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On estimating a stress-strength type reliability

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Abstract

This article deals with estimating an extension of the well-known stress-strength reliability in nonparametric setup. By means of Monte Carlo simulations, the proposed estimator is compared with its parametric analogs in the case of exponential distribution. The results show that the estimator could be highly efficient in many situations considered.

Keywords: Judgment ranking; ℓ-out-of-k system; Robustness.

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1. Introduction

In the reliability literature, the stress-strength term refers to a component with random strength Y which is subjected to random stress X. The component functions if the strength exceeds the stress applied, while it fails otherwise. Thus, $\theta = P(X < Y)$ is a measure of component reliability.

The estimation of θ has been extensively investigated in the literature when X and Y are independent variables belonging to the same univariate family of distributions. An exhaustive account of this topic is given by [4].

Suppose $Y_1, \ldots, Y_n$ are random strength of n components which are subjected to random stresses $X_1, \ldots, X_m$. It is further assumed that X_i's and Y_j's are independent with density functions f and g, respectively. A generalized reliability measure can be defined as

$$\theta_{r,s} = P(X_{r:m} < Y_{s:n}), \tag{1.1}$$

where $X_{r:m}$ ($Y_{s:n}$) is the rth (sth) order statistic of $X_1, \ldots, X_m$ ($Y_1, \ldots, Y_n$). The standard stress-strength reliability θ is obtained when $m = n = 1$. Some other especial cases are listed below:

- $r = 1$ and $s = 1$: minimum strength component is subjected to minimum stress component.

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- $r = 1$ and $s = n$: maximum strength component is subjected to minimum stress component.
- $r = m$ and $s = 1$: minimum strength component is subjected to maximum stress component.
- $r = m$ and $s = n$: maximum strength component is subjected to maximum stress component.

[7] considered estimation of $\theta_{r,s}$ when

$$f(x) = \alpha \exp\{-\alpha x\}, \quad x > 0,$$

and

$$g(y) = \beta \exp\{-\beta y\}, \quad y > 0.$$

This will be referred to as model I. Under this setup, it can be shown that

$$\theta_{r,s} = s\binom{n}{s}\sum_{j=r}^{m}\sum_{i=0}^{j}\sum_{\ell=0}^{s-1}\binom{m}{j}\binom{j}{i}\binom{s-1}{\ell}(-1)^{i+\ell}\left[(n-s+1+\ell)+\frac{\alpha}{\beta}(m+i-j)\right]^{-1}. \tag{1.2}$$

Mathematical form of the exponential distribution allows one to derive maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE) and Bayesian estimator of $\theta_{r,s}$. For example, MLE is given by

$$\hat{\theta}_{r,s} = s\binom{n}{s}\sum_{j=r}^{m}\sum_{i=0}^{j}\sum_{\ell=0}^{s-1}\binom{m}{j}\binom{j}{i}\binom{s-1}{\ell}(-1)^{i+\ell}\left[(n-s+1+\ell)+\frac{\hat{\alpha}}{\hat{\beta}}(m+i-j)\right]^{-1}, \tag{1.3}$$

where $\hat{\alpha} = m/\sum_{i=1}^{m} X_i$ and $\hat{\beta} = n/\sum_{i=1}^{n} Y_i$. Also, UMVUE is obtained as

$$\tilde{\theta}_{r,s} = s\begin{pmatrix} n \\ s \end{pmatrix}\sum_{j=r}^{m}\sum_{i=0}^{j}\sum_{\ell=0}^{s-1}\begin{pmatrix} m \\ j \end{pmatrix}\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} s-1 \\ \ell \end{pmatrix}(-1)^{i+\ell}\widehat{\phi}_{i,j,\ell}(\alpha,\beta), \tag{1.4}$$

