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Abelian product of free Abelian and free Lie algebras

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Abstract

Let F_n be a free Lie Algebra of finite rank n and A be a free abelian Lie algebra of finite rank $m \geq 0$. We investigate the properties of the generating sets and subalgebras of the abelian product $A *_{ab} F_n$. Moreover these properties are used to solve the membership problem for $A *_{ab} F_n$.

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1. Introduction

Let F_n be a free Lie algebra with a free generating set $X = \{x_1, x_2, ..., x_n\}$ over a field of K of characteristic zero and A be a free abelian Lie algebra generated by a set $Y = \{y_1, ..., y_m\}$ over K. Let D be the cartesian subalgebra of the free product $A * F_n$ of A and F_n, that is ,the kernel of the canonical homomorphism from $A * F_n$ onto the direct sum $A \oplus F_n$. The $k-th$ solvable product of A and F_n is defined as $(A*F_n)/\delta^k(A*F_n) \cap D$, where $\delta^k(A * F_n)$is the $k - th$ term of the derived series of $A * F_n$. In the case $k = 0$ the $k - th$ solvable product becomes $(A * F_n)/D$, which is isomorphic to the direct sum $A \oplus F_n$. However for the sake of compliance with the sprit of our work we shall refer to $0-th$ solvable product of A and F_n as abelian product and we denote it by $A *_{ab} F_n$. Free abelian Lie algebras and free Lie algebras have been extensively studied in the literature. Many questions that admit simple solutions when dealing with A and F_n individually, require far more involved solutions over abelian product $A *_{ab} F_n$ of A and F_n. This is the case, when one considers subalgebras and generating sets: Generating sets of $A *_{ab} F_n$ have common properties with generating sets of free abelian and free Lie algebras. It is well known that every subalgebra of a free abelian Lie algebra is free abelian and every subalgebra of free Lie algebra is again free. These two facts lead to the same property

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for abelian product $A *_{ab} F_n$. As another example we may consider the study of test elements and test sets. The Lie algebra $A *_{ab} F_n$ doesn't have test elements but in the free Lie algebra F_n there are many test elements. Interest in the test ranks of the abelian product $A *_{ab} F_n$ is explained in [4]. In [5,6] test sets and test ranks of solvable and metabelian products of groups were studied.

In this paper we investigate subalgebras of abelian products of the form $A *_{ab} F_n$ and we define there several properties shared by both families of free abelian and free Lie algebras. As a consequence we observe that the Lie algebras of the form $A *_{ab} F_n$ have solvable membership problem. The motivation of this work is based on the studies in groups. We use the methods introduced in [3] to prove our results.

For any nonempty subset Z of a Lie algebra by $< Z >$ we mean the subalgebra generated by Z . For the rank of any free Lie algebra B we write $rank(B)$.

2. Abelian Product of A and F_n

Let $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$ be disjoint sets, where $m, n \geqslant 0$. The ideas of this section are similar to the corresponding ideas in group theory [3], but have subtle differences. Consider the Lie algebra L defined by the presentation

$$L =< Y \cup X : [y_i, y_j], [y_i, x_s], 1 \leq i, j \leq m, 1 \leq s \leq n, y_i \in Y, x_s \in X > .$$

Let A and F_n be the subalgebras of L generated, respectively, by Y and X. We shall refer to $A =< Y >$ and $F_n =< X >$ as the free-abelian and free parts of L, respectively. It is clear that the Lie algebra L is the abelian product of A and F_n, namely $A *_{ab} F_n$ .Therefore every element g of L can be written as $g = u + v$ in a unique way, where $u \in A$, $v \in F_n$. Naturally we can write g as $\sum \alpha_i y_i + v$ where $\alpha_i \in K$, $y_i \in Y, v \in F_n$. By straightforward computations we see that the center of L is A.

