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On certain multivalent analytic functions starlike with respect to k-symmetric points

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Abstract

Two new subclasses $Q_{p,k}(\lambda, A, B)$ and $G_{p,k}(\lambda, A, B)$ of analytic and p-valent functions which are starlike with respect to k-symmetric points are introduced. Distortion bounds, inclusion relations, integral transforms and convolution properties for these classes are studied.

Keywords: Analytic function, convolution, subordination, distortion bound, inclusion relation, integral transform, symmetric point.

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1. Introduction and preliminaries

In this article we assume that

(1.1) $\quad N = \{1, 2, 3, \cdots\}, k \in N \setminus \{1\}, -1 \leq B < A \leq 1, B \leq 0$ and $\lambda \geq 1$.

Let $A(p)$ be the class of functions of the form

(1.2) $$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N)$$

which are analytic and p-valent in the open unit disk $U = \{z : |z| < 1\}$.

For functions f and g analytic in U, the function f is said to be subordinate to g, written $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

Let

$$f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \in A(p) \quad (j = 1, 2).$$

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Then the Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,1} a_{n,2} z^n.$$

The following lemma will be required in our investigation.

1.1. Lemma. *Let $f \in A(p)$ defined by (1.2) satisfy*

$$\sum_{n=p+1}^{\infty} [\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| \le p(A-B). \tag{1.3}$$

Then

$$\frac{p(1-\lambda)z^p + \lambda z f'(z)}{f_{p,k}(z)} \prec p\frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.4}$$

where

$$\delta_{n,p,k} = \begin{cases} 0 & \left(\frac{n-p}{k} \notin N\right), \\ 1 & \left(\frac{n-p}{k} \in N\right) \end{cases} \tag{1.5}$$

for $n \ge p+1$,

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-jp} f(\varepsilon_k^j z) \quad \textit{and} \quad \varepsilon_k = exp\left(\frac{2\pi i}{k}\right). \tag{1.6}$$

Proof. For $f \in A(p)$ defined by (1.2), the function $f_{p,k}(z)$ in (1.6) can be expressed as

$$f_{p,k}(z) = z^p + \sum_{n=p+1}^{\infty} \delta_{n,p,k} a_n z^n, \tag{1.7}$$

where

$$\delta_{n,p,k} = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_n^{j(n-p)} = \begin{cases} 0 & \left(\frac{n-p}{k} \notin N\right), \\ 1 & \left(\frac{n-p}{k} \in N\right) \end{cases}$$

for $n \ge p+1$. Also, by (1.1) and (1.5), we see that

$$\lambda n - p\delta_{n,p,k} > 0 \quad (n \ge p+1) \tag{1.8}$$

and

$$pA\delta_{n,p,k} - \lambda Bn > 0 \quad (n \ge p+1). \tag{1.9}$$

Let the inequality (1.3) hold. Then from (1.7), (1.8) and (1.9) we deduce that

$$\begin{aligned} &\left|\frac{\frac{p(1-\lambda)z^p+\lambda z f'(z)}{f_{p,k}(z)} - p}{pA - B\frac{p(1-\lambda)z^p+\lambda z f'(z)}{f_{p,k}(z)}}\right| \\ &= \left|\frac{\sum_{n=p+1}^{\infty}(\lambda n - p\delta_{n,p,k})a_n z^{n-p}}{p(A-B) + \sum_{n=p+1}^{\infty}[pA\delta_{n,p,k} - \lambda Bn]a_n z^{n-p}}\right| \\ &\le \frac{\sum_{n=p+1}^{\infty}(\lambda n - p\delta_{n,p,k})|a_n|}{p(A-B) - \sum_{n=p+1}^{\infty}[pA\delta_{n,p,k} - \lambda Bn]|a_n|} \\ &\le 1 \quad (|z| = 1). \end{aligned}$$

Hence, by the maximum modulus theorem, we arrive at (1.4). □

We now consider the following two subclasses of $A(p)$.

1.2. Definition. A function $f \in A(p)$ defined by (1.2) is said to be in the class $Q_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

It follows from Lemma 1.1 that, if $f \in Q_{p,k}(\lambda, A, B)$, then the subordination relation (1.4) holds.

1.3. Definition. A function $f \in A(p)$ defined by (1.2) is said to be in the class $G_{p,k}(\lambda, A, B)$ if and only if it satisfies

$$\sum_{n=p+1}^{\infty} n[\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| \leq p^2(A-B). \tag{1.10}$$

It is clear that

$$f(z) \in G_{p,k}(\lambda, A, B) \Longleftrightarrow \frac{zf'(z)}{p} \in Q_{p,k}(\lambda, A, B). \tag{1.11}$$

If we write

$$\alpha_n = \alpha_{n,p,k}(\lambda, A, B) = \frac{\lambda n(1-B) - p\delta_{n,p,k}(1-A)}{p(A-B)}$$

and

$$\beta_n = \frac{n}{p}\alpha_n > \alpha_n \quad (n \geq p+1),$$

then it is easy to verify that

$$\frac{\partial \beta_n}{\partial \lambda} = \frac{n}{p}\frac{\partial \alpha_n}{\partial \lambda} > 0, \ \frac{\partial \beta_n}{\partial A} = \frac{n}{p}\frac{\partial \alpha_n}{\partial A} < 0 \ \text{ and } \ \frac{\partial \beta_n}{\partial B} = \frac{n}{p}\frac{\partial \alpha_n}{\partial B} \geq 0.$$

Therefore we have the following inclusion relations. If

$$1 \leq \lambda_0 < \lambda, -1 \leq B_0 \leq B < A \leq A_0 \leq 1 \text{ and } B \leq 0,$$

then

$$\begin{aligned} G_{p,k}(\lambda, A, B) &\subset Q_{p,k}(\lambda, A, B) \subset Q_{p,k}(\lambda_0, A_0, B_0) \\ &\subset Q_{p,k}(1, 1, -1) \subset \left\{ f \in A(p) : \operatorname{Re}\frac{zf'(z)}{f_{p,k}(z)} > 0 \quad (z \in U) \right\}. \end{aligned}$$

Thus, by Lemma 1.1, we see that each function in the classes $Q_{p,k}(\lambda, A, B)$ and $G_{p,k}(\lambda, A, B)$ is p-valent starlike with respect to k-symmetric points. Analytic (and meromorphic) functions which are starlike with respect to symmetric points and related functions have been extensively studied by several authors (see, e.g., [1]-[14]).