where

$$\widehat{\phi}_{i,j,\ell}(\alpha,\beta) = \begin{cases} Q(V,1) & (m+i-j)V \le 1, (n-s+\ell) \le 1 \\ Q(V, \frac{1}{n-s+\ell}) & (m+i-j)V \le 1, (n-s+\ell) > 1 \\ Q(V, \frac{1}{(m+i-j)V}) & (m+i-j)V > 1, (n-s+\ell) \le 1 \\ Q(V, \min\{\frac{1}{n-s+\ell}, \frac{1}{(m+i-j)V}\}) & (m+i-j)V > 1, (n-s+\ell) > 1 \end{cases}$$

with $V = \sum_{i=1}^{n} Y_i / \sum_{i=1}^{m} X_i$ and

$$Q(V,p) = \int_0^p \left[1-(m+i-j)Vu\right]^{m-1}\left[1-(n-s+\ell)u\right]^{n-1}(n-1)(1-u)^{n-2}\mathrm{d}u.$$

Even in this case the estimators have complicated form and their variances are computed numerically. The reader is referred to [7] for details. There are similar concerns about the Bayesian estimator. So we do not consider it in the sequel.

As mentioned before, the above developments are possible owing to the tractable mathematical form of the exponential distribution. This is not an easy job for many other distributions. Moreover, estimation in parametric settings is sensitive to violation of distributional assumptions. In this work, nonparametric estimation of $\theta_{r,s}$ in an especial case is studied. We consider the situation that $X_{r:m}$ are $Y_{s:n}$ are extreme order statistics, i.e. $r = 1, m$ and $s = 1, n$. This setup is particularly important from practical point of view. For example, information about $\theta_{m,1}$ and $\theta_{m,n}$ are vital for planning a reliability experiment.

Section 2 reviews the design under which we study nonparametric estimation of $\theta_{r,s}$. The estimator is presented in Section 3. Section 4 contains results of Monte Carlo simulations conducted to compare our estimator with its parametric competitors in the case of exponential distribution. We conclude the paper with a summary in Section 5.

2. Sampling designs

Ranked set sampling (RSS), introduced by [6], is a technique designed for situations where the sampling units are difficult or expensive to measure, but can be easily ordered by some means without actual quantification. Inference procedures based on RSS are often superior to their counterpart based on simple random sampling (SRS), given a fixed sample size. Although McIntyre's work was motivated by the problem of estimating the average yields from plots of cropland, RSS has also been applied in areas such as environmental science, reliability and medicine. [1] provides a review of nonparametric RSS methodology. For a book-length treatment of RSS and its applications, see [2].

To implement basic RSS scheme, a set size k and a number of cycles t are specified at first. Select k random samples of size k from the target population. The units within each sample are (judgment) ranked with respect to the variable of interest without making any formal quantification. The ranking can be done based on expert opinion, concomitant variable, or a combination of them. From the ith $(i = 1, \dots, k)$ sample, actual measurement is made from the unit with ith smallest rank. This forms a cycle of RSS which yields k measured units. The cycle may be repeated t times to obtain tk units.

There is a connection between RSS and reliability theory. In each cycle of RSS, the ith $(i = 1, \dots, k)$ observation, measured from the ith sample, may be viewed as the lifetime of a $(k - i + 1)$-out-of-k system consisting of k components. Recall that a ℓ-out-of-k system functions if at least ℓ $(\ell = 1, \dots, k)$ of its components are working (see [5]). Two recent papers in the context of reliability estimation from exponential populations based on RSS are [9] and [3].

Many authors have introduced extensions of RSS to construct improved estimators of different population attributes. Extreme ranked set sampling (ERSS) is one such a design proposed by [8]. In the ERSS, one only identifies extreme order statistics, and thereby errors in ranking process are reduced.

The ERSS procedure in a single cycle can be summarized as follows. First, draw k random samples of size k from the target population. The units within each sample are ranked with respect to the variable of interest. If the set size k is even, select the smallest unit from $k/2$ samples, and the largest unit from the other $k/2$ samples, for actual measurement. If the set size is odd, select the smallest unit from $(k-1)/2$ samples, the largest unit from the other $(k-1)/2$ samples, and the median of the last sample, for actual measurement. In the next section, we build on ERSS to construct an efficient estimator of (1.1).

