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A NOTE ON THE DEPTH OF A SOURCE ALGEBRA OVER ITS DEFECT GROUP

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Abstract. By results of Boltje and Külshammer, if a source algebra A of a principal p-block of a finite group with a defect group P with inertial quotient E is a depth two extension of the group algebra of P, then A is isomorphic to a twisted group algebra of the group $P \rtimes E$. We show in this note that this is true for arbitrary blocks. We observe further that the results of Boltje and Külshammer imply that A is a depth two extension of its hyperfocal subalgebra, with a criterion for when this is a depth one extension. By a result of Watanabe, this criterion is satisfied if the defect groups are abelian.

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Let p be a prime and $\mathcal{O}$ a complete local principal ideal domain with an algebraically closed residue field k of characteristic p, allowing the case $\mathcal{O} = k$. We will make without further comment use of the fact that by [9, II, Prop. 8], the canonical group homomorphism $\mathcal{O}^\times \to k^\times$ splits canonically, and hence group cohomology with coefficients in $k^\times$ can be viewed as cohomology with coefficients in $\mathcal{O}^\times$. Following terminology in [4], a ring extension $B \to A$ is called *of depth one* if A is isomorphic, as a B-B-bimodule, to a direct summand of B^n for some positive integer n, and a ring extension $B \to A$ is called of *of depth two* if $A \otimes_B A$ is isomorphic, as an A-B-bimodule, to a direct summand of A^n, for some positive integer n. Tensoring by $A \otimes_B -$ shows that a ring extension of depth one is also an extension of depth two.

Let A be a source algebra of a block algebra over $\mathcal{O}$ of a finite group, with a defect group P. Boltje and Külshammer showed in [2, 2.4] that if A is isomorphic to a twisted group algebra of the form $\mathcal{O}_\alpha(P \rtimes E)$ for some p'-subgroup E of $\mathrm{Aut}(P)$ and some $\alpha \in H^2(E; k^\times)$, inflated trivially to $P \rtimes E$, then the canonical map $\mathcal{O}P \to A$ is an extension of depth two. Moreover, they showed that the converse holds for principal blocks. The following result shows that this converse holds for arbitrary blocks. See for instance [10, §11, §38] and [5, §6, §7] for background material on the Brauer homomorphism Br_P and fusion in source algebras.

Theorem 1. *Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b and $A = i\mathcal{O}Gi$ a source algebra of b, where i is a primitive idempotent in the P-fixed point algebra $(\mathcal{O}Gb)^P$ such that $\mathrm{Br}_P(i) \neq 0$. The following are equivalent:*

(i) *The ring extension $\mathcal{O}P \to A$ induced by the canonical map $P \to A^\times$ is of depth two.*

(ii) *The ring extension $kP \to k \otimes_{\mathcal{O}} A$ induced by the canonical map $P \to A^\times$ is of depth two.*

(iii) *There is an isomorphism of interior P-algebras $A \cong \mathcal{O}_\alpha(P \rtimes E)$ for some p'-subgroup E of $\mathrm{Aut}(P)$ and some $\alpha \in H^2(E; k^\times)$ inflated trivially to $P \rtimes E$.*

(iv) *There is an isomorphism of interior P-algebras $k \otimes_{\mathcal{O}} A \cong k_\alpha(P \rtimes E)$ for some p'-subgroup E of $\mathrm{Aut}(P)$ and some $\alpha \in H^2(E; k^\times)$ inflated trivially to $P \rtimes E$.*

Proof. The equivalence of (iii) and (iv) is an immediate consequence of results of Puig (either apply [7, 14.6] over both $\mathcal{O}$ and k, or use the lifting property [6, 7.8] for source algebras). Statement (iv) implies (i) and (ii) by Boltje and Külshammer [2, 2.4]. The implication (i) $\Rightarrow$ (ii) is trivial. It suffices to show that (ii) implies (iv). We may therefore assume that $\mathcal{O} = k$. Suppose that (ii) holds but that (iv) does not hold. As an A-kP-bimodule, A is indecomposable since $1_A = i$ is primitive in A^P. Thus, if (ii) holds, then the Krull-Schmidt theorem implies that any indecomposable direct summand of $A \otimes_{kP} A$ as an A-kP-bimodule is isomorphic to A as an A-kP-bimodule. Now if (iv) does not hold, then by [7, 14.6], there is a proper subgroup Q of P and an injective group homomorphism φ from Q to P such that the indecomposable kP-kP-bimodule $kP \otimes_{kQ} ({}_\varphi kP)$ is isomorphic to a direct summand of A as a kP-kP-bimodule. Thus $A \otimes_{kQ} ({}_\varphi kP)$ is isomorphic to a direct summand of $A \otimes_{kP} A$ as an A-kP-bimodule, and hence so is $Aj \otimes_{kQ} ({}_\varphi kP)$, where j is a primitive idempotent in A^Q. Since Aj is indecomposable as an A-kQ-bimodule, so is the $k(G \times Q)$-module kGj. Green's indecomposability theorem implies that the $k(G \times P)$-module $kGj \otimes_{kQ} ({}_\varphi kP)$ is indecomposable. Using that multiplication by i yields a Morita equivalence between kGb and A it follows that the A-kP-bimodule $Aj \otimes_{kQ} ({}_\varphi kP)$ is also indecomposable, hence isomorphic to A as an A-kP-bimodule, by the above. Since $\mathrm{Br}_P(i) \neq 0$ this is, however, only possible if $Q = P$, a contradiction. $\square$

