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Some Generating Functions for a Class of Hypergeometric Polynomials

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Abstract

This article concerned with some new features for hypergeometric polynomials $S_k^{(\alpha,\beta)}(x)$. The results obtained here contain the various families of multilinear and multilateral generating functions, various features and some exceptions for these polynomials. We will also give a theorem to families giving certain bilateral generating functions for hypergeometric polynomials $S_k^{(\alpha,\beta)}(x)$ and generalized Lauricella functions. Finally, we get a few interesting results for this given theorem.

1. INTRODUCTION

The hypergeometric polynomials $S_k^{(\alpha,\beta)}(x)$ defined by [1]

$$S_k^{(\alpha,\beta)}(x) := \binom{\alpha+k-1}{k} {}_2F_1(-k,\beta;\alpha;x), \ (x,\alpha,\beta \in C). \tag{1}$$

These polynomials have the following linear generating function [2]:

$$\sum_{k=0}^{\infty} S_k^{(\alpha,\beta)}(x)t^k = (1-t)^{-\alpha}\left(1+\frac{xt}{1-t}\right)^{-\beta} = (1-t)^{\beta-\alpha}[1-(1-x)t]^{-\beta}, \quad \left(|t| < \min\{1, |1-x|^{-1}\}\right) \tag{2}$$

and the following extended generating function holds true [1]:

$$\sum_{k=0}^{\infty}\binom{m+k}{k} S_{m+k}^{(\alpha,\beta)}(x)t^k = (1-t)^{\beta-\alpha-m}[1-(1-x)t]^{-\beta} S_m^{(\alpha,\beta)}\left(\frac{x}{1-(1-x)t}\right). \quad \left(|t| < \min\{1, |1-x|^{-1}\} \ ; \ m \in \mathrm{N}_0\right) \tag{3}$$

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Here, we expanded the validity region based on the principle of analytic continuation on t.
The definitions and icons used in last section are presented here as follows:
Before everything, the four Appell functions of two variables, indicated by F_1, F_2, F_3 and F_4, (see [2], p. 53), were generalized by Lauricella [3] to functions of n variables, staded by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ (see [3], p. 60), where and

$$F_A^{(2)} = F_2,\ F_B^{(2)} = F_3,\ F_C^{(2)} = F_4,\ F_D^{(2)} = F_1.$$

Further generalization of two-variables hypergeometric function is defined generalized Lauricella function as follows (see [3-7]):

$$F^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}\left(\begin{matrix}\left[(a):\theta^{(1)},\dots,\theta^{(n)}\right]: & \left[(b^{(1)}):\ \phi^{(1)}\right]; & \dots; & \left[(b^{(n)}):\ \phi^{(n)}\right]; \\ \left[(c):\psi^{(1)},\dots,\psi^{(n)}\right]: & \left[(d^{(1)}):\ \delta^{(1)}\right]; & \dots; & \left[(d^{(n)}):\ \delta^{(n)}\right];\end{matrix}\ z_1,\dots,z_n\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \Omega(m_1,\dots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},$$

where for convenience,

$$\Omega(m_1,\dots,m_n) := \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\dots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\dots+m_n\psi_j^{(n)}}}\frac{\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}}\cdots\frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$$\theta_j^{(k)}\ (j=1,\dots,A;\ k=1,\dots,n),\ \phi_j^{(k)}\ (j=1,\dots,B^{(k)};\ k=1,\dots,n),$$

$$\psi_j^{(k)}\ (j=1,\dots,C;\ k=1,\dots,n),\ \delta_j^{(k)}\ (j=1,\dots,D^{(k)};\ k=1,\dots,n)$$

are real constants and $\left(b^{(k)}_{B^{(k)}}\right)$ abbreviates with similar comments for other parameters [8].
Here, $(\lambda)_v$ denotes the Pochhammer symbol, since

$$(1)_n = n!\ \ (n \in N_0)$$

which is defined by

$$(\lambda)_v = \begin{cases} 1, & \text{if } v=0;\ \lambda \in C\setminus\{0\} \\ \lambda(\lambda+1)\dots(\lambda+n-1), & \text{if } v=n\in N;\ \lambda\in C \end{cases},$$

this should be understood as conventional $(0)_0 := 1$.