3. Proposed estimator

Let $\theta_{r,s}$ be defined as in (1.1), $r = 1, m$ and $s = 1, n$. Nonparametric estimation of $\theta_{r,s}$ based on SRS involves drawing M samples of size m from f, and N samples of size n from g. From each sample of size $m(n)$, one measures rth (sth) order statistic. The natural estimator is given by

$$\hat{\theta}^{\text{SRS}}_{r,s} = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} I(X^i_{r:m} < Y^j_{s:n}),$$

where $X^i_{r:m}(Y^j_{s:n})$ is the rth (sth) order statistic from the ith (jth) sample of size $m(n)$. In doing so, a total of $mM + nN$ measurements are made which could be too large for an accurate estimation. The situation will deteriorate if measurement is costly.

The above argument led us to resort to RSS. Particularly, we employ a modification of ERSS in which the first/last order statistic is quantified from any sub-sample of size k. Suppose $X^1_{[r:m]}, \dots, X^m_{[r:m]}$ and $Y^1_{[s:n]}, \dots, Y^n_{[s:n]}$ are drawn from f and g using the aforesaid

ERSS scheme with set sizes m and n, respectively. It is emphasized that $r = 1, m$ and $s = 1, n$. Then, we suggest to estimate $\theta_{r,s}$ by

$$\hat{\theta}_{r,s}^{\mathrm{ERSS}} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} I(X_{[r:m]}^{i} < Y_{[s:n]}^{j}). \tag{3.1}$$

It is worth noting that the above mentioned variation of ERSS allows to draw m (n) independent copies of $X_{[r:m]}^{i}(Y_{[s:n]}^{j})$ only based on $m+n$ actual quantifications. Therefore, comparing (3.1) with (1.3) or (1.4) is meaningful. In simulation studies conducted, it is assumed that judgment rankings needed to collect $X_{[r:m]}^{i}$'s and $Y_{[s:n]}^{j}$'s are free of errors. This is not a strict assumption as one can identify the smallest/largest order statistic in a sample of size m or n.

We close this section by a result concerning distributional properties of the proposed estimator. Let F_r (G_s) be the distribution function of $X_{r:m}^{1}$ ($Y_{s:n}^{1}$), and $\bar{G}_s = 1 - G_s$.

Proposition 1 *Suppose $\hat{\theta}_{r,s}^{ERSS}$ is defined as in (3.1). Under perfect ranking assumption, we have $E\left(\hat{\theta}_{r,s}^{ERSS}\right) = \theta_{r,s}$ and*

$$Var\left(\hat{\theta}_{r,s}^{ERSS}\right) = \frac{1}{mn}\left[(m-1)\int F_r^2(y)dG_s(y) + (n-1)\int \bar{G}_s^2(x)dF_r(x) + \theta_{r,s} - (m+n-1)\theta_{r,s}^2\right].$$

Proof. The unbiasedness of $\hat{\theta}_{r,s}^{\mathrm{ERSS}}$ is readily verified. To obtain the variance expression, we have

$$\begin{aligned} m^2n^2E\left(\hat{\theta}_{r,s}^{\mathrm{ERSS}}\right)^2 &= E\Bigg\{\sum_{i\neq i'}\sum_{j\neq j'} I(X_{r:m}^{i} < Y_{s:n}^{j})\, I(X_{r:m}^{i'} < Y_{s:n}^{j'}) \\ &+ \sum_{j}\sum_{i\neq i'} I(X_{r:m}^{i} < Y_{s:n}^{j})\, I(X_{r:m}^{i'} < Y_{s:n}^{j}) \\ &+ \sum_{i}\sum_{j\neq j'} I(X_{r:m}^{i} < Y_{s:n}^{j})\, I(X_{r:m}^{i} < Y_{s:n}^{j'}) \\ &+ \sum_{i}\sum_{j} I(X_{r:m}^{i} < Y_{s:n}^{j})\Bigg\} \\ &= m(m-1)n(n-1)\theta_{r,s}^2 + nm(m-1)\int F_r^2(y)\mathrm{d}G_s(y) \\ &+ mn(n-1)\int \bar{G}_s^2(x)\mathrm{d}F_r(x) + mn\theta_{r,s}. \end{aligned} \tag{3.2}$$