The object of this article is to investigate distortion bounds, inclusion relations and convolution properties for the classes $Q_{p,k}(\lambda, A, B)$ and $G_{p,k}(\lambda, A, B)$.

2. Distortion bounds

2.1. Theorem. *Suppose that either*

(a) $(k-1)(1-B) \geq p(1-A)$ *and* $\lambda \geq 1$, *or*

(b) $(k-1)(1-B) < p(1-A)$ *and* $\lambda \geq \frac{p(1-A)}{(k-1)(1-B)} > 1$.

(i) If $f \in Q_{p,k}(\lambda, A, B)$, *then for* $z \in U$,

$$|z|^p - \frac{p(A-B)}{\lambda(p+1)(1-B)}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{p(A-B)}{\lambda(p+1)(1-B)}|z|^{p+1}. \tag{2.1}$$

The bounds in (2.1) *are sharp for the function*

$$f(z) = z^p - \frac{p(A-B)}{\lambda(p+1)(1-B)}z^{p+1} \in Q_{p,k}(\lambda, A, B). \tag{2.2}$$

(ii) If $f \in G_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$p\left(|z|^{p-1} - \frac{p(A-B)}{\lambda(p+1)(1-B)}|z|^p\right) \leq |f'(z)| \leq p\left(|z|^{p-1} + \frac{p(A-B)}{\lambda(p+1)(1-B)}|z|^p\right). \tag{2.3}$$

The bounds in (2.3) are sharp for the function

$$f(z) = z^p - \frac{p^2(A-B)}{\lambda(p+1)^2(1-B)}z^{p+1} \in G_{p,k}(\lambda, A, B). \tag{2.4}$$

Proof. For $n \geq p+1$ and $\frac{n-p}{k} \in N$, we have $n = p + mk$ $(m \in N)$, $\delta_{n,p,k} = 1$, and

$$\lambda n(1-B) - p\delta_{n,p,k}(1-A) \geq \lambda(p+k)(1-B) - p(1-A). \tag{2.5}$$

For $n \geq p+1$ and $\frac{n-p}{k} \notin N$, we have $\delta_{n,p,k} = 0, \delta_{p+m,p,k} = 0$ $(1 \leq m \leq k-1)$ and

$$\lambda n(1-B) - p\delta_{n,p,k}(1-A) \geq \lambda(p+1)(1-B). \tag{2.6}$$

If either (a) or (b) is satisfied, then

$$\lambda(p+k)(1-B) - p(1-A) \geq \lambda(p+1)(1-B). \tag{2.7}$$

(i) If

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B),$$

then it follows from (2.5) to (2.7) that

$$\lambda(p+1)(1-B)\sum_{n=p+1}^{\infty}|a_n| \leq \sum_{n=p+1}^{\infty}[\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| \leq p(A-B).$$

Hence we have

$$|f(z)| \leq |z|^p + |z|^{p+1}\sum_{n=p+1}^{\infty}|a_n| \leq |z|^p + \frac{p(A-B)}{\lambda(p+1)(1-B)}|z|^{p+1}$$

and

$$|f(z)| \geq |z|^p - |z|^{p+1}\sum_{n=p+1}^{\infty}|a_n| \geq |z|^p - \frac{p(A-B)}{\lambda(p+1)(1-B)}|z|^{p+1} \geq 0$$

for $z \in U$.

(ii) If

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in G_{p,k}(\lambda, A, B),$$

then (2.5) to (2.7) yield

$$\lambda(p+1)(1-B)\sum_{n=p+1}^{\infty} n|a_n| \leq p^2(A-B).$$

From this we easily have (2.3). □

2.2. Theorem. *Let*

$$(k-1)(1-B) < p(1-A) \text{ and } 1 \leq \lambda < \frac{p(1-A)}{(k-1)(1-B)}. \tag{2.8}$$

(i) If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$|f(z)| \geq |z|^p - \sum_{n=p+1}^{p+k-1}|a_n||z|^n - \frac{p(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n|a_n|}{\lambda(p+k)(1-B) - p(1-A)}|z|^{p+k} \tag{2.9}$$

and

$$|f(z)| \le |z|^p + \sum_{n=p+1}^{p+k-1} |a_n||z|^n + \frac{p(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n|a_n|}{\lambda(p+k)(1-B) - p(1-A)} |z|^{p+k}. \tag{2.10}$$

Equalities in (2.9) *and* (2.10) *are attained, for example, by the function*

$$f(z) = z^p - \frac{p(A-B)}{\lambda(p+k)(1-B) - p(1-A)} z^{p+k} \in Q_{p,k}(\lambda, A, B). \tag{2.11}$$

(ii) If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in G_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$\begin{aligned} |f'(z)| \ge & p|z|^{p-1} - \sum_{n=p+1}^{p+k-1} n|a_n||z|^{n-1} \\ & - \frac{p^2(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n^2|a_n|}{\lambda(p+k)(1-B) - p(1-A)} |z|^{p+k-1} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} |f'(z)| \le & p|z|^{p-1} + \sum_{n=p+1}^{p+k-1} n|a_n||z|^{n-1} \\ & + \frac{p^2(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n^2|a_n|}{\lambda(p+k)(1-B) - p(1-A)} |z|^{p+k-1}. \end{aligned} \tag{2.13}$$

