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Maximal accretive singular quasi-differential operators

Pembe Ipek*† and Zameddin I. Ismailov‡

Abstract

In this paper firstly all maximal accretive extensions of the minimal operator generated by a first order linear singular quasi-differential expression in the weighted Hilbert space of vector-functions on right semi-axis are described. Later on, the structure of spectrum set of these extensions has been researched.

Keywords: Accretive operator, Quasi-differential operator, Spectrum.

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1. Introduction

It is known that a linear closed densely defined operator $T : D(T) \subset H \to H$ in Hilbert space H is called accretive(dissipative) if for all $f \in D(T)$ the inequality $Re < Tf, f >_H \geq 0\,(Im < Tf, f >_H \geq 0)$ is satisfied. Also it is called maximal accretive(maximal dissipative) if it is accretive(dissipative) and does not have any proper accretive(dissipative) extension [3], [1]. The class of accretive operators is an important class of non-selfadjoint operators in the operator theory. Note that the spectrum set of accretive operators lies in right half-plane.

The maximal accretive extensions and their spectral analysis of the minimal operator generated by regular differential-operator expression in Hilbert space of vector-functions defined in one finite interval case have been studied by V.V. Levchuk [4].

This work is organised as follows: In Section 3, all maximal accretive extensions of the minimal operator generated by a linear singular quasi-differential operator expression in the weighted Hilbert spaces of the vector functions defined at right semi-axis are examined. In Section 4, the structure of the spectrum of these type extensions has been investigated.

*Institute of Natural Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey, Email: ipekpembe@gmail.com

†Corresponding Author.

‡Department of Mathematics, Karadeniz Technical University, 61080, Trabzon, Turkey, Email: zameddin.ismailov@gmail.com

2. Statement of the problem

Let H be a separable Hilbert space and $a \in \mathbb{R}$. Moreover assumed that $\alpha : (a, \infty) \to (0, \infty)$, $\alpha \in C(a, \infty)$ and $\alpha^{-1} \in L^1(a, \infty)$. In the weighted Hilbert space $L^2_\alpha(H, (a, \infty))$ of $H-$ valued vector-functions defined on the right semi-axis consider the following linear first order quasi-differential expression with operator coefficient

$$l(u) = (\alpha u)' + Au,$$

where $A : H \to H$ is a selfadjoint operator with condition $A \geq 0$.

By a standard way the minimal L_0 and maximal L operators corresponding to quasi-differential expression $l(\,\cdot\,)$ in $L^2_\alpha(H, (a, \infty))$ can be defined (see [2]). In this case the minimal operator L_0 is accretive, but it is not maximal in $L^2_\alpha(H, (a, \infty))$.

The main goal of this work is to describe of all maximal accretive extensions of the minimal operator L_0 in terms of boundary condition in $L^2_\alpha(H, (a, \infty))$. Secondly, the structure of the spectrum set of these extensions will be investigated.

3. Description of maximal accretive extensions

Note that in similar way the minimal operator L_0^+ generated by a quasi-operator expression

$$l^+(v) = -(\alpha v)' + Av$$

can be defined in $L^2_\alpha(H, (a, \infty))$ (see [2]). In this case the operator $L^+ = (L_0)^*$ in $L^2_\alpha(H, (a, \infty))$ is called the maximal operator generated by $l^+(\,\cdot\,)$. It is clear that $L_0 \subset L$ and $L_0^+ \subset L^+$.