The result then follows from (3.2) and unbiasedness of $\hat{\theta}_{r,s}^{\mathrm{ERSS}}$. □

4. Numerical results

This section reports results of simulation studies carried out to compare the performance of $\hat{\theta}_{r,s}^{\mathrm{ERSS}}$ with $\hat{\theta}_{r,s}$ and $\tilde{\theta}_{r,s}$. To this end, for some configurations of the involved parameters, values of $\theta_{r,s}$ in (1.2) were computed that appear in Table 1. For each combination of (m, n) and (r, s), four choices of β were used which are marked with asterisks. The parameter α was always set to unity.

The efficiency of $\hat{\theta}_{r,s}^{\mathrm{ERSS}}$ relative to the parametric rivals, defined as ratio of the corresponding mean squared errors (MSEs), were estimated based on 5,000 replications. Also, biases of the three estimators were computed. The results are given in Tables 2 and 3. In each case, the three entries show bias of $\hat{\theta}_{r,s}$ or $\tilde{\theta}_{r,s}$, bias of $\hat{\theta}_{r,s}^{\mathrm{ERSS}}$, and the relative efficiency (RE), respectively. For convenience, the REs are given in bold. The values of β are not reported as they can be read from Table 1.

Table 1. Values of $\theta_{r,s}$ under model I

(m,n)	(r,s)	β_1	β_2	β_3	β_4
(5,5)	(5,1)	0.15*	0.1*	0.05*	0.02*
		0.24345	0.36941	0.58838	0.80141
	(5,5)	2*	1.5*	1*	0.7*
		0.17682	0.29326	0.5	0.68388
(10,10)	(10,1)	0.06*	0.04*	0.02*	0.01*
		0.21428	0.34375	0.57257	0.75162
	(10,10)	1.5*	1.3*	1*	0.9*
		0.24371	0.32654	0.5	0.57163
(20,20)	(20,1)	0.025*	0.015*	0.01*	0.005*
		0.19455	0.36185	0.50135	0.70316
	(20,20)	1.2*	1.1*	1*	0.95*
		0.35044	0.42041	0.5	0.54303

Table 2. Monte Carlo biases of $\hat{\theta}_{r,s}$ and $\hat{\theta}_{r,s}^{\text{ERSS}}$, and the relative efficiency under model I

(m,n)	(r,s)	β_1	β_2	β_3	β_4
(5,5)	(5,1)	0.02586	0.00855	-0.0177	-0.02504
		-0.001	-0.0011	-0.00359	-0.00241
		1.151	**1.01**	**0.859**	**0.635**
	(5,5)	0.06808	0.05059	0.00188	-0.04264
		-0.00016	-0.00076	-0.00092	-0.00261
		2.771	**2.266**	**2.034**	**2.224**
(10,10)	(10,1)	0.01459	0.00373	-0.01144	-0.01454
		-0.00126	-0.00222	-0.00204	-0.00093
		1.354	**1.187**	**0.915**	**0.654**
	(10,10)	0.04616	0.03265	-0.00241	-0.01736
		0.00068	5e-04	0.00146	0.0014
		4.118	**3.726**	**3.507**	**3.563**
(20,20)	(20,1)	0.00805	6e-05	-0.00521	-0.00824
		-0.00039	-0.00039	-0.00058	-0.00053
		1.437	**1.238**	**1.057**	**0.734**
	(20,20)	0.02157	0.01052	-0.00275	-0.00993
		0.00068	0.00078	0.00062	0.00077
		5.884	**5.721**	**5.666**	**5.696**

It is observed that $\hat{\theta}_{r,s}^{\text{ERSS}}$ has less absolute bias than $\hat{\theta}_{r,s}$ and $\tilde{\theta}_{r,s}$. Also, there are few cases that MLE or UMVUE is more efficient than the new estimator. Most of the RE values are in the range $(1,6)$ confirming that the nonparametric estimator could be highly efficient in some cases.