The main purpose of this manuscript is to obtain families of multilinear and multilateral generating functions, various features and also various special cases for hypergeometric polynomials $S_k^{(\alpha,\beta)}(x)$. Furthermore, for these polynomials, a theorem is given that gives certain families of the bilateral generating functions and generalized Lauricella functions. There are a few interesting results of this theorem.

2. BILINEAR AND BILATERAL GENERATING FUNCTIONS

In this part, bilinear and bilateral generating functions of various families will be obtained for hypergeometric polynomials $S_k^{(\alpha,\beta)}(x)$ given by (1). Similar studies of the method used in this section are [9-12].

Lemma 1. *The following addition formula applies to hypergeometric polynomials* $S_k^{(\alpha,\beta)}(x)$:

$$S_k^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x)=\sum_{m=0}^{k}S_{k-m}^{(\alpha,\beta)}(x)S_m^{(\alpha,\beta)}(x). \tag{4}$$

Proof: *Replacing* α *by* $\alpha_1+\alpha_2$ *and* β *by* $\beta_1+\beta_2$ *in (2), we obtain*

$$\begin{aligned}\sum_{k=0}^{\infty}S_k^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x)t^k &= (1-t)^{\beta_1+\beta_2-\alpha_1-\alpha_2}\left[1-(1-x)t\right]^{-\beta_1-\beta_2}\\ &=\left\{(1-t)^{\beta_1-\alpha_1}\left[1-(1-x)t\right]^{-\beta_1}\right\}\left\{(1-t)^{\beta_2-\alpha_2}\left[1-(1-x)t\right]^{-\beta_2}\right\}\\ &=\sum_{k=0}^{\infty}S_k^{(\alpha,\beta)}(x)t^k\sum_{m=0}^{\infty}S_m^{(\alpha,\beta)}(x)t^m\\ &=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}S_k^{(\alpha,\beta)}(x)S_m^{(\alpha,\beta)}(x)t^{k+m}\\ &=\sum_{k=0}^{\infty}\sum_{m=0}^{k}S_{k-m}^{(\alpha,\beta)}(x)S_m^{(\alpha,\beta)}(x)t^k.\end{aligned}$$

Comparing of the coefficients of t^k, *lemma is proved.*

Theorem 1. *For a non-vanishing function* $\Omega_\mu(y_1,\ldots,y_r)$ *of* r *complex variables* $y_1,\ldots,y_r$ $(r\in N)$ *and of complex order* μ, *let*

$$\Lambda_{\mu,\psi}(y_1,\ldots,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,\ldots,y_r)\zeta^k$$
$$(a_k\neq 0,\ \mu,\psi\in C)$$

and

$$\Theta_{n,p}^{\mu,\psi}\left(x;y_1,\ldots,y_r;\xi\right):=\sum_{k=0}^{[n/p]}a_kS_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,\ldots,y_r)\xi^k.$$

Then,

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\frac{\xi}{t^p}\right)t^n=(1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\Lambda_{\mu,\psi}(y_1,...,y_r;\xi). \tag{5}$$

Proof: *Let S show the first member of the claim (5). Then,*

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k S_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\frac{\xi^k}{t^{pk}}t^n.$$

Replacing n by $n+pk$, then

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\, S_n^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\frac{\xi^k}{t^{pk}}t^{n+pk}$$

$$=\sum_{n=0}^{\infty}S_n^{(\alpha,\beta)}(x)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k$$

$$=(1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\Lambda_{\mu,\psi}(y_1,...,y_r;\xi)$$

which completes the proof.