Equalities in (2.12) *and* (2.13) *are attained, for example, by the function*

$$f(z) = z^p - \frac{p^2(A-B)}{(p+k)[\lambda(p+k)(1-B) - p(1-A)]} z^{p+k} \in G_{p,k}(\lambda, A, B). \tag{2.14}$$

Proof. (i) If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B)$, then from (2.5), (2.6) and (2.8) we find that

$$\begin{aligned} p(A-B) \ge & \sum_{n=p+1}^{\infty} [\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| \\ \ge & \sum_{n=p+1}^{p+k-1} \lambda n(1-B)|a_n| \\ & + [\lambda(p+k)(1-B) - p(1-A)] \sum_{n=p+k}^{\infty} |a_n|. \end{aligned}$$

From this we easily have (2.9) and (2.10).

(ii) If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in G_{p,k}(\lambda, A, B)$, then we have

$$\begin{aligned} p^2(A-B) \ge & \sum_{n=p+1}^{\infty} n[\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| \\ \ge & \sum_{n=p+1}^{p+k-1} \lambda n^2(1-B)|a_n| \\ & + [\lambda(p+k)(1-B) - p(1-A)] \sum_{n=p+k}^{\infty} n|a_n|. \end{aligned}$$

This leads to (2.12) and (2.13). □

2.3. Theorem. *Let $f(z)$ given by (1.2) be in the class $Q_{p,k}(\lambda, A, B)$.*

(i) If $A = 1$, then for $z \in U$,

$$p\left(|z|^{p-1} - \frac{1}{\lambda}|z|^p\right) \leq |f'(z)| \leq p\left(|z|^{p-1} + \frac{1}{\lambda}|z|^p\right). \tag{2.15}$$

The bounds in (2.15) are sharp for the function

$$f(z) = z^p + \frac{p}{\lambda(p+1)}z^{p+1} \in Q_{p,k}(\lambda, A, B). \tag{2.16}$$

(ii) If $A < 1$, then for $z \in U$,

$$\begin{aligned} |f'(z)| \geq & p|z|^{p-1} - \sum_{n=p+1}^{p+k-1} n|a_n||z|^{n-1} \\ & - \frac{p(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n|a_n|}{\lambda(p+k)(1-B) - p(1-A)}(p+k)|z|^{p+k-1} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} |f'(z)| \leq & p|z|^{p-1} + \sum_{n=p+1}^{p+k-1} n|a_n||z|^{n-1} \\ & + \frac{p(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n|a_n|}{\lambda(p+k)(1-B) - p(1-A)}(p+k)|z|^{p+k-1}. \end{aligned} \tag{2.18}$$

Equalities in (2.17) and (2.18) are attained, for example, by the function $f(z)$ in (2.11).

Proof. (i) If $A = 1$, then

$$\sum_{n=p+1}^{\infty} [\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| = \lambda(1-B)\sum_{n=p+1}^{\infty} n|a_n| \leq p(1-B),$$

which gives (2.15).

(ii) If $A < 1$, then it is seen that

$$\frac{\lambda n(1-B) - p\delta_{n,p,k}(1-A)}{n} \geq \lambda(1-B) - \frac{p(1-A)}{p+k} \qquad \left(n \geq p+1, \frac{n-p}{k} \in N\right) \tag{2.19}$$

and

$$\frac{\lambda n(1-B) - p\delta_{n,p,k}(1-A)}{n} = \lambda(1-B) \qquad \left(n \geq p+1, \frac{n-p}{k} \notin N\right). \tag{2.20}$$

Using (2.19) and (2.20) we obtain

$$\begin{aligned} p(A-B) &\geq \sum_{n=p+1}^{\infty} [\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| \\ &\geq \lambda(1-B)\sum_{n=p+1}^{p+k-1} n|a_n| + \left[\lambda(1-B) - \frac{p(1-A)}{p+k}\right]\sum_{n=p+k}^{\infty} n|a_n|. \end{aligned}$$

From this we easily have (2.17) and (2.18). $\square$

2.4. Theorem. *Let $f(z)$ given by (1.2) be in the class $G_{p,k}(\lambda, A, B)$.*

(i) If either

$$(k-1)(2p+k+1)(1-B) \geq p(p+k)(1-A) \text{ and } \lambda \geq 1, \tag{2.21}$$

or

$$(2.22)\quad (k-1)(2p+k+1)(1-B) < p(p+k)(1-A)$$
$$\text{and } \lambda \geq \frac{p(p+k)(1-A)}{(k-1)(2p+k+1)(1-B)},$$

then for $z \in U$,

$$(2.23)\quad |z|^p - \frac{p^2(A-B)}{\lambda(p+1)^2(1-B)}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{p^2(A-B)}{\lambda(p+1)^2(1-B)}|z|^{p+1}.$$

The bounds in (2.23) *are sharp for the function given by* (2.4).

(ii) If

$$(2.24)\quad (k-1)(2p+k+1)(1-B) < p(p+k)(1-A)$$
$$\text{and } \ 1 \leq \lambda < \frac{p(p+k)(1-A)}{(k-1)(2p+k+1)(1-B)},$$

then for $z \in U$,

$$(2.25)\quad |f(z)| \geq |z|^p - \sum_{n=p+1}^{p+k-1} |a_n||z|^n - \frac{p^2(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n^2|a_n|}{(p+k)[\lambda(p+k)(1-B) - p(1-A)]}|z|^{p+k}$$

and

$$(2.26)\quad |f(z)| \leq |z|^p + \sum_{n=p+1}^{p+k-1} |a_n||z|^n + \frac{p^2(A-B) - \lambda(1-B)\sum_{n=p+1}^{p+k-1} n^2|a_n|}{(p+k)[\lambda(p+k)(1-B) - p(1-A)]}|z|^{p+k}.$$

The bounds in (2.25) *and* (2.26) *are attained, for example, by the function in* (2.14).