For the sake of completeness, we mention that the depth of an extension $D \to A$, where D is a hyperfocal subalgebra (cf. [8]) in a source algebra A of a block of a finite group, can be determined essentially as an application of the methods from [1] and [2]. The first statement of the following proposition is a special case of [1, 1.5].

Proposition 2. *Let A be a source algebra of a block of a finite group algebra over $\mathcal{O}$ with defect group P, and let D be a hyperfocal subalgebra of A. The following hold.*

(i) *The extension $D \to A$ is of depth two.*
(ii) *The extension $D \to A$ is of depth one if and only if P acts by inner automorphisms on D.*

Proof. As mentioned above, statement (i) is a special case of [1, 1.5], as A is P/Q-graded, with D as 1-component. Since the argument is short and some parts of the notation will be useful in the proof of (ii), we sketch this briefly. We identify P with its canonical image in $A^\times$. The following definitions and facts on the hyperfocal subalgebra D of A are from [8]. The subalgebra D is P-stable, and the group $Q = P \cap D^\times$ is the $\mathcal{F}$-hyperfocal subgroup of P, where $\mathcal{F}$ is the fusion system of A on P. An immediate consequence of these properties is that D is indecomposable as an $\mathcal{O}$-algebra. Indeed, we have $D^P \subseteq A^P$, which is local, and hence P permutes the blocks of D transitively. But we also have $\mathrm{Br}_P(1_A) \neq 0$, and hence D has a unique block. By [8, Theorem 1.8] we have $A = \oplus_{u \in [P/Q]} Du$, where $[P/Q]$ is a set of representatives in P of P/Q. Since D is P-stable, this is a decomposition of A as a D-D-bimodule. Thus $A \otimes_D A = \oplus_{u \in [P/Q]} A \otimes_D Du$ is a decomposition of $A \otimes_D A$ as an A-D-bimodule. For $u \in P$, a trivial verification shows that the A-D-bimodule $A \otimes_D Du$ is isomorphic to A via the map sending $a \otimes du$ to adu, where $a \in A$ and $d \in D$. Thus any indecomposable direct summand of the A-D-bimodule $A \otimes_D A$ is isomorphic to a direct summand of A as an A-D-bimodule. This proves (i). The summands Du in the D-D-bimodule decomposition $A = \oplus_{u \in [P/Q]} Du$ are all indecomposable as D-D-bimodules. Indeed, D is indecomposable by the above, and Du is isomorphic to the image of D under the Morita equivalence on $\mathrm{mod}(D \otimes_{\mathcal{O}} D^{\mathrm{op}})$ obtained from twisting the right D-module structure by the automorphism induced by conjugation with u. Thus the extension $D \to A$ is of depth one if and only if $Du \cong D$ as D-D-bimodules, for all $u \in [P/Q]$, hence for all $u \in P$. By standard facts on automorphisms (cf. [3, §55A]) this is equivalent to the condition that u induces an inner automorphism of D, for all $u \in P$. This proves (ii). $\square$

In conjunction with a result of Watanabe [11], this yields the following consequence.

Corollary 3. *With the notation of Proposition 2, if P is abelian, then the extension $D \to A$ is of depth one.*

Proof. By [11, Theorem 2], if P is abelian, then P acts as inner automorphisms on D. Thus the result follows from Proposition 2 (ii). □

Remark 4. *What we have called depth two in this note is called right D2 in* [4, 3.1]*, with left D2 being the obvious analogue, requiring $A \otimes_B A$ to be a direct summand, as a B-A-bimodule, of A^n for some positive integer n. It is easy to see directly that left and right D2 are equivalent conditions for the extensions $\mathcal{O}P \to A$ and $D \to A$ considered in the results above; this follows also from a more general result in* [4, 6.4]*. See* [2, §2.3] *for a related discussion.*

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