Theorem 2. *For a non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in N)$ and of complex order μ, let*

$$\Lambda_{n,\,p,\,\mu,\psi}^{(\alpha_1+\alpha_2,\,\beta_1+\beta_2)}(x;y_1,...,y_r;\eta):=\sum_{k=0}^{[n/p]}a_k S_{n-pk}^{(\alpha_1+\alpha_2,\,\beta_1+\beta_2)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k,$$
$$(a_k\neq 0,\ \mu,\ \psi\in C,\ n,p\in N).$$

Then,

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l S_{n-k}^{(\alpha_1,\beta_1)}(x)S_{k-pl}^{(\alpha_2,\beta_2)}(x)\Omega_{\mu+\psi l}(y_1,...,y_r)\eta^l$$
$$=\Lambda_{n,\,p,\,\mu,\psi}^{(\alpha_1+\alpha_2,\,\beta_1+\beta_2)}(x;y_1,...,y_r;\eta). \tag{6}$$

Proof: *Let T denote the first member of the assetion (6). Then, upon subsituting for the polynomials $S_n^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x)$ from the (4) of Lemma 1 into the left-hand side of (6), we obtain*

$$T=\sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl}a_l S_{n-k-pl}^{(\alpha_1,\beta_1)}(x)S_k^{(\alpha_2,\beta_2)}(x)\Omega_{\mu+\psi l}(y_1,...,y_r)\eta^l$$

$$=\sum_{l=0}^{[n/p]}a_l\left(\sum_{k=0}^{n-pl}S_{n-k-pl}^{(\alpha_1,\beta_1)}(x)S_k^{(\alpha_2,\beta_2)}(x)\right)\Omega_{\mu+\psi l}(y_1,...,y_r)\eta^l$$

$$=\sum_{l=0}^{[n/p]}a_l S_{n-pl}^{(\alpha_1+\alpha_2,\,\beta_1+\beta_2)}(x)\Omega_{\mu+\psi l}(y_1,...,y_r)\eta^l$$

$$=\Lambda^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}_{n,p,\mu,\psi}(x;y_1,...,y_r;\eta).$$

Theorem 3. *For a non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in N)$ and of complex order μ, let*

$$\Lambda_{\mu,p,q}(x;y_1,...,y_r;t):=\sum_{n=0}^{\infty}a_n S^{(\alpha,\beta)}_{n+qn}(x)\Omega_{\mu+pn}(y_1,...,y_r)\frac{t^n}{(nq)!}$$

where $a_n\neq 0$, $\mu\in C$ *and*

$$\theta_{n,p,q}(y_1,...,y_r;\eta):=\sum_{k=0}^{[n/q]}\binom{n+m}{n-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r)\eta^k.$$

We have

$$\sum_{n=0}^{\infty}S^{(\alpha,\beta)}_{n+m}(x)\theta_{n,p,q}(y_1,...,y_r;\eta)\frac{t^n}{n!}$$

$$=(1-t)^{\beta-\alpha-m}\left[1-(1-x)t\right]^{-\beta}\Lambda_{\mu,p,q}\left(\frac{x}{1-(1-x)t};y_1,...,y_r;\eta\left(\frac{t}{1-t}\right)^q\right). \tag{7}$$

Proof: *Let T denote the first member of the assertion (7). Then,*

$$T=\sum_{n=0}^{\infty}S^{(\alpha,\beta)}_{n+m}(x)\sum_{k=0}^{[n/q]}\binom{n+m}{n-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r)\eta^k\frac{t^n}{n!}.$$