If $\widetilde{L}$ is any maximal accretive extension of the minimal operator L_0 in $L^2_\alpha(H, (a, \infty))$ and $\widetilde{M}$ is corresponding extension of the minimal operator M_0 generated by a quasi-differential expression

$$m(u) = i(\alpha u)'$$

in $L^2_\alpha(H, (a, \infty))$, then it is clear that

$$\begin{aligned}
\widetilde{L}u &= (\alpha u)'(t) + Au(t) \\
&= i(-i(\alpha u)')(t) + Au(t) \\
&= i(-\widetilde{M})(t) + Au(t) \\
&= i\left(-\left(Re\widetilde{M} + iIm\widetilde{M}\right)\right) u(t) + Au(t) \\
&= \left(Im\widetilde{M}\right) u(t) - i\left(Re\widetilde{M}\right) u(t) + Au(t) \\
&= \left[\left(Im\widetilde{M}\right) + A\right] u(t) - i\left(Re\widetilde{M}\right) u(t).
\end{aligned}$$

Therefore

$$\left(Re\widetilde{L}\right) = \left(Im\widetilde{M}\right) + A.$$

On the other hand it is clear that

$$\left(Re\widetilde{L}\right) = \left(Im\widetilde{M}\right) + A = Im\left(\widetilde{M} + A\right).$$

Hence to describe all maximal accretive extension of the minimal operator L_0 in $L^2_\alpha(H, (a, \infty))$ it is sufficiently to describe all maximal dissipative extensions of the minimal operator S_0 generated by quasi-differential expression

$$s(u) = i(\alpha u)' + Au$$

in $L^2_\alpha(H, (a, \infty))$.

Furthermore, we will denote the maximal operator generated by the quasi-differential expression $s(\,\cdot\,)$ in $L^2_\alpha(H, (a, \infty))$ by S.

In this section, we will investigate the general representation of all maximal dissipative extensions of the minimal operator S_0 in $L^2_\alpha(H,(a,\infty))$ by using Calkin-Gorbachuk method. Let us prove the following proposition.

3.1. Lemma. *The deficiency indices of the minimal operator* S_0 *in* $L^2_\alpha(H,(a,\infty))$ *are given in the form*

$$(n_+(S_0), n_-(S_0)) = (dimH, dimH).$$

Proof. For the simplicity of calculations, we will take $A = 0$. It is clear that the general solutions of differential equations

$$i(\alpha u_\pm)'(t) \pm iu_\pm(t) = 0,\ t > a$$

in $L^2_\alpha(H,(a,\infty))$

$$u_\pm(t) = \frac{1}{\alpha(t)} exp\left(\mp\int\limits_a^t \frac{ds}{\alpha(s)}\right) f,\ f \in H,\ t > a.$$

From these representations, we have

$$\begin{aligned}
\|u_+\|^2_{L^2_\alpha(H,(a,\infty))} &= \int\limits_a^\infty \|u_+(t)\|^2_H dt \\
&= \int\limits_a^\infty \|\frac{1}{\alpha(t)} exp\left(-\int\limits_a^t \frac{ds}{\alpha(s)}\right) f\|^2_H \alpha(t) dt \\
&= \int\limits_a^\infty \frac{1}{\alpha(t)} exp\left(-2\int\limits_a^t \frac{ds}{\alpha(s)}\right) dt \|f\|^2_H \\
&= \int\limits_a^\infty exp\left(-2\int\limits_a^t \frac{ds}{\alpha(s)}\right) d\left(\int\limits_a^t \frac{ds}{\alpha(s)}\right) \|f\|^2_H \\
&= \frac{1}{2}\left(1 - exp\left(-2\int\limits_a^\infty \frac{ds}{\alpha(s)}\right)\right) \|f\|^2_H < \infty.
\end{aligned}$$

Consequently $n_+(S_0) = dim\ ker(S + iE) = dimH$.