Proof. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in G_{p,k}(\lambda, A, B)$. For $n \geq p+1$ and $\frac{n-p}{k} \in N$, we have

$$(2.27)\quad n[\lambda n(1-B) - p\delta_{n,p,k}(1-A)] \geq (p+k)[\lambda(p+k)(1-B) - p(1-A)].$$

For $n \geq p+1$ and $\frac{n-p}{k} \notin N$, we have

$$(2.28)\quad n[\lambda n(1-B) - p\delta_{n,p,k}(1-A)] \geq \lambda(p+1)^2(1-B).$$

(i) If either (2.21) or (2.22) is satisfied, then

$$(2.29)\quad (p+k)[\lambda(p+k)(1-B) - p(1-A)] \geq \lambda(p+1)^2(1-B)$$

and it follows from (2.27) to (2.29) that

$$\lambda(p+1)^2(1-B)\sum_{n=p+1}^{\infty}|a_n| \leq \sum_{n=p+1}^{\infty} n[\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n| \leq p^2(A-B).$$

Thus we arrive at (2.23).

(ii) If (2.24) is satisfied, then we get

$$(p+k)[\lambda(p+k)(1-B) - p(1-A)] < \lambda(p+1)^2(1-B)$$

and so

$$p^2(A-B) \geq \sum_{n=p+1}^{\infty} n[\lambda n(1-B) - p\delta_{n,p,k}(1-A)]|a_n|$$

$$\geq \lambda(1-B) \sum_{n=p+1}^{p+k-1} n^2|a_n|$$

$$+ (p+k)[\lambda(p+k)(1-B) - p(1-A)] \sum_{n=p+k}^{\infty} |a_n|.$$

Now we easily have (2.25) and (2.26). □

3. Inclusion relation

In this section we generalize and improve the above mentioned inclusion relation

$$G_{p,k}(\lambda, A, B) \subset Q_{p,k}(\lambda, A, B). \tag{3.1}$$

3.1. Theorem. *If* $-1 \leq D \leq 0$*, then*

$$G_{p,k}(\lambda, A, B) \subset Q_{p,k}(\lambda, C(D), D), \tag{3.2}$$

where

$$C(D) = D + \frac{p(1-D)(A-B)}{(p+1)(1-B)}, \tag{3.3}$$

and the number $C(D)$ *cannot be decreased for each* D.

Proof. It is clear that $D < C(D) < 1$. Let $f(z) \in G_{p,k}(\lambda, A, B)$. In order to prove that $f(z) \in Q_{p,k}(\lambda, C(D), D)$, we need only to find the smallest C $(D < C < 1)$ such that

$$\frac{\lambda n(1-D) - p\delta_{n,p,k}(1-C)}{p(C-D)} \leq \frac{n[\lambda n(1-B) - p\delta_{n,p,k}(1-A)]}{p^2(A-B)} \tag{3.4}$$

for all $n \geq p+1$, that is, that

$$\frac{(\lambda n - p\delta_{n,p,k})(1-D)}{p(C-D)} + \delta_{n,p,k} \leq \frac{n}{p}\left\{\frac{(\lambda n - p\delta_{n,p,k})(1-B)}{p(A-B)} + \delta_{n,p,k}\right\} \quad (n \geq p+1). \tag{3.5}$$

For $n \geq p+1$ and $\frac{n-p}{k} \in N$, (3.5) is equivalent to

$$C \geq D + \frac{1-D}{\frac{n(1-B)}{p(A-B)} + \frac{n-p}{\lambda n - p}} = \varphi(n). \tag{3.6}$$

Noting that (1.1), a simple calculation shows that $\varphi(n)$ $(n \geq p+1, \lambda \geq 1)$ is decreasing in n. Hence

$$\varphi(n) \leq \varphi(p+k) = D + \frac{1-D}{\frac{(p+k)(1-B)}{p(A-B)} + \frac{k}{\lambda(p+k)-p}}. \tag{3.7}$$

For $n \geq p+1$ and $\frac{n-p}{k} \notin N$, (3.5) reduces to

$$C \geq D + \frac{1-D}{\frac{n(1-B)}{p(A-B)}} = \psi(n) \tag{3.8}$$

and we have

$$\psi(n) \leq \psi(p+1) = D + \frac{1-D}{\frac{(p+1)(1-B)}{p(A-B)}}. \tag{3.9}$$

It is obvious that $\varphi(p+k) < \psi(p+1)$. Thus, by taking $C = \psi(p+1) = C(D)$, it follows from (3.4) to (3.9) that $f(z) \in Q_{p,k}(\lambda, C(D), D)$.

Furthermore, for $D < C_0 < C(D)$, we have

$$\frac{\lambda(p+1)(1-D)}{p(C_0-D)} \cdot \frac{p^2(A-B)}{\lambda(p+1)^2(1-B)} > \frac{\lambda(p+1)(1-D)}{p(C(D)-D)} \cdot \frac{p^2(A-B)}{\lambda(p+1)^2(1-B)} = 1,$$

which implies that the function $f(z) \in G_{p,k}(\lambda, A, B)$ defined by (2.4) is not in the class $Q_{p,k}(\lambda, C_0, D)$. The proof of Theorem 3.1 is thus completed. □

Taking $D = B$, Theorem 3.1 reduces to the following result.

