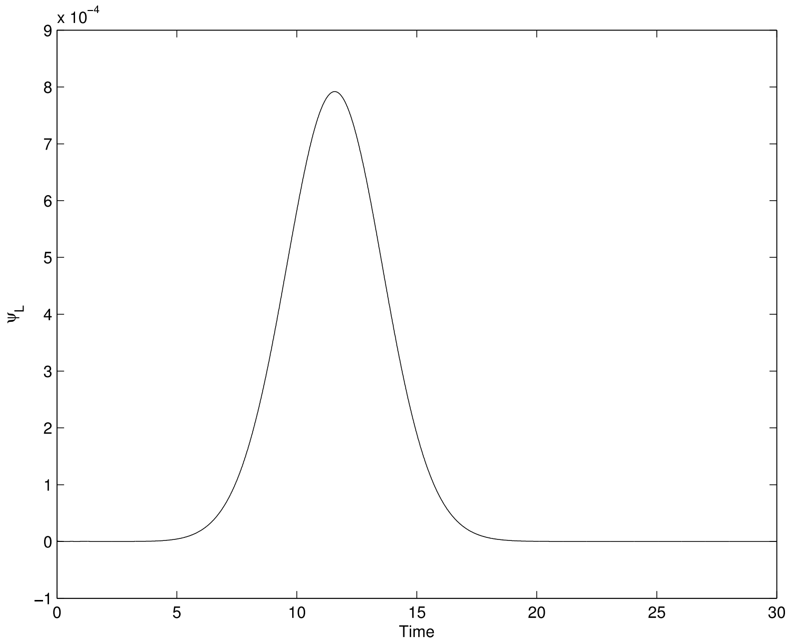


***Figure 8.*** *Cable direction tracking error*

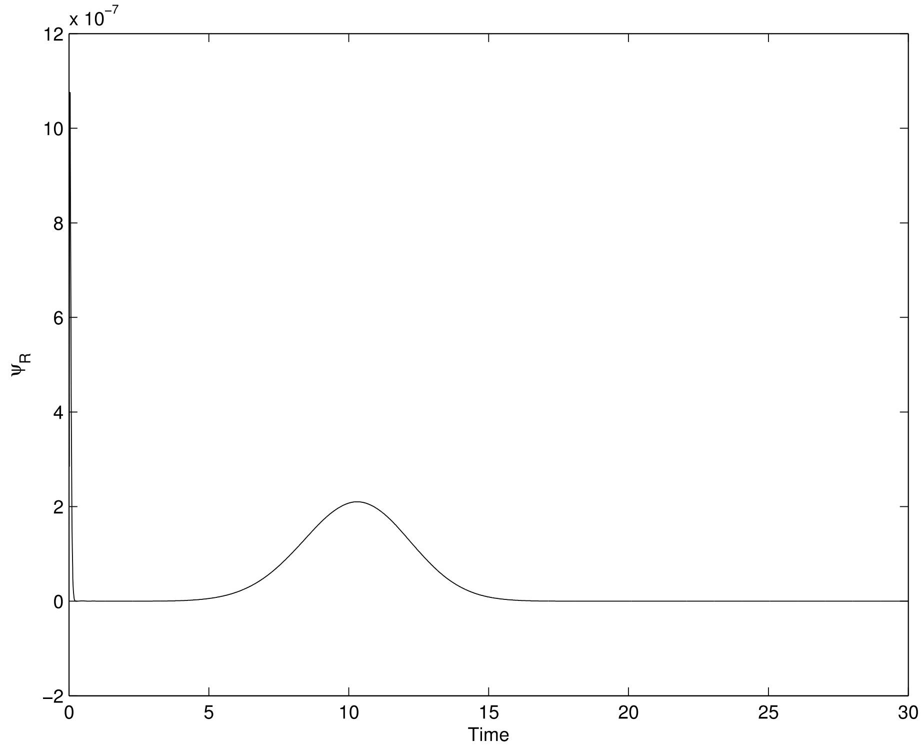


***Figure 9.*** *Quadrotor attitude tracking error*

Also, one can observe in ***Figure 10*** that during the mission, rotors speeds are only a bit more than hovering speeds because desired trajectory has been generated such that control inputs are minimum.

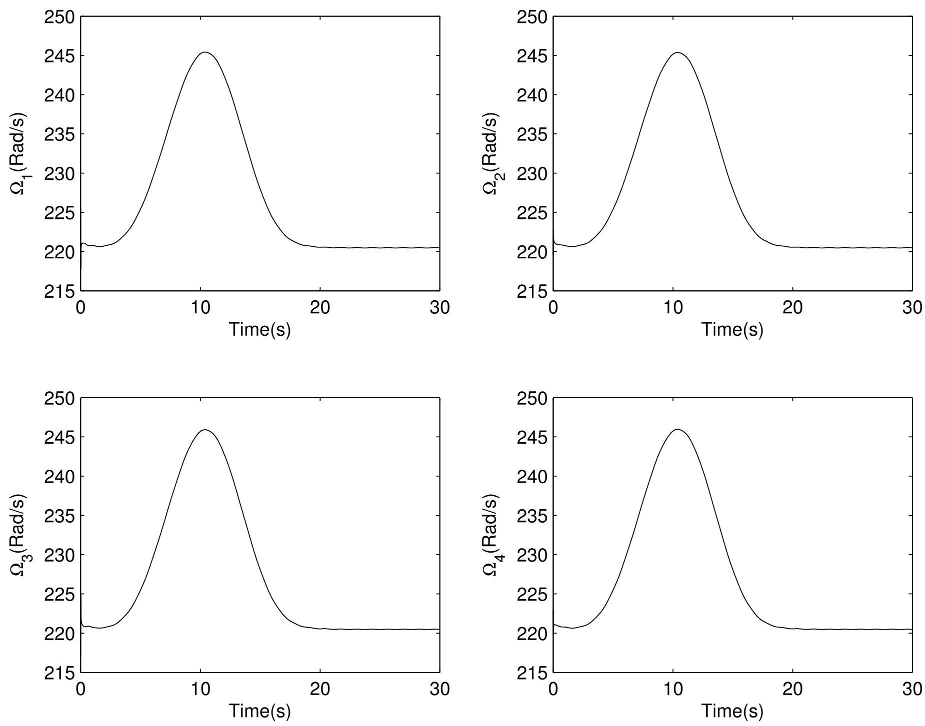


***Figure 10.*** *Speed of rotors in transport mission*

## 6. CONCLUSION

In this paper, a special transport mission of a point-mass load suspended from a quadrotor through a cable has been studied. This mission includes picking up the load from ground and locating it on a specified place in space at a specified time. Proper trajectories for the quadrotor and the load have been generated to minimize torque control inputs of the quadrotor. Trajectory tracking controllers have been designed by Backstepping and configuration error functions defined on $S^2$ and SO(3) groups. Designed attitude controller of the quadrotor is not subjected to singularities of common attitude controllers. Moreover, sufficient conditions for exponential convergence of tracking errors of system to zero have been derived by using Lyapunov exponential stability theorem together with an argument about singularly perturbed system. Simulations performed on an experimentally verified model of OS4 quadrotor have examined accuracy of the proposed trajectory generation procedures and capability of the presented control laws in tracking trajectories completely. Experimental implementation of the proposed trajectory generation methods and designed controllers in a real mission is interesting and may be concerned in future studies.

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