2.1. Definition. Let (Z, T) be a pair of subsets of L. If

i) Z is an abelian basis of the center of L,

ii) T is a free generating set of a free subalgebra of L,

iii) $L =< Z \cup T >$,

then the pair (Z, T) of subsets of L is called a minimal generating pair of L. In this case we shall say that $Z \cup T$ is a minimal generating set of L.

Let (Z, T) be a minimal generating pair of L. From definition it follows that $< Z > \cap < T >= \{0\}$ and $Z \cap T = \emptyset$, since $< Z > \cap < T >$ is contained in the center of L, but no nontrivial element of $< T >$ belongs to it. Since we can consider the set Z as the elements in $Z \cup T$ which belong to the center of L, and T as the remaining elements then the set $Z \cup T$ is linearly independent modulo the derived subalgebra of L . So L cannot be generated by less than $|Z| + |T|$ elements, where $|Z|$ and $|T|$ are cardinalities of Z and T , respectively. Therefore the set $Z \cup T$ is a minimal generating set of L defined by the minimal generating pair (Z, T). We shall refer to $|Z| + |T|$ as the rank of L and we denote it by $rank(L)$.

The following lemma shows that the ranks of the free abelian and free parts of L are invariants of L.

2.2. Lemma. *Let A, B be free abelian Lie algebras and F and G be free Lie algebras of ranks at least 2. Then the Lie algebra $A *_{ab} F$ is isomorphic to $B *_{ab} G$ if and only if $rank(A) = rank(B)$ and $rank(F) = rank(G)$*

The proof of Lemma 2.2 is straightforward. So we omit it.

2.3. Corollary. *Every abelian product M of a nonzero free abelian Lie algebra with a free Lie algebra has a minimal generating pair. Moreover, every minimal generating pair*

(Z,T) satisfies $rank(M) = |Z| + |T|$, where $|Z|$ and $|T|$ are cardinalities of Z and T , respectively.

Proof. Without loss of generality we consider the Lie algebra $L = A *_{ab} F_n$, where $A \neq \{0\}$. It is clear that $Y \cup X$ is a minimal generating set and (Y, X) is a minimal generating pair. If (Z,T) is a minimal generating pair of L then by definition, $L =< Z > *_{ab} < T >$, $< Z >$ is a free abelian Lie algebra of rank $|Z|$ and $< T >$ is a free Lie algebra of rank $|T|$. Hence by Lemma 2.2, $rank(L) = |Z| + |T|$. □

2.4. Proposition. *Every nontrivial subalgebra of L is an abelian product of a free abelian Lie algebra and a free Lie algebra.*

Proof. Let H be a subalgebra of L. Clearly if $|X| = 0, 1$ then L is free abelian and so H is free abelian. if $|Y| = 0$ then H is a free subalgebra. So the result follows. Assume $|X| \geq 2$. Consider the inclusion map $i : A \to L$ and the projection $\pi : L \to F_n$. Since for every $a \in A$ and $f \in F_n$, $i(a) = a$ and $\pi(a + f) = f$ we have

$$Ker\pi = Imi = A.$$

Restricting π to the subalgebra H we get $\{0\} \subseteq Ker\pi_{|H} = Ker\pi \cap H = A \cap H \subseteq A$ and $\{0\} \subseteq \pi(H) \subseteq F_n$. Therefore $Ker\ \pi_{|H}$ is a free abelian Lie algebra and $\pi(H)$ is a free Lie algebra. Since $\pi(H)$ is free, $\pi_{|H}$ has a splitting $\alpha : \pi(H) \to H$ sending back each element of a chosen free generating set of $\pi(H)$ to an arbitrary preimage. Hence α is injective and $\alpha\pi(H) \cong \pi(H)$. Every element h of H can be decomposed as $h = h - \alpha\pi(h) + \alpha\pi(h)$. Clearly $h - \alpha\pi(h) \in Ker\pi_{|H}$. Thus

$$H = Ker\pi_{|H} *_{ab} \alpha\pi(H)$$

is the free abelian product of $Ker\pi_{|H}$ with $\alpha\pi(H)$. This completes the proof. □

2.5. Remark. Proposition 2.4 gives us a way of decomposing H into an abelian product of a free abelian subalgebra and a free subalgebra:

$$H = (H \cap A) *_{ab} \alpha\pi(H) \tag{2.1}$$

where $H \cap A$ and $\alpha\pi(H)$ are the free abelian and free parts of H, respectively. This decomposition gives a characterization of minimal generating sets and ranks of an arbitrary subalgebra H of L.