3.2. Corollary. *$G_{p,k}(\lambda, A, B) \subset Q_{p,k}(\lambda, C(B), B)$, where*

$$C(B) = B + \frac{p(A-B)}{p+1} \in (B, A)$$

cannot be decreased for each B.

Note that Corollary 3.2 refines the inclusion relation (3.1).

4. Integral transforms

4.1. Theorem. *Let $f(z) \in Q_{p,k}(\lambda, A, B)$ and*

$$I_\mu(z) = \frac{\mu+p}{z^\mu} \int_0^z t^{\mu-1} f(t)dt \quad (\mu > -p). \tag{4.1}$$

Then $I_\mu(z) \in Q_{p,k}(\lambda, C_1(D), D)$, where $-1 \le D \le 0$ and

$$C_1(D) = D + \frac{(\mu+p)(A-B)(1-D)}{(\mu+p+1)(1-B)}. \tag{4.2}$$

The number $C_1(D)$ cannot be decreased for each D.

Proof. Clearly $D < C_1(D) < 1$. For

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B),$$

it follows from (4.1) that

$$I_\mu(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\mu+p}{\mu+n} a_n z^n. \tag{4.3}$$

In order to prove that $I_\mu(z) \in Q_{p,k}(\lambda, C_1(D), D)$, we need only to find the smallest C $(D < C \le 1)$ such that

$$\frac{\lambda n(1-D) - p\delta_{n,p,k}(1-C)}{p(C-D)} \cdot \frac{\mu+p}{\mu+n} \le \frac{\lambda n(1-B) - p\delta_{n,p,k}(1-A)}{p(A-B)} \tag{4.4}$$

for all $n \ge p+1$.

For $n \ge p+1$ and $\frac{n-p}{k} \in N$, (4.4) reduces to

$$C \ge D + \frac{1-D}{\frac{(\mu+n)(1-B)}{(\mu+p)(A-B)} + \frac{p(n-p)}{(\mu+p)(\lambda n-p)}} = \varphi_1(n). \tag{4.5}$$

It is easy to show that $\varphi_1(n)$ $(n \ge p+1, \lambda \ge 1)$ is decreasing in n and so

$$\varphi_1(n) \le \varphi_1(p+k) = D + \frac{1-D}{\frac{(\mu+p+k)(1-B)}{(\mu+p)(A-B)} + \frac{pk}{(\mu+p)(\lambda p+\lambda k-p)}}. \tag{4.6}$$

For $n \geq p+1$ and $\frac{n-p}{k} \notin N$, (4.4) becomes

$$C \geq D + \frac{1-D}{\frac{(\mu+n)(1-B)}{(\mu+p)(A-B)}} = \psi_1(n) \tag{4.7}$$

and we have

$$\psi_1(n) \leq \psi_1(p+1) = D + \frac{1-D}{\frac{(\mu+p+1)(1-B)}{(\mu+p)(A-B)}}. \tag{4.8}$$

It is clear that $\varphi_1(p+k) < \psi_1(p+1)$. Therefore, by taking $C = \psi_1(p+1) = C_1(D)$, it follows from (4.4) to (4.8) that $I_\mu(z) \in Q_{p,k}(\lambda, C_1(D), D)$.

Furthermore the number $C_1(D)$ is best possible for the function $f(z)$ defined by (2.2). The proof of the theorem is completed. □

4.2. Theorem. *Let $I_\mu(z)$ $(\mu > -p)$ and $C_1(D)$ $(-1 \leq D \leq 0)$ be the same as in Theorem 4.1. If $f(z) \in G_{p,k}(\lambda, A, B)$, then $I_\mu(z) \in G_{p,k}(\lambda, C_1(D), D)$ and the number $C_1(D)$ cannot be decreased for each D.*

Proof. In view of (4.3) we have

$$I_\mu(z) = \left(z^p + \sum_{n=p+1}^{\infty} \frac{\mu+p}{\mu+n} z^n \right) * f(z)$$

and so

$$\frac{zI'_\mu(z)}{p} = \left(z^p + \sum_{n=p+1}^{\infty} \frac{\mu+p}{\mu+n} z^n \right) * \left(\frac{zf'(z)}{p} \right). \tag{4.9}$$

From (4.9) and (1.11), an application of Theorem 4.1 yields Theorem 4.2. □

4.3. Corollary. *Let $f(z) \in G_{p,k}(\lambda, A, B)$ and $I_\mu(z)$ $(\mu > -p)$ be the same as in Theorem 4.1. Then $I_\mu(z) \in Q_{p,k}(\lambda, C_2(D), D)$, where $-1 \leq D \leq 0$ and*

$$C_2(D) = D + \frac{p(\mu+p)(A-B)(1-D)}{(p+1)(\mu+p+1)(1-B)}.$$

The number $C_2(D)$ cannot be decreased for each D.

Proof. Clearly $D < C_2(D) < 1$. Let $f(z) \in G_{p,k}(\lambda, A, B)$. Then it follows from Theorem 4.2 and Corollary 3.2 that $I_\mu(z) \in Q_{p,k}(\lambda, C(D), D)$, where $-1 \leq D \leq 0$ and

$$C(D) = D + \frac{p(C_1(D) - D)}{p+1} = D + \frac{p(\mu+p)(A-B)(1-D)}{(p+1)(\mu+p+1)(1-B)} = C_2(D).$$

Furthermore, for the function $f(z) \in G_{p,k}(\lambda, A, B)$ given by (2.4) and $D < C_0 < C_2(D)$, we have

$$I_\mu(z) = z^p - \frac{p^2(A-B)(\mu+p)}{\lambda(p+1)^2(1-B)(\mu+p+1)} z^{p+1}$$

and

$$\begin{aligned} &\frac{\lambda(p+1)(1-D)}{p(C_0-D)} \cdot \frac{p^2(A-B)(\mu+p)}{\lambda(p+1)^2(1-B)(\mu+p+1)} \\ &> \frac{1-D}{C_2(D)-D} \cdot \frac{p(A-B)(\mu+p)}{(p+1)(1-B)(\mu+p+1)} \\ &= 1. \end{aligned}$$