To assess robustness properties of the parametric estimators, a partial simulation study was conducted. To do so, we assume X has exponential distribution with mean $1/\alpha$, and Y has Weibull distribution with shape parameter γ, and scale parameter β.

Table 3. Monte Carlo biases of $\tilde{\theta}_{r,s}$ and $\hat{\theta}_{r,s}^{\mathrm{ERSS}}$, and the relative efficiency under model I

(m,n)	(r,s)	β_1	β_2	β_3	β_4
(5,5)	(5,1)	-0.04301	-0.06301	-0.09157	-0.10981
		-0.001	-0.0011	-0.00359	-0.00241
		1.131	**1.111**	**1.046**	**1.004**
	(5,5)	-0.03919	-0.06777	-0.12284	-0.17316
		-0.00016	-0.00076	-0.00092	-0.00261
		2.236	**2.257**	**2.505**	**3.083**
(10,10)	(10,1)	-0.03336	-0.04725	-0.06433	-0.07212
		-0.00126	-0.00222	-0.00204	-0.00093
		1.335	**1.296**	**1.098**	**0.941**
	(10,10)	-0.05565	-0.07792	-0.12749	-0.14893
		0.00068	5e-04	0.00146	0.0014
		3.398	**3.428**	**3.832**	**4.164**
(20,20)	(20,1)	-0.02031	-0.03044	-0.03624	-0.04139
		-0.00039	-0.00039	-0.00058	-0.00053
		1.424	**1.329**	**1.175**	**0.905**
	(20,20)	-0.06921	-0.08622	-0.10651	-0.11789
		0.00068	0.00078	0.00062	0.00077
		5.367	**5.521**	**5.817**	**6.059**

Table 4. Values of $\theta_{r,s}$ under model II

(m,n)	(r,s)	β_1	β_2	β_3	β_4
(5,5)	(5,1)	0.15*	0.1*	0.05*	0.02*
		0.5813	0.75631	0.92396	0.98686
	(5,5)	2*	1.5*	1*	0.7*
		0.04564	0.10536	0.27258	0.49585
(10,10)	(10,1)	0.06*	0.04*	0.02*	0.01*
		0.72634	0.85913	0.96098	0.98997
	(10,10)	1.5*	1.3*	1*	0.9*
		0.02907	0.03026	0.13985	0.17725

That is

$$f(x) = \alpha \exp\{-\alpha x\}, \quad x > 0,$$

and

$$g(y) = \gamma\beta^{\gamma} y^{\gamma-1} \exp\{-(\beta y)^{\gamma}\}, \quad y > 0.$$

This will be referred to as model II. Under this setup, it can be seen that

$$\begin{aligned} \theta_{r,s} &= s\binom{n}{s}\gamma\beta^{\gamma}\sum_{j=r}^{m}\sum_{i=0}^{j}\sum_{\ell=0}^{s-1}\binom{m}{j}\binom{j}{i}\binom{s-1}{\ell}(-1)^{i+\ell} \\ &\times \int_0^{\infty} y^{\gamma-1}\exp\Big\{-\alpha(m+i-j)y-(n-s+1+\ell)(\beta y)^{\gamma}\Big\}\,\mathrm{d}y. \end{aligned} \tag{4.1}$$

It is to be noted that model I corresponds to the case $\gamma = 1$.