Using the relation (3) and replacing n by $n+qk$, then

$$T=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+m+qk}{n}S^{(\alpha,\beta)}_{n+m+qk}(x)a_k\,\Omega_{\mu+pk}(y_1,...,y_r)\eta^k\frac{t^{n+qk}}{(n+qk)!}$$

$$=\sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\binom{n+m+qk}{n}S^{(\alpha,\beta)}_{n+m+qk}(x)t^n\right)\Omega_{\mu+pk}(y_1,...,y_r)\frac{(\eta t^q)^k}{(kq)!}$$

$$=\sum_{k=0}^{\infty}a_k\left[(1-t)^{\beta-\alpha-m+qk}\left[1-(1-x)t\right]^{-\beta}S^{(\alpha,\beta)}_{m+qk}\left(\frac{x}{1-(1-x)t}\right)\right]\Omega_{\mu+pk}(y_1,...,y_r)\frac{(\eta t^q)^k}{(kq)!}$$

$$=(1-t)^{\beta-\alpha-m}\left[1-(1-x)t\right]^{-\beta}\sum_{k=0}^{\infty}a_kS^{(\alpha,\beta)}_{m+qk}\left(\frac{x}{1-(1-x)t}\right)\Omega_{\mu+pk}(y_1,...,y_r)\frac{\left(\eta\left(\frac{t}{1-t}\right)^q\right)^k}{(kq)!}$$

$$=(1-t)^{\beta-\alpha-m}\left[1-(1-x)t\right]^{-\beta}\Lambda_{\mu,p,q}\left(\frac{x}{1-(1-x)t};y_1,...,y_r;\eta\left(\frac{t}{1-t}\right)^q\right),$$

which completes the proof.

3. SPECIAL CASES

$\Omega_{\mu+\psi k}(y_1,...,y_r)$ $(k \in N_0,\ r \in N)$ multivariable function, when a variable is expressed in terms of more and more simple functions, then we can provide the further embodiment of the above theorems.

By choosing $r=1$ and

$$\Omega_{\mu+\psi k}(y_1) = c_{\mu+\psi k}(\beta; y)$$

in Theorem 1, where Poisson-Charlier polynomials $c_n(\alpha; x)$ are generated by [13, 14]

$$\sum_{n=0}^{\infty} c_n(\beta; x)\frac{t^n}{n!} = (1-\frac{t}{x})^{\beta}\exp(t), \quad \left(x>0,\ \beta \in N_0 := N \cup \{0\}\right). \tag{8}$$

We are thus leading to the following conclusion that we provide of bilateral generating functions for Poisson-Charlier polynomials $c_n(\alpha; x)$ and hypergeometric polynomials $S_n^{(\alpha,\beta)}(x)$.

Corollary 1. *If*

$$\Lambda_{\mu,\psi}(\beta; y; \zeta) := \sum_{k=0}^{\infty} a_k c_{\mu+\psi k}(\beta; y)\zeta^k \quad \left(a_k \neq 0,\ \mu,\ \psi \in C\right),$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k S_{n-pk}^{(\alpha,\beta)}(x) c_{\mu+\psi k}(\beta; x)\zeta^k t^{n-pk} = (1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\Lambda_{\mu,\psi}(\beta; y; \zeta). \tag{9}$$

Remark 1. *Using the generating relation (8) for Poisson-Charlier polynomials* $c_n(\beta; y)$ *and getting* $a_k = \frac{1}{k!},\ \mu = 0,\ \psi = 1$ *in (9), we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} S_{n-pk}^{(\alpha,\beta)}(x) c_k(\beta; y)\frac{\zeta^k}{k!} t^{n-pk} = (1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}(1-\frac{\zeta}{y})^{\beta}\exp(\zeta),$$

$$|t| < \min\left\{1,\ |1-x|^{-1}\right\}.$$

By choosing $r=1$ and

$$\Omega_{\mu+\psi k}(y_1) = S_{\mu+\psi k}^{(\alpha_3,\beta_3)}(y)$$

in Theorem 2, for hypergeometric polynomials $S_k^{(\alpha,\beta)}(x)$, we have the following bilinear generating functions.