Hence $I_\mu(z) \notin Q_{p,k}(\lambda, C_0, D)$ and the proof of Corollary 4.3 is completed. □

5. Convolution properties

In this section, we assume that

$$-1 \le B_j < A_j \le 1 \text{ and } B_j \le 0 \quad (j=1,2). \tag{5.1}$$

5.1. Theorem. *Let $f_j(z) \in Q_{p,k}(\lambda, A_j, B_j)$ $(j=1,2)$.*

(i) If

$$p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] \ge (p-k+1)(1-B_1)(1-B_2) \quad \text{and } \lambda \ge 1, \tag{5.2}$$

then $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, A(B), B)$, where

$$A(B) = B + \frac{p(1-B)}{\lambda(p+1)} \prod_{j=1}^{2} \frac{A_j - B_j}{1-B_j} \tag{5.3}$$

and the number $A(B)$ cannot be decreased for each B.

(ii) If

$$p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] < (p-k+1)(1-B_1)(1-B_2) \quad \text{and } \lambda \ge \lambda_1, \tag{5.4}$$

where

$$1 < \lambda_1 = \frac{p(1-B_1)(1-B_2) - p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)]}{(k-1)(1-B_1)(1-B_2)}, \tag{5.5}$$

then $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, A(B), B)$ and the number $A(B)$ cannot be decreased for each B.

(iii) If

$$p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] < (p-k+1)(1-B_1)(1-B_2) \quad \text{and } 1 \le \lambda < \lambda_1, \tag{5.6}$$

then $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, \widetilde{A}(B), B)$, where

$$\widetilde{A}(B) = B + \frac{1-B}{\frac{\lambda(p+k)-p}{p} \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j}}, \tag{5.7}$$

and the number $\widetilde{A}(B)$ cannot be decreased for each B.

Proof. Clearly $B < A(B) < 1$ and $B < \widetilde{A}(B) < 1$. Let

$$f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \in Q_{p,k}(\lambda, A_j, B_j) \quad (j=1,2).$$

Then

$$\begin{aligned} &\sum_{n=p+1}^{\infty} \left\{ \prod_{j=1}^{2} \frac{\lambda n(1-B_j) - p\delta_{n,p,k}(1-A_j)}{p(A_j-B_j)} \right\} |a_{n,1}a_{n,2}| \\ &\le \prod_{j=1}^{2} \left\{ \sum_{n=p+1}^{\infty} \frac{\lambda n(1-B_j) - p\delta_{n,p,k}(1-A_j)}{p(A_j-B_j)} |a_{n,j}| \right\} \le 1. \end{aligned} \tag{5.8}$$

Also, $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, A, B)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{\lambda n(1-B) - p\delta_{n,p,k}(1-A)}{p(A-B)} |a_{n,1}a_{n,2}| \le 1. \tag{5.9}$$

In order to prove the theorem, it follows from (5.8) and (5.9) that we need only to find the smallest A such that

$$\frac{\lambda n(1-B)-p\delta_{n,p,k}(1-A)}{p(A-B)} \le \prod_{j=1}^{2} \frac{\lambda n(1-B_j)-p\delta_{n,p,k}(1-A_j)}{p(A_j-B_j)} \tag{5.10}$$

for all $n \ge p+1$, that is, that

$$A \ge B + \frac{(1-B)\left(\frac{\lambda n-p\delta_{n,p,k}}{p}\right)}{\prod_{j=1}^{2}\left\{\frac{(\lambda n-p\delta_{n,p,k})(1-B_j)}{p(A_j-B_j)}+\delta_{n,p,k}\right\}-\delta_{n,p,k}} \quad (n \ge p+1). \tag{5.11}$$

For $n \ge p+1$ and $\frac{n-p}{k} \in N$, (5.11) becomes

$$A \ge B + \frac{1-B}{\frac{\lambda n-p}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}+\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \varphi_2(n). \tag{5.12}$$

The function $\varphi_2(n)$ $(n \ge p+1, \lambda \ge 1)$ is decreasing in n and hence

$$\varphi_2(n) \le \varphi_2(p+k) = B + \frac{1-B}{\frac{\lambda(p+k)-p}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}+\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j}}. \tag{5.13}$$

For $n \ge p+1$ and $\frac{n-p}{k} \notin N$, (5.11) becomes

$$A \ge B + \frac{1-B}{\frac{\lambda n}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \psi_2(n)$$

and we have

$$\psi_2(n) \le \psi_2(p+1) = B + \frac{1-B}{\frac{\lambda(p+1)}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}}. \tag{5.14}$$

Now

$$\begin{aligned}\frac{\lambda(p+k)-p}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}+\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j}-\frac{\lambda(p+1)}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}\\ =\frac{h(\lambda)}{p(A_1-B_1)(A_2-B_2)},\end{aligned} \tag{5.15}$$

where

$$\begin{aligned} h(\lambda) &= [\lambda(p+k)-p](1-B_1)(1-B_2)+p[(1-B_1)(A_2-B_2)\\ &\qquad +(1-B_2)(A_1-B_1)]-\lambda(p+1)(1-B_1)(1-B_2)\\ &= a\lambda+b,\\ a &= (k-1)(1-B_1)(1-B_2)\end{aligned} \tag{5.16}$$

and

$$b = p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)]-p(1-B_1)(1-B_2).$$

Note that $a>0$ and

$$h(1) = (k-1-p)(1-B_1)(1-B_2)+p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)]. \tag{5.17}$$