Table 5. Monte Carlo biases and MSEs of $\hat{\theta}_{r,s}$ under models I and II

(m,n)	(r,s)	β_1	β_2	β_3	β_4
(5,5)	(5,1)	-0.33114	-0.39367	-0.35882	-0.20807
		0.02586	**0.00855**	**-0.0177**	**-0.02504**
		0.12987	0.1791	0.14965	0.05205
		0.02899	**0.03427**	**0.03228**	**0.01629**
	(5,5)	0.16638	0.20695	0.2084	0.1388
		0.06808	**0.05059**	**0.00188**	**-0.04264**
		0.06115	0.08834	0.09685	0.06571
		0.05125	**0.06389**	**0.07182**	**0.06776**
(10,10)	(10,1)	-0.52309	-0.53737	-0.41914	-0.26433
		0.01459	**0.00373**	**-0.01144**	**-0.01454**
		0.28384	0.30226	0.1878	0.07625
		0.01664	**0.02076**	**0.01863**	**0.01027**
	(10,10)	0.21147	0.27871	0.31252	0.33599
		0.04616	**0.03265**	**-0.00241**	**-0.01736**
		0.0761	0.11583	0.14287	0.15799
		0.05222	**0.05855**	**0.06322**	**0.06235**

Table 6. Monte Carlo biases and MSEs of $\tilde{\theta}_{r,s}$ under models I and II

(m,n)	(r,s)	β_1	β_2	β_3	β_4
(5,5)	(5,1)	-0.40301	-0.46682	-0.43119	-0.29174
		-0.04301	**-0.06301**	**-0.09157**	**-0.10981**
		0.18196	0.24257	0.20613	0.09228
		0.02887	**0.03824**	**0.03982**	**0.02585**
	(5,5)	0.05398	0.07893	0.07933	0.01492
		-0.03919	**-0.06777**	**-0.12284**	**-0.17316**
		0.03031	0.05196	0.06745	0.04925
		0.04178	**0.0649**	**0.08957**	**0.09474**
(10,10)	(10,1)	-0.57129	-0.58869	-0.47144	-0.3211
		-0.03336	**-0.04725**	**-0.06433**	**-0.07212**
		0.33582	0.35989	0.23411	0.10897
		0.01617	**0.02236**	**0.02209**	**0.0145**
	(10,10)	0.11092	0.16861	0.1913	0.21093
		-0.05565	**-0.07792**	**-0.12749**	**-0.14893**
		0.03718	0.06113	0.07711	0.08406
		0.04242	**0.05277**	**0.06741**	**0.07119**

Again, for some configurations of the involved parameters, values of $\theta_{r,s}$ in (4.1) were computed which are given in Table 4. The choices of (m,n), (r,s), α and β are nearly as in Table 1. We only set $\gamma = 2$ to deviate from model I. Now, biases and MSEs for $\hat{\theta}_{r,s}$ and $\tilde{\theta}_{r,s}$ were estimated based on 5,000 replications. Tables 5 and 6 show the results. In each case, the four entries are two biases, and then two MSEs. To facilitate comparisons, the biases and MSEs under model I are given in bold.

With a few exceptions, the absolute biases, and MSEs of both estimators increase under model II, as expected. The amount of increase will be more pronounced if the departure from model I is not so mild. This supports the use of nonparametric estimator whenever possible.

Computer codes used to compare $\hat{\theta}_{r,s}^{\mathrm{ERSS}}$ with $\hat{\theta}_{r,s}$ are given in the appendix. They can be easily modified for comparing the suggested estimator and $\tilde{\theta}_{r,s}$.

5. Conclusion

This article attends to estimation of a reliability measure which extends the usual stress-strength reliability. Estimating this index in parametric settings is generally a difficult task. Moreover, the resulting estimators are prone to violation of distributional assumptions. Therefore, a nonparametric approach merits investigation. Toward this end, we employ a variation of a sampling design which often leads to improved inference procedures as compared with the usual SRS scheme. The aforesaid design called RSS combines measurement with judgment ranking information for statistical inference purpose. Monte Carlo simulations are conducted to compare the proposed estimator with its parametric rivals in the case of exponential distribution. The results confirm preference of the estimator in many situations considered.