On the other hand, it is clear that for any $f \in H$,

$$\begin{aligned}
\|u_-\|^2_{L^2_\alpha(H,(a,\infty))} &= \int_a^\infty \|u_-(t)\|^2_H dt \\
&= \int_a^\infty \|\frac{1}{\alpha(t)} exp\left(\int_a^t \frac{ds}{\alpha(s)}\right) f\|^2_H \alpha(t) dt \\
&= \int_a^\infty \frac{1}{\alpha(t)} exp\left(2\int_a^t \frac{ds}{\alpha(s)}\right) dt \|f\|^2_H \\
&= \int_a^\infty exp\left(2\int_a^t \frac{ds}{\alpha(s)}\right) d\left(\int_a^t \frac{ds}{\alpha(s)}\right) \|f\|^2_H \\
&= \frac{1}{2}\left(exp\left(2\int_a^\infty \frac{ds}{\alpha(s)}\right) - 1\right) \|f\|^2_H < \infty.
\end{aligned}$$

It follows from that $n_-(S_0) = dim\ ker(S - iE) = dim H$. This completes the proof of the theorem. □

Consequently, the minimal operator S_0 has a maximal dissipative extension (see [1]).

In order to describe these extensions, we need to obtain the space of boundary values.

3.2. Definition. [1] Let $\mathfrak{H}$ be any Hilbert space and $S : D(S) \subset \mathfrak{H} \to \mathfrak{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$ having equal finite or infinite deficiency indices. A triplet $(\mathbf{H}, \gamma_1, \gamma_2)$, where $\mathbf{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into $\mathbf{H}$, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$

$$< S^* f, g >_{\mathfrak{H}} - < f, S^* g >_{\mathfrak{H}} = < \gamma_1(f), \gamma_2(g) >_{\mathbf{H}} - < \gamma_2(f), \gamma_1(g) >_{\mathbf{H}}$$

while for any $F_1, F_2 \in \mathbf{H}$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

3.3. Lemma. *Define*

$$\begin{aligned}
\gamma_1 : D(S) \to H,\ \gamma_1(u) &= \frac{1}{\sqrt{2}}\left((\alpha u)(\infty) - (\alpha u)(a)\right) \quad and \\
\gamma_2 : D(S) \to H,\ \gamma_2(u) &= \frac{1}{i\sqrt{2}}\left((\alpha u)(\infty) + (\alpha u)(a)\right),\ u \in D(S).
\end{aligned}$$

Then the triplet (H, γ_1, γ_2) is a space of boundary values of the minimal operator S_0 in $L^2_\alpha(H,(a,\infty))$.

Proof. For any $u,\ v \in D(S)$

$$\begin{aligned}
& < Su, v >_{L^2_\alpha(H,(a,\infty))} - < u, Sv >_{L^2_\alpha(H,(a,\infty))} \\
= \ & < i(\alpha u)' + Au, v >_{L^2_\alpha(H,(a,\infty))} - < u, i(\alpha v)' + Av >_{L^2_\alpha(H,(a,\infty))} \\
= \ & < i(\alpha u)', v >_{L^2_\alpha(H,(a,\infty))} - < u, i(\alpha v)' >_{L^2_\alpha(H,(a,\infty))} \\
= \ & \int_a^\infty < i(\alpha u)'(t), v(t) >_H \alpha(t)dt - \int_a^\infty < u(t), i(\alpha v)'(t) >_H \alpha(t)dt \\
= \ & i\left[\int_a^\infty < (\alpha u)'(t), (\alpha v)(t) >_H dt + \int_a^\infty < (\alpha u)(t), (\alpha v)'(t) >_H dt\right] \\
= \ & i\int_a^\infty < (\alpha u)(t), (\alpha v)(t) >'_H dt \\
= \ & i\,[< (\alpha u)(\infty), (\alpha v)(\infty) >_H - < (\alpha u)(a), (\alpha v)(a) >_H] \\
= \ & < \gamma_1(u), \gamma_2(v) >_H - < \gamma_2(u), \gamma_1(v) >_H \ .
\end{aligned}$$

Now for any given elements $f,\ g \in H$, we can find the function $u \in D(S)$ such that

$$\gamma_1(u) = \frac{1}{\sqrt{2}}((\alpha u)(\infty) - (\alpha u)(a)) = f \ \text{ and } \ \gamma_2(u) = \frac{1}{i\sqrt{2}}((\alpha u)(\infty) + (\alpha u)(a)) = g.$$