Corollary 2. *If*

$$\Lambda_{n,p,\mu,\psi}^{(\alpha_1+\alpha_2,\beta_1+\beta_2,\alpha_3,\beta_3)}(x;y;\eta) := \sum_{k=0}^{[n/p]} a_k S_{n-pk}^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x) S_{\mu+\psi k}^{(\alpha_3,\beta_3)}(y)\eta^k$$

$$(a_k \neq 0,\ \mu, \psi \in C),$$

then, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l S_{n-k}^{(\alpha_1,\beta_1)}(x) S_{k-pl}^{(\alpha_2,\beta_2)}(x) S_{\mu+\psi l}^{(\alpha_3,\beta_3)}(y)\eta^l = \Lambda_{n,p,\mu,\psi}^{(\alpha_1+\alpha_2,\beta_1+\beta_2,\alpha_3,\beta_3)}(x;y;\eta). \tag{10}$$

Remark 2. *Using (4) and taking* $a_l = 1,\ p = 1,\ \mu = 0,\ \psi = 1,\ \eta = 1,\ x = y$ *in (10), we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{k} S_{n-k}^{(\alpha_1,\beta_1)}(x) S_{k-l}^{(\alpha_2,\beta_2)}(x) S_{l}^{(\alpha_3,\beta_3)}(x) = S_n^{(\alpha_1+\alpha_2+\alpha_3,\beta_1+\beta_2+\beta_3)}(x).$$

By choosing $r = 1$ and

$$\Omega_{\mu+pn}(y_1) = g_{\mu+pn}^{(s)}(\lambda, y)$$

in Theorem 3, where generalized Cesàro polynomials $g_{\mu+\psi k}^{(s)}(\lambda, y)$ are generated by [15]

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, y)t^n = (1-t)^{-s-1}(1-yt)^{-\lambda}, \tag{11}$$

for generalized Cesàro polynomials $g_n^{(s)}(\lambda, y)$ and hypergeometric polynomials $S_k^{(\alpha,\beta)}(x)$, we give a family of bilateral generating functions as follows:

Corollary 3. *If*

$$\Lambda_{\mu,p,q}(x;\lambda,y;t) := \sum_{n=0}^{\infty} a_n S_{m+qn}^{(\alpha,\beta)}(x) g_{\mu+pn}^{(s)}(\lambda, y)\frac{t^n}{(nq)!},$$

$$\left(a_n \neq 0,\ m \in N_0,\ n \neq 0\right)$$

and

$$\theta_{n,p,q}(\lambda, y;\eta) := \sum_{k=0}^{[n/q]} \binom{n+m}{n-qk} a_k g_{\mu+pk}^{(s)}(\lambda, y)\eta^k,\ \ (n, p \in N).$$

Then, we get

$$\sum_{n=0}^{\infty} S_{n+m}^{(\alpha,\beta)}(x)\theta_{n,p,q}(y;\eta)\frac{t^n}{n!} = (1-t)^{\beta-\alpha-m}\left[1-(1-x)t\right]^{-\beta}\Lambda_{\mu,p,q}\left(\frac{x}{1-(1-x)t};\lambda,y;\eta\left(\frac{t}{1-t}\right)^q\right). \qquad (12)$$

For the family of hypergeometric polynomials $S_n^{(\alpha,\beta)}(x)$ explicitly provided by (1), in addition, for each suitable selection of the coefficients a_k $(k \in N_0)$, the claims of Theorem 1, Theorem 2, Theorem 3 can be applied to obtain the various families of multilinear, multilateral generating functions if $\Omega_{\mu+\psi k}(y_1,\ldots,y_r)$ $(r \in N)$ is expressed as an appropriate product of multivariable functions.

4. SOME PROPERTIES OF THE HYPERGEOMETRIC POLYNOMIALS

In this part, we investigate some properties for hypergeometric polynomials given by (1).

Theorem 4. *The hypergeometric polynomials* $S_n^{(\alpha,\beta)}(x)$ *have the following integral representation:*

$$S_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\alpha-\beta)\Gamma(\beta)}\int_0^{\infty}\int_0^{\infty} e^{-(u_1+u_2)}\frac{\left(u_1+(1-x)u_2\right)^n}{n!}u_1^{\alpha-\beta-1}u_2^{\beta-1}du_1du_2.$$

Proof: *With the help of the*

$$a^{-v} = \frac{1}{\Gamma(v)}\int_0^{\infty} e^{-at}t^{v-1}dt, \qquad \left(\mathrm{Re}(v) > 0\right)$$

from (2), we get

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\beta)}(x)t^n = (1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}$$

$$= \frac{1}{\Gamma(\alpha-\beta)}\int_0^{\infty} e^{-(1-t)u_1}u_1^{\alpha-\beta-1}du_1\,\frac{1}{\Gamma(\beta)}\int_0^{\infty} e^{-[1-(1-x)t]u_2}u_2^{\beta-1}du_2$$

$$= \frac{1}{\Gamma(\alpha-\beta)\Gamma(\beta)}\int_0^{\infty}\int_0^{\infty} e^{-(u_1+u_2)}e^{\left(u_1+(1-x)u_2\right)t}u_1^{\alpha-\beta-1}u_2^{\beta-1}du_1du_2$$

$$= \frac{1}{\Gamma(\alpha-\beta)\Gamma(\beta)}\int_0^{\infty}\int_0^{\infty} e^{-(u_1+u_2)}\sum_{n=0}^{\infty}\frac{\left(u_1+(1-x)u_2\right)^n}{n!}t^n u_1^{\alpha-\beta-1}u_2^{\beta-1}du_1du_2.$$

Comparing coefficents of t^n *in last equation, we have desired relation.*

We now discusses several recurrence relationship of some of hypergeometric polynomials $S_n^{(\alpha,\beta)}(x)$. If we differentiate each member of the generating function (2) with respect to x and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k),$$

we arrive at the following (diffrerential) recurrence relations for hypergeometric polynomials $S_n^{(\alpha,\beta)}(x)$, respectively:

$$\left(\frac{d}{dx}S_n^{(\alpha,\beta)}(x)\right)=\beta\sum_{m=0}^{n-1}(1-x)^m S_{n-m-1}^{(\alpha,\beta)}(x), \quad n\geq 1.$$

Besides, by differentiating each member of the generating function relation (2) according to t, for these polynomials, we have another recurrence relationship below:

$$S_{n+1}^{(\alpha,\beta)}(x)=(\alpha-\beta)\sum_{m=0}^{n}S_{n-m}^{(\alpha,\beta)}(x)+\beta\sum_{p=0}^{n}(1-x)^{p+1}S_{n-p}^{(\alpha,\beta)}(x).$$

5. BILATERAL GENERATING FUNCTIONS

This section, we will derive several families of bilateral generating functions for the hypergeometric polynomials $S_n^{(\alpha,\beta)}(x)$ and the generalized Lauricella functions.

For a suitably bounded non-vanishing multiple sequence $\{\Omega(m_1,m_2,\ldots,m_s)\}_{m_1,m_2,\ldots,m_s\in N_0}$ of real or complex parameters, let $\phi_n(u_1;u_2,\ldots,u_s)$ of s (real or complex) variables $u_1;u_2,\ldots,u_s$ defined by

$$\phi_n(u_1;u_2,\ldots,u_s):=\sum_{m_1=0}^{n}\sum_{m_2,\ldots,m_s=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\big(f(m_1,\ldots,m_s),m_2,\ldots,m_s\big)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!} \tag{13}$$

where, for convenience,

$$((b))_{m_1\phi}=\prod_{j=1}^{B}(b_j)_{m_1\phi_j},\ ((d))_{m_1\delta}=\prod_{j=1}^{D}(d_j)_{m_1\delta_j}.$$

Theorem 5. *The hypergeometric polynomials* $S_n^{(\alpha,\beta)}(x)$ *have the following bilateral generating function representation:*

$$\sum_{n=0}^{\infty}S_n^{(\alpha,\beta)}(x)\phi_n(u_1;u_2,\ldots,u_s)t^n$$

$$=(1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\sum_{m_1,p,m_2,\ldots,m_s=0}^{\infty}\frac{(\alpha)_{m_1+p}(\beta)_p}{(\alpha)_p}\frac{((b))_{(m_1+p)\phi}}{((d))_{(m_1+p)\delta}}$$

$$\times\Omega\big(f(m_1+p,m_2,\ldots,m_s),m_2,\ldots,m_s\big)\frac{(-\frac{u_1t}{1-t})^{m_1}}{m_1!}\frac{\left(\frac{u_1xt}{(1-t)(1-(1-x)t)}\right)^p}{p!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}$$

where $\phi_n(u_1;u_2,\ldots,u_s)$ *is given by (13).*

Proof: *By using the relationship (3), easily seen*

$$\sum_{n=0}^{\infty}S_n^{(\alpha,\beta)}(x)\phi_n(u_1;u_2,\ldots,u_s)t^n$$

$$=\sum_{n=0}^{\infty} S_n^{(\alpha,\beta)}(x) \sum_{m_1=0}^{n} \sum_{m_2,\ldots,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!}t^n$$

$$=\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{n+m_1}{n} S_n^{(\alpha,\beta)}(x)t^n\right)\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)(-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}$$

$$=\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\left((1-t)^{\beta-\alpha-m_1}\left[1-(1-x)t\right]^{-\beta} S_{m_1}^{(\alpha,\beta)}\left(\frac{x}{1-(1-x)t}\right)\right)\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}$$
$$\times\Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)(-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}$$

$$=(1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty} S_{m_1}^{(\alpha,\beta)}\left(\frac{x}{1-(1-x)t}\right)$$
$$\times\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)(-\frac{u_1t}{1-t})^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}$$

$$=(1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\left\{\binom{\alpha+m_1-1}{m_1}\,{}_2F_1\left(-m_1,\beta;\alpha;\frac{x}{1-(1-x)t}\right)\right\}$$
$$\times\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)(-\frac{u_1t}{1-t})^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}$$

$$=(1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\frac{(\alpha+m_1-1)!}{(\alpha-1)!m_1!}\sum_{p=0}^{m_1}\frac{(-m_1)_p(\beta)_p}{(\alpha)_p}\frac{\left(\frac{x}{1-(1-x)t}\right)^p}{p!}$$
$$\times\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)(-\frac{u_1t}{1-t})^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}$$

$$=(1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\frac{(\alpha+m_1-1)!}{(\alpha-1)!m_1!}\sum_{p=0}^{m_1}\frac{(-m_1)_p(\beta)_p}{(\alpha)_p}\frac{\left(\frac{x}{1-(1-x)t}\right)^p}{p!}$$
$$\times\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)(-\frac{u_1t}{1-t})^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}.$$

By appropriately choosing the multiple sequence $\Omega(m_1,m_2,\ldots,m_s)$ in Theorem 5, for hypergeometric polynomials $S_n^{(\alpha,\beta)}(x)$ and generalized Lauricella functions, we obtain interesting results that give bilateral generating functions.

I. Upon setting

$$\Omega\big(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\big)=\frac{(a)_{m_1+\ldots+m_s}(b_2)_{m_2}\cdots(b_s)_{m_s}}{(\alpha)_{m_1}(c_2)_{m_2}\cdots(c_s)_{m_s}},$$

$$\phi_1 = \ldots = \phi_B \text{ and } \delta_1 = \ldots = \delta_D = 0$$

in Theorem 5, we arrive at the following result.

Corollary 4. *The following bilateral generating function applies:*

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\beta)}(x) F_A^{(s)}\left[a,-n,b_2,\ldots,b_s;\alpha,c_2,\ldots,c_s;u_1,u_2,\ldots,u_s\right]t^n$$
$$= (1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta} F^{1:0;1;1;\ldots;1}_{0:0;1;1;\ldots;1}$$

$$\left(\begin{array}{l} \left[(a):\psi^{(1)},\ldots,\psi^{(s+1)}\right]: \;-;\; [\beta:1];\; [b_2:1