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Appendix

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### R code used for computing entries of Table 2 ###

B=5000
m=5;  n=5
r=m;  s=1
alpha=1;  beta=0.15
ML.srs=rss=c()

g=0
for(j in r:m){
  for(i in 0:j){
    for(k in 0:(s-1)){
      g=g+choose(m,j)*choose(j,i)*choose(s-1,k)*(-1)^(i+k)/((n-s+1+k)+alpha/beta*(m+i-j))
    }
  }
}
theta=s*choose(n,s)*g

set.seed(1)
for(b in 1:B){
  x=rexp(m,alpha);  y=rexp(n,beta)
  xh=1/mean(x);  yh=1/mean(y)

  h=0
  for(j in r:m){
    for(i in 0:j){
      for(k in 0:(s-1)){
        h=h+choose(m,j)*choose(j,i)*choose(s-1,k)*(-1)^(i+k)/((n-s+1+k)+xh/yh*(m+i-j))
      }
    }
  }
  ML.srs[b]=s*choose(n,s)*h

  X=Y=c()
  for(i in 1:m){
    u=rexp(m,alpha);  u=sort(u)
    X[i]=u[r]
  }
  for(j in 1:n){
    v=rexp(n,beta);  v=sort(v)
    Y[j]=v[s]
  }

  cn=0
  for(i in 1:m){  cn=cn+sum(X[i]<Y)  }
  rss[b]=cn/(m*n)
}
```

```
round(theta,5)

round(mean(ML.srs)-theta,5)

round(mean(rss)-theta,5)

N=mean((ML.srs-theta)^2);  D=mean((rss-theta)^2)

round(N/D,3)


### R code used for computing entries of Table 5 ###

B=5000
m=5;  n=5
r=m;  s=1
alpha=1;  c=2;  beta=0.15
ML.srs=MLR.srs=c()

g=0
for(j in r:m){
  for(i in 0:j){
    for(k in 0:(s-1)){
      g=g+choose(m,j)*choose(j,i)*choose(s-1,k)*(-1)^(i+k)/((n-s+1+k)+alpha/beta*(m+i-j))
    }
  }
}
theta=s*choose(n,s)*g

set.seed(1)
for(b in 1:B){
  x=rexp(m,alpha);  y=rexp(n,beta)
  xh=1/mean(x);  yh=1/mean(y)

  h=0
  for(j in r:m){
    for(i in 0:j){
      for(k in 0:(s-1)){
        h=h+choose(m,j)*choose(j,i)*choose(s-1,k)*(-1)^(i+k)/((n-s+1+k)+xh/yh*(m+i-j))
      }
    }
  }
  ML.srs[b]=s*choose(n,s)*h
}

g=0
for(j in r:m){
  for(i in 0:j){
    for(k in 0:(s-1)){
      g=g+choose(m,j)*choose(j,i)*choose(s-1,k)*(-1)^(i+k)*
      integrate(function(t) t^(c-1)*exp(-alpha*(m+i-j)*t-(n-s+1+k)*(beta*t)^c),
```

```
      lower=0,upper=Inf)$value
    }
  }
}
true=s*choose(n,s)*c*beta^c*g

for(b in 1:B){
  x=rexp(m,alpha);  y=rweibull(n,c,1/beta)
  xh=1/mean(x);  yh=1/mean(y)

  h=0
  for(j in r:m){
    for(i in 0:j){
      for(k in 0:(s-1)){
        h=h+choose(m,j)*choose(j,i)*choose(s-1,k)*(-1)^(i+k)/((n-s+1+k)+xh/yh*(m+i-j))
      }
    }
  }
  MLR.srs[b]=s*choose(n,s)*h
}

round(theta,5)

round(true,5)

round(mean(MLR.srs)-true,5)

round(mean(ML.srs)-theta,5)

round(mean((MLR.srs-true)^2),5)

round(mean((ML.srs-theta)^2),5)
```