If either (5.2) or (5.4) is satisfied, then it follows from (5.10) to (5.17) that $h(\lambda) \ge 0$, $\varphi_2(p+k) \le \psi_2(p+1) = A(B)$ and $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, A(B), B)$. Furthermore, for

$B < A_0 < A(B)$, we have

$$\frac{\lambda(p+1)(1-B)}{p(A_0-B)}\prod_{j=1}^{2}\frac{p(A_j-B_j)}{\lambda(p+1)(1-B_j)}$$
$$> \frac{\lambda(p+1)(1-B)}{p(A(B)-B)}\prod_{j=1}^{2}\frac{p(A_j-B_j)}{\lambda(p+1)(1-B_j)} = 1.$$

Therefore the functions

$$f_j(z) = z^p - \frac{p(A_j-B_j)}{\lambda(p+1)(1-B_j)}z^{p+1} \in Q_{p,k}(\lambda, A_j, B_j) \quad (j=1,2)$$

show that $(f_1 * f_2)(z) \notin Q_{p,k}(\lambda, A_0, B)$. This proves (i) and (ii).

(iii) If (5.6) is satisfied, then we have $h(\lambda) < 0$ $(1 \le \lambda < \lambda_1)$, $\psi_2(p+1) < \varphi_2(p+k) = \widetilde{A}(B)$ and $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, \widetilde{A}(B), B)$. Furthermore, the number $\widetilde{A}(B)$ cannot be decreased as can be seen from the functions

$$f_j(z) = z^p - \frac{p(A_j-B_j)}{\lambda(p+k)(1-B_j)-p(1-A_j)}z^{p+k} \in Q_{p,k}(\lambda, A_j, B_j) \quad (j=1,2).$$

□

5.2. Corollary. *Let $f_1(z) \in Q_{p,k}(\lambda, A_1, B_1)$, $f_2(z) \in G_{p,k}(\lambda, A_2, B_2)$ and let $A(B), \widetilde{A}(B), \lambda_1$ be given as in Theorem 5.1.*

(i) If $p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] \ge (p-k+1)(1-B_1)(1-B_2)$ and $\lambda \ge 1$, then $(f_1 * f_2)(z) \in G_{p,k}(\lambda, A(B), B)$ and the number $A(B)$ cannot be decreased for each B.

(ii) If $p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] < (p-k+1)(1-B_1)(1-B_2)$ and $\lambda \ge \lambda_1$, then $(f_1 * f_2)(z) \in G_{p,k}(\lambda, A(B), B)$ and the number $A(B)$ cannot be decreased for each B.

(iii) If $p[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] < (p-k+1)(1-B_1)(1-B_2)$ and $1 \le \lambda < \lambda_1$, then $(f_1 * f_2)(z) \in G_{p,k}(\lambda, \widetilde{A}(B), B)$ and the number $\widetilde{A}(B)$ cannot be decreased for each B.

Proof. Since

$$f_1(z) \in Q_{p,k}(\lambda, A_1, B_1), \quad \frac{zf_2'(z)}{p} \in Q_{p,k}(\lambda, A_2, B_2)$$

(see (1.11)), and

$$f_1(z) * \frac{zf_2'(z)}{p} = \frac{z(f_1 * f_2)'(z)}{p} \quad (z \in U),$$

an application of Theorem 5.1 yields Corollary 5.2. □

5.3. Theorem. *Let $f_1(z) \in Q_{p,k}(\lambda, A_1, B_1)$ and $f_2(z) \in G_{p,k}(\lambda, A_2, B_2)$.*

(i) If $p(p+k)[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] + p^2(A_1-B_1)(A_2-B_2) \ge [(p+1)^2 - k(p+k)](1-B_1)(1-B_2)$ and $\lambda \ge 1$, then $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, A_1(B), B)$, where

$$A_1(B) = B + \frac{p^2(1-B)}{\lambda(p+1)^2}\prod_{j=1}^{2}\frac{A_j-B_j}{1-B_j} \tag{5.18}$$

and the number $A_1(B)$ cannot be decreased for each B.

(ii) If $p(p+k)[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] + p^2(A_1-B_1)(A_2-B_2) < [(p+1)^2 - k(p+k)](1-B_1)(1-B_2)$ and $\lambda \ge \lambda_2$, where λ_2 is the root in $(1, +\infty)$ of the

equation $h_1(\lambda) = a_1\lambda^2 + b_1\lambda + c_1 = 0$,

$$\tag{5.19} \begin{cases} a_1 = (p+k)[(p+k)^2 - (p+1)^2], \\ b_1 = p[(p+1)^2 - 2(p+k)^2](1-B_1)(1-B_2) \\ \qquad +p(p+k)^2[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)] \\ c_1 = p^2(p+k)(1-B_1)(1-B_2) + kp^2(A_1-B_1)(A_2-B_2) \\ \qquad -p^2(p+k)[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)], \end{cases}$$

then $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, A_1(B), B)$ *and the number* $A_1(B)$ *cannot be decreased for each* B.

(iii) If $p(p+k)[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)] + p^2(A_1-B_1)(A_2-B_2) < [(p+1)^2 - k(p+k)](1-B_1)(1-B_2)$ *and* $1 \le \lambda < \lambda_2$, *then* $(f_1 * f_2)(z) \in Q_{p,k}(\lambda, \widetilde{A_1}(B), B)$, *where*

$$\tag{5.20} \widetilde{A_1}(B) = B + \frac{1-B}{\frac{(p+k)(\lambda(p+k)-p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{p+k}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{k}{\lambda(p+k)-p}},$$

and the number $\widetilde{A_1}(B)$ *cannot be decreased for each* B.