From this, we obtain

$$(\alpha u)(\infty) = (ig+f)/\sqrt{2} \ \text{ and } \ (\alpha u)(a) = (ig-f)/\sqrt{2}.$$

If we choose the function $u(\ \cdot\ )$ in following form

$$u(t) = \frac{1}{\alpha(t)}(1-e^{a-t})(ig+f)/\sqrt{2} + \frac{1}{\alpha(t)}e^{a-t}(ig-f)/\sqrt{2},$$

then it is clear that $u \in D(S)$ and $\gamma_1(u) = f,\ \gamma_2(u) = g$. □

The following result can be established by using the method given in [1].

3.4. Theorem. *If $\widetilde{S}$ is a maximal dissipative extension of the minimal operator S_0 in $L^2_\alpha(H,(a,\infty))$, then it is generated by the differential-operator expression $s(\ \cdot\ )$ and boundary condition*

$$(\alpha u)(a) = K(\alpha u)(\infty),$$

where $K : H \to H$ is a contraction operator. Moreover, the contraction operator K in H is determined uniquely by the extension $\widetilde{S}$, i.e. $\widetilde{S} = S_K$ and vice versa.

Proof. It is known that each maximal dissipative extension $\widetilde{S}$ of the minimal operator S_0 is described by the differential-operator expression $s(\ \cdot\ )$ and the boundary condition

$$(V-E)\gamma_1(u) + i(V+E)\gamma_2(u) = 0,$$

where $V : H \to H$ is a contraction operator. Therefore from Lemma 3.3, we obtain

$$(V-E)\left((\alpha u)(\infty) - (\alpha u)(a)\right) + (V+E)\left((\alpha u)(\infty) + (\alpha u)(a)\right) = 0,\ u \in D(\widetilde{S}).$$

From this, it implies that

$$(\alpha u)(a) = -V(\alpha u)(\infty).$$

Choosing $K = -V$ in last boundary condition, we have

$$(\alpha u)(a) = K(\alpha u)(\infty).$$

□

From this theorem and the note mentioned above, it implies the validity of the following result.

3.5. Theorem. *Each maximal accretive extension $\widetilde{L}$ of the minimal operator L_0 is generated by linear singular quasi-differential expression $l(\,\cdot\,)$ and boundary condition*

$$(\alpha u)(a) = K(\alpha u)(\infty),$$

where $K : H \to H$ is a contraction operator such that this operator is determined uniquely by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_K$ and vice versa.

4. The spectrum of the maximal accretive extensions

In this section the structure of the spectrum set of the maximal accretive extensions of the minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$ will be researched.

4.1. Theorem. *The spectrum of any maximal accretive extension L_K has the form*

$$\begin{aligned}\sigma(L_K) \quad &= \quad \left\{\lambda \in \mathbb{C} : \lambda = \left(\int_a^\infty \frac{ds}{\alpha(s)}\right)^{-1} \left(ln\left(|\mu|^{-1}\right) + iarg(\overline{\mu}) + 2n\pi i\right),\right.\\ &\left.\mu \in \sigma\left(Kexp\left(-A\int_a^\infty \frac{ds}{\alpha(s)}\right)\right),\ n \in \mathbb{Z}\right\}.\end{aligned}$$

Proof. Consider the following problem to get the spectrum of the extension L_K, i.e.

$$L_K(u) = \lambda u + f,\ \lambda \in \mathbb{C},\ \lambda_r = Re\lambda \geq 0.$$

Then we have

$$\begin{aligned}&(\alpha u)'(t) + Au(t) = \lambda u(t) + f(t),\ t > a,\\ &(\alpha u)(a) = K(\alpha u)(\infty).\end{aligned}$$