2.6. Corollary. *Let H be a subalgebra of L and E a subset of H. Then E is a minimal generating set of H in the sense of Definition 2.1 if and only if $E = E_A \cup E_{\alpha\pi(H)}$ or E can be transformed into a set of the form $E_A \cup E_{\alpha\pi(H)}$,where E_A is a basis of $H \cap A$ and $E_{\alpha\pi(H)}$ is a free generating set of $\alpha\pi(H)$, for a certain splitting α as in the proof of Proposition 2.4.*

Proof. By Proposition 2.4, H is a free abelian Lie algebra or a free Lie algebra or it is of the form $H = (H \cap A) *_{ab} \alpha\pi(H)$. Let E be a subset of the subalgebra H of L.If E can be transformed into a set of the form $E_A \cup E_{\alpha\pi(H)}$ then without loss of generality we can assume that $E = E_A \cup E_{\alpha\pi(H)}$,where E_A is a basis of $H \cap A$ and $E_{\alpha\pi(H)}$ is a free generating set of $\alpha\pi(H)$.Then $(E_A, E_{\alpha\pi(H)})$ is a minimal generating pair by Definition 2.1 and $E = E_A \cup E_{\alpha\pi(H)}$ is a minimal generating set. Suppose now that E is a minimal generating set of H in the sense of Definition 2.1. Then it is defined by a minimal generating pair (Z,T). Thus $E = Z \cup T$. We now consider the decomposition (2.1) for a choosen splitting α. If $rank(\pi(H)) = 0$, then H is abelian, Z is an abelian basis of H and T is empty. In this case $H \cap A$ having Z as an abelian basis. Taking $Z = E_A$

and $T = E_{\alpha\pi(H)} = \emptyset$ leads the result. If $rank(\pi(H)) = 1$ then H is again abelian and Z is an abelian basis for H. Let $Z = \{z_1, ..., z_r\}$. Since $rank(\pi(H)) = 1$, each z_i is in the form $c_i + \alpha_i w$, where c_i is a linear combination of elements of Y, $\alpha_i \in K$ and $w \in \pi(H)$. Since $rank(\pi(H)) = 1$, at least one of the elements Z is of the form $c_i + \alpha_i w$ where $\alpha_i \neq 0$. If exactly one element in Z is in the form $c_i + \alpha_i w$,where $\alpha_i \neq 0$, then take $E_{\alpha\pi(H)} = \{\ c_i + \alpha_i w\}$and $E_A = Z\backslash\{\ c_i + \alpha_i w\}$. If at least two α_i's nonzero then without loss of generality we may assume that the set Z is in the form $\{c_1+\alpha_1 w, ..., c_s+\alpha_s w, c_{s+1}, ..., c_r\}, s \geq 2, \alpha_i \neq 0, 1 \leq i \leq s$. Applying the transformation θ defined as

$$\begin{aligned} \theta \quad : \quad & z_i \to \alpha_{i+1} z_i - \alpha_i z_{i+1}, i = 1, .., s-1, \\ & z_j \to z_j,\ j \geq s \end{aligned}$$

we can transform the set Z into the form $\{a_1, ..., a_{s-1}, c_s + \alpha_s w, c_{s+1}, ..., c_r\}$, where $a_1, ..., a_{s-1}$ are linear combinations of elements of Y. Choosing $E_A = \{a_1, ..., a_{s-1}, c_{s+1}, ..., c_r\}$ and $E_{\alpha\pi(H)} = \{c_s + \alpha_s w\}$, we obtain the result. In the case $rank(\pi(H)) > 1$ the restriction $\pi_{|<T>} :< T >\to \pi(H)$ is an isomorphism. Now we take the pull back α as $\alpha = \pi^{-1}_{|<T>}$. Then $E_{\alpha\pi(H)}$ is a free generating set of $\alpha\pi(H)$. Hence the result follows. □

2.7. Corollary. *Let H be a subalgebra of L and (Z, T) be a minimal generating pair of*

H. Then $rank(H) = |Z| + |T|$, where $0 \leq |Z| \leq m$ and
i) in case of $n = 0, 1 : 0 \leq |T| \leq n$
ii)in case of $n \geq 2\ :\ 0 \leq |T| \leq \varkappa_0$.

2.8. Corollary. *Let H be a subalgebra of L. Then H is finitely generated if and only if $\pi(H)$ is finitely generated.*

Proof. Let H be a subalgebra of L. Consider the decomposition $H = Ker\pi_{|H} *_{ab} \alpha\pi(H)$. If H is finitely generated then by Corollary 2.6 it has a finite minimal generating set E which is of the form $E = E_{Ker\pi_{|H}} \cup E_{\alpha\pi(H)}$,where $E_{Ker\pi_{|H}}$ is a basis of $Ker\pi_{|H}$ and $E_{\alpha\pi(H)}$ is a free generating set of $\alpha\pi(H)$. Thus $\alpha\pi(H)$ is finitely generated. Since $\pi(H)$ is isomorphic to $\alpha\pi(H)$ then $\pi(H)$ is also finitely generated.

Now assume that $\pi(H)$ is finitely generated. Any minimal generating set E of H is of the form $E = E_{Ker\pi_{|H}} \cup E_{\alpha\pi(H)}$ or it can be transformed into this form, where $E_{Ker\pi_{|H}}$ is a generating set of $Ker\pi_{|H}$ and $E_{\alpha\pi(H)}$ is a free generating set of $\alpha\pi(H)$. Any subalgebra of a finitely generated free abelian Lie algebra is finitely generated. Using the fact

$$< E_{Ker\pi_{|H}} >= Ker\pi_{|H} = H \cap A \subseteq A$$

we obtain that $E_{Ker\pi_{|H}}$ is finite. Therefore E is finite. □

2.9. Proposition. *Let L be finitely generated and H be a subalgebra of L which is given by a finite set of generators. Then there is an algorithm computing a generating set for H and writing the new and old generators in terms of each other.*

Proof. Let H be a subalgebra of L which is given by a finite set of generators $c_1 + w_1, ..., c_p + w_p$, where $p \geq 1$, $c_i \in A$, $w_i \in F_n, i = 1, 2, ..., p$. If all w_i's are zero then, H is free abelian and $c_1, ..., c_p$ are generators of H. Applying elementary Lie transformations to the set $\{c_1, ..., c_p\}$,(see[2]), we get a minimal generating set for H.

Now assume that $w_i \neq 0$ for at least one i. We can obtain a free subset $\{u_1, ..., u_r\}$ of F_n by applying suitable elementary Lie transformations to the set $\{w_1, ..., w_p\}$ (see [7]), where $0 \leq r \leq p$. Clearly $\{u_1, ..., u_r\}$ is a free generating set of $\pi(H) =< w_1, ..., w_p >$. We have an effective way to express the elements $u_1, ..., u_r$ as words on $w_1, ..., w_p$, say

$$u_j = u_j(w_1, ..., w_p),\ j = 1, ..., r$$

as well as express the elements $w_1, ..., w_p$ in terms of $u_1, ..., u_r$, say

$$w_i = f_i(u_1, ..., u_r), i = 1, ..., p.$$

We now consider the map

$$\alpha : \pi(H) \to H \ , \ \alpha(u_j) = u_j(c_1 + w_1, ..., c_p + w_p), \ j = 1, ..., r.$$

Since

$$u_j(c_1 + w_1, ..., c_p + w_p) = a_j + u_j(w_1, ..., w_p),$$

where a_j, $j = 1, ..., r$, are linear combinations of $c_1, ..., c_p$, then

$$\begin{aligned} \alpha\pi(c_i + w_i) &= \alpha(w_i) \\ &= \alpha(f_i(u_1, ..., u_r)) \\ &= f_i(a_1 + u_1, ..., a_r + u_r) \\ &= d_i + f_i(u_1, ..., u_r) \end{aligned}$$

where d_i is a linear combination of $a_1, ..., a_r$, $i = 1, ..., p$. Hence the mapping α can serve as a splitting.

We now determine a generating set for $Ker\pi/_H = A \cap H$. For each given generator $c_i + w_i$, calculate $c_i + w_i - \alpha\pi(c_i + w_i)$:

$$\begin{aligned} c_i + w_i - \alpha\pi(c_i + w_i) &= c_i + w_i - d_i - f_i(u_1, ..., u_r) \\ &= c_i - d_i \end{aligned}$$

Thus $c_i + w_i - \alpha\pi(c_i + w_i) \in Ker\pi/_H$.

Since $H = Ker\pi_{|H} *_{ab} \alpha\pi(H)$ then the set $\{s_1, ..., s_p\}$ generates $Ker\pi_{|H} = H \cap A$,where $s_i = c_i - d_i$, $i = 1, ..., p$. Applying elementary transformations to the set $\{s_1, ..., s_p\}$, we obtain a linearly independent generating set $b_1, ..., b_l$ for $H \cap A$, where $0 \leq l \leq p$.

We get a minimal generating pair (B, C) for H by considering the following cases:

i) If $r = 1$ then take $B = \{b_1, ..., b_l, a_1 + u_1\}$ and $C = \emptyset$,

ii) If $r \neq 1$ then take $B = \{b_1, ..., b_l\}$ and $C = \{a_1 + u_1, ..., a_r + u_r\}$

Now we are going to compute the expressions of the new and old generators in terms of each other as the following:

We have

$$a_j + u_j = u_j(c_1 + w_1, ..., c_p + w_p), j = 1, ..., r.$$

We can also compute expressions of $b_1, ..., b_l$ in terms of $s_1, ..., s_r$ and of the $s_1, ..., s_r$ in terms of the $c_1 + w_1, ..., c_p + w_p$. Therefore we can compute the expressions of the new generators in terms of the old generators.

For the other direction we have

$$w_i = f_i(u_1, ..., u_r), i = 1, ..., p$$

Hence

$$f_i(a_1 + u_1, ..., a_r + u_r) = d_i + f_i(u_1, ..., u_r) = d_i + w_i.$$

But

$$c_i + w_i - (d_i + w_i) = c_i - d_i \in A \cap H.$$

So we can write the elements $c_i - d_i$ as

$$c_i - d_i = \beta_1 b_1 + ... + \beta_l b_l, i = 1, ..., p.$$

Thus

$$\begin{aligned} c_i + w_i &= c_i - d_i + d_i + w_i \\ &= \beta_1 b_1 + ... + \beta_l b_l + d_i + w_i \\ &= \beta_1 b_1 + ... + \beta_l b_l + f_i(a_1 + u_1, ..., a_r + u_r), \end{aligned}$$

where $i = 1, ..., p$. □

2.10. Proposition. *Given elements $g, h_1, ..., h_p \in L$, it is decidable whether $g \in H =< h_1, ..., h_p >$.*

Proof. Let $g \in L$. Write $g = a + w$, where $a \in A, w \in F_n$. Let $\{b_1, ..., b_l, a_1 + u_1, ..., a_r + u_r\}$ be a generating set of H. Now check whether $\pi(g) = w \in \pi(H) =< u_1, ..., u_r >$. We can derive from [1] that the subalgebra membership problem is decidable for free Lie algebras. Since $\pi(H)$ is a free Lie algebra we can decide whether $\pi(g) \in \pi(H)$. Now if $\pi(g) \notin \pi(H)$, then $g \notin H$. If $\pi(g) \in \pi(H)$, then $\pi(g) = w$ can be expressed in terms of free generators $u_1, ..., u_r$, say $w = f(u_1, ..., u_r)$. Computing $f(a_1 + u_1, ..., a_r + u_r)$ we get

$$f(a_1 + u_1, ..., a_r + u_r) = c + f(u_1, ..., u_r) = c + w \in H.$$

It is clear that

$$g = a + w \in H \text{ if and only if } a - c = a + w - (c + w) \in H \ ,$$

that is

$$g = a + w \in H \text{ if and only if } a - c \in < b_1, ..., b_l > \subseteq A.$$

In the affirmative case we can compute g in terms of the elements of the set $\{b_1, ..., b_l, a_1 + u_1, ..., a_r + u_r\}$. Using expressions we already have for the elements of the set $\{b_1, ..., b_l, a_1 + u_1, ..., a_r + u_r\}$ in terms of the h_i's we find an expression for g in terms of the h_i's. □

2.11. Remark. Let H be a subalgebra of L and (B, C) be a minimal generating pair of H, where $B = \{b_1, ..., b_l\}$, $C = \{a_1 + u_1, ..., a_r + u_r\}, 0 \leq l \leq m$, $a_i \in A$, $u_i \in F_n, i = 1, ..., r$. Then $\{u_1, ..., u_r\}$ is a free generating set of $\pi(H)$ (see the proof of Proposition 2.9) and B is an abelian basis of $H \cap A$. Every element h of H can be written as

$$h = \sum_{i=1}^{l} \beta_i b_i + \sum_{i=1}^{r} \gamma_i a_i + w(u_1, ..., u_r),$$

where $\beta_i, \gamma_i \in K$, $1 \leq i \leq l, 1 \leq j \leq r$ and γ_i 's are defined uniquely by w . We can write the elements $a_i \in A$ as

$$a_i = \sum_{j=1}^{m} \delta_{ij} y_j, \delta_{ij} \in K, i = 1, ..., r.$$

Denote by $\overline{a_i}$ the vector $(\delta_{i1}, ..., \delta_{im})$. Let $R = (\gamma_1, ..., \gamma_r)$, $S = \begin{pmatrix} y_1 \\ . \\ . \\ y_m \end{pmatrix}$ and let M be the $r \times m$ matrix

$$M = \begin{pmatrix} \overline{a_1} \\ . \\ . \\ . \\ \overline{a_r} \end{pmatrix}.$$

Then straightforward calculations show that the element h is in the form

$$h = a + w(u_1, ..., u_r),$$

where $w(u_1, ..., u_r) \in F_r$ and $a \in RMS+ < B >$ (here $w \in F_r$ and $w(u_1, ..., u_r)$ is the element of $\pi(H)$ obtained by replacing the i th letter in w by u_i, $i = 1, ..., r$).

We have proved the following.

2.12. Lemma. *With previous notations, we have*

$$H = \{a + w(u_1, ..., u_r) : w(u_1, ..., u_r) \in \pi(H), a \in RMS+ < B >\}.$$

2.13. Definition. Let H be a subalgebra of L and $w \in F_n$. The set

$$C = \{a : a + w \in H\} \subseteq A$$

is called the abelian completion of w.

2.14. Corollary. *With the above notation, for every $w \in F_n$ we have*

$i)$ If $w \notin \pi(H)$, then $C = \emptyset$,
$ii)$ If $w \in \pi(H)$, then $C = RMS+ < B >$.

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