Proof. It is clear that $B < A_1(B) < 1$ and $B < \widetilde{A_1}(B) < 1$. In order to prove Theorem 5.3, we need only to find the smallest A such that

$$\tag{5.21} \frac{\lambda n(1-B) - p\delta_{n,p,k}(1-A)}{p(A-B)} \le \frac{n}{p}\prod_{j=1}^{2}\frac{\lambda n(1-B_j) - p\delta_{n,p,k}(1-A_j)}{p(A_j-B_j)}$$

for all $n \ge p+1$, that is, that

$$\tag{5.22} A \ge B + \frac{(1-B)\left(\frac{\lambda n - p\delta_{n,p,k}}{p}\right)}{\frac{n}{p}\prod_{j=1}^{2}\left\{\frac{(\lambda n - p\delta_{n,p,k})(1-B_j)}{p(A_j-B_j)} + \delta_{n,p,k}\right\} - \delta_{n,p,k}} \quad (n \ge p+1).$$

For $n \ge p+1$ and $\frac{n-p}{k} \in N$, (5.22) becomes

$$\tag{5.23} A \ge B + \frac{1-B}{\frac{n(\lambda n-p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{n}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{n-p}{\lambda n-p}} = \varphi_3(n).$$

It is easy to verify that $\varphi_3(n)$ $(n \ge p+1, \lambda \ge 1)$ is decreasing in n and hence

$$\tag{5.24} \begin{aligned} \varphi_3(n) &\le \varphi_3(p+k) \\ &= B + \frac{1-B}{\frac{(p+k)(\lambda(p+k)-p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{p+k}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{k}{\lambda(p+k)-p}} \\ &= \widetilde{A_1}(B). \end{aligned}$$

For $n \ge p+1$ and $\frac{n-p}{k} \notin N$, (5.22) reduces to

$$\tag{5.25} A \ge B + \frac{1-B}{\frac{\lambda n^2}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \psi_3(n)$$

and we have

$$\tag{5.26} \psi_3(n) \le \psi_3(p+1) = B + \frac{1-B}{\frac{\lambda(p+1)^2}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = A_1(B).$$

Now

$$\frac{(p+k)[\lambda(p+k)-p]-\lambda(p+1)^2}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} \tag{5.27}$$

$$+\frac{p+k}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j}+\frac{k}{\lambda(p+k)-p}$$

$$=\frac{h_1(\lambda)}{p^2[\lambda(p+k)-p](A_1-B_1)(A_2-B_2)},$$

where

$$\begin{aligned} h_1(\lambda) = &(p+k)[\lambda(p+k)-p]^2(1-B_1)(1-B_2) \\ &+p(p+k)[\lambda(p+k)-p][(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] \\ &-\lambda(p+1)^2[\lambda(p+k)-p](1-B_1)(1-B_2) \\ &+kp^2(A_1-B_1)(A_2-B_2) \\ = &a_1\lambda^2+b_1\lambda+c_1 \end{aligned}$$

and a_1, b_1, c_1 are given by (5.19), Note that $a_1 > 0$ and

$$\begin{aligned} h_1(1) = &k[k(p+k)-(p+1)^2](1-B_1)(1-B_2) \\ &+kp(p+k)[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)] \\ &+kp^2(A_1-B_1)(A_2-B_2). \end{aligned}$$

The remaining part of the proof of Theorem 5.3 is much akin to that of Theorem 5.1.

Furthermore, the number $A_1(B)$ is best possible for the functions

$$f_1(z)=z^p-\frac{p(A_1-B_1)}{\lambda(p+1)(1-B_1)}z^{p+1}\in Q_{p,k}(\lambda,A_1,B_1),$$

$$f_2(z)=z^p-\frac{p^2(A_2-B_2)}{\lambda(p+1)^2(1-B_2)}z^{p+1}\in G_{p,k}(\lambda,A_2,B_2),$$

and the number $\widetilde{A_1}(B)$ is best possible for the functions

$$f_1(z)=z^p-\frac{p(A_1-B_1)}{\lambda(p+k)(1-B_1)-p(1-A_1)}z^{p+k}\in Q_{p,k}(\lambda,A_1,B_1),$$

$$f_2(z)=z^p-\frac{p^2(A_2-B_2)}{(p+k)[\lambda(p+k)(1-B_2)-p(1-A_2)]}z^{p+k}\in G_{p,k}(\lambda,A_2,B_2).$$

□

5.4. Corollary. *Let* $f_j(z)\in G_{p,k}(\lambda,A_j,B_j)$ $(j=1,2)$ *and let* $A_1(B),\widetilde{A_1}(B),\lambda_2$ *be given as in Theorem 5.3.*

(i) If $p(p+k)[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)]+p^2(A_1-B_1)(A_2-B_2)\geq[(p+1)^2-k(p+k)](1-B_1)(1-B_2)$ *and* $\lambda\geq 1$, *then* $(f_1*f_2)(z)\in G_{p,k}(\lambda,A_1(B),B)$ *and the number* $A_1(B)$ *cannot be decreased for each* B.

(ii) If $p(p+k)[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)]+p^2(A_1-B_1)(A_2-B_2)<[(p+1)^2-k(p+k)](1-B_1)(1-B_2)$ *and* $\lambda\geq\lambda_2$, *then* $(f_1*f_2)(z)\in G_{p,k}(\lambda,A_1(B),B)$ *and the number* $A_1(B)$ *cannot be decreased for each* B.

(iii) If $p(p+k)[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)]+p^2(A_1-B_1)(A_2-B_2)<[(p+1)^2-k(p+k)](1-B_1)(1-B_2)$ *and* $1\leq\lambda<\lambda_2$, *then* $(f_1*f_2)(z)\in Q_{p,k}(\lambda,\widetilde{A_1}(B),B)$ *and the number* $\widetilde{A_1}(B)$ *cannot be decreased for each* B.

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