The general solution of the last differential equation

$$(\alpha u)'(t) = \frac{1}{\alpha(t)}(\lambda E - A)(\alpha u)(t) + f(t),\ t > a$$

is

$$\begin{aligned}u(t;\lambda) \quad &= \quad \frac{1}{\alpha(t)} exp\left((\lambda E - A)\int_a^t \frac{ds}{\alpha(s)}\right) f_\lambda\\ &- \quad \frac{1}{\alpha(t)}\int_t^\infty \exp\left((\lambda E - A)\int_s^t \frac{d\tau}{\alpha(\tau)}\right) f(s)ds,\ f_\lambda \in H,\ t > a.\end{aligned}$$

In this case

$$
\begin{aligned}
& \| \frac{1}{\alpha(t)} exp \left((\lambda E - A) \int\limits_a^t \frac{ds}{\alpha(s)} \right) f_\lambda \|^2_{L^2_\alpha (H,(a,\infty))} \\
= & \int\limits_a^\infty \| \frac{1}{\alpha(t)} exp \left((\lambda E - A) \int\limits_a^t \frac{ds}{\alpha(s)} \right) f_\lambda \|^2_H \alpha(t) dt \\
= & \int\limits_a^\infty < \frac{1}{\alpha(t)} exp \left((\lambda E - A) \int\limits_a^t \frac{ds}{\alpha(s)} \right) f_\lambda , \frac{1}{\alpha(t)} exp \left((\lambda E - A) \int\limits_a^t \frac{ds}{\alpha(s)} \right) f_\lambda >_H \alpha(t) dt \\
= & \int\limits_a^\infty \frac{1}{\alpha(t)} exp \left(2\lambda_r \int\limits_a^t \frac{ds}{\alpha(s)} \right) < exp \left(-A \int\limits_a^t \frac{ds}{\alpha(s)} \right) f_\lambda , exp \left(-A \int\limits_a^t \frac{ds}{\alpha(s)} \right) f_\lambda >_H dt \\
= & \int\limits_a^\infty \frac{1}{\alpha(t)} exp \left(2\lambda_r \int\limits_a^t \frac{ds}{\alpha(s)} \right) \| exp \left(-A \int\limits_a^t \frac{ds}{\alpha(s)} \right) f_\lambda \|^2_H dt \\
\leq & \int\limits_a^\infty \frac{1}{\alpha(t)} exp \left(2\lambda_r \int\limits_a^t \frac{ds}{\alpha(s)} \right) dt \| f_\lambda \|^2_H \\
= & \frac{1}{2\lambda_r} \left(exp \left(2\lambda_r \int\limits_a^\infty \frac{ds}{\alpha(s)} \right) - 1 \right) \| f_\lambda \|^2_H < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
& \| - \frac{1}{\alpha(t)} \int\limits_t^\infty \exp \left((\lambda E - A) \int\limits_s^t \frac{d\tau}{\alpha(\tau)} \right) f(s) ds \|^2_{L^2_\alpha (H,(a,\infty))} \\
= & \int\limits_a^\infty \| \frac{1}{\alpha(t)} \int\limits_t^\infty \exp \left((\lambda E - A) \int\limits_s^t \frac{d\tau}{\alpha(\tau)} \right) f(s) ds \|^2_H \alpha(t) dt \\
= & \int\limits_a^\infty \frac{1}{\alpha(t)} \| \int\limits_t^\infty \exp \left((\lambda E - A) \int\limits_s^t \frac{d\tau}{\alpha(\tau)} \right) f(s) ds \|^2_H dt \\
= & \int\limits_a^\infty \frac{1}{\alpha(t)} \| \int\limits_t^\infty \exp \left(\lambda E \int\limits_s^t \frac{d\tau}{\alpha(\tau)} \right) \left[\exp \left(-A \int\limits_s^t \frac{d\tau}{\alpha(\tau)} \right) f(s) \right] ds \|^2_H dt
\end{aligned}
$$

$$
= \int\limits_a^\infty \frac{1}{\alpha(t)} \| \int\limits_t^\infty \exp \left((\lambda_r + i\lambda_i) \int\limits_s^t \frac{d\tau}{\alpha(\tau)} \right) \left[\exp \left(-A \int\limits_s^t \frac{d\tau}{\alpha(\tau)} \right) \frac{1}{\sqrt{\alpha(s)}} \left(\sqrt{\alpha(s)} f(s) \right) \right] ds \|^2_H dt
$$

$$\leq \int_a^\infty \frac{1}{\alpha(t)} \left(\int_a^\infty \frac{1}{\alpha(s)} \exp\left(2\lambda_r \int_s^t \frac{d\tau}{\alpha(\tau)}\right) ds \right) \left(\int_a^\infty \alpha(s) \|f\|_H^2 ds \right) dt$$

$$\leq \int_a^\infty \frac{1}{\alpha(t)} \left(\int_a^\infty \frac{1}{\alpha(s)} \exp\left(2\lambda_r \int_a^\infty \frac{d\tau}{\alpha(\tau)}\right) ds \right) \|f\|^2_{L^2_\alpha(H,(a,\infty))}$$

$$= \exp\left(2\lambda_r \int_a^\infty \frac{d\tau}{\alpha(\tau)}\right) \left(\int_a^\infty \frac{ds}{\alpha(s)} \right)^2 \|f\|^2_{L^2_\alpha(H,(a,\infty))} < \infty.$$

Hence $u(\,\cdot\,,\lambda) \in L^2_\alpha(H,(a,\infty))$ for $\lambda \in \mathbb{C},\ Re\lambda \geq 0$.
Furthermore from boundary condition, we get

$$\left(E - Kexp\left((\lambda E - A) \int_a^\infty \frac{ds}{\alpha(s)} \right) \right) f_\lambda = \int_a^\infty exp\left((\lambda E - A) \int_s^a \frac{d\tau}{\alpha(\tau)} \right) f(s) ds.$$

Therefore in order to obtain $\lambda \in \sigma(L_K)$ the necessary and sufficient condition is

$$exp\left(-\lambda \int_a^\infty \frac{ds}{\alpha(s)} \right) = \mu \in \sigma \left(Kexp\left(-A \int_a^\infty \frac{ds}{\alpha(s)} \right) \right).$$

Hence

$$-\lambda \int_a^\infty \frac{ds}{\alpha(s)} = ln|\mu| + iarg\mu + 2m\pi i,\ m \in \mathbb{Z},$$

that is, $\lambda = \left(\int_a^\infty \frac{ds}{\alpha(s)} \right)^{-1} \left(ln\left(|\mu|^{-1}\right) + iarg(\overline{\mu}) + 2n\pi i \right),\ n \in \mathbb{Z},\ \mu \in \sigma\left(Kexp\left(-A \int_a^\infty \frac{ds}{\alpha(s)} \right) \right).$ □

Example. All maximal accretive extensions L_r of the minimal operator L_0 generated by a differential expression

$$l(u) = (t^\alpha u(t,x))' + Au(t,x),\ \alpha > 1,$$

in Hilbert space $L^2_{t^\alpha}((0,1) \times (1,\infty))$ in terms of boundary conditions are described by the following form

$$(t^\alpha u(t,x))(1) = r\,(t^\alpha u(t,x))(\infty),\ 0 < r < 1,\ 0 < x < 1,$$

where

$$A : L^2(0,1) \to L^2(0,1),\ Av(x) = xv(x).$$

Moreover, the spectrum of such extensions is

$$\sigma(L_r) = \left\{ \lambda \in \mathbb{C} : \lambda = (1-\alpha)\left(ln\left(|\mu|^{-1}\right) + iarg(\overline{\mu}) + 2n\pi i \right), \mu \in \sigma\left(rexp\left(\frac{A}{\alpha-1} \right) \right),\ \ n \in \mathbb{Z} \right\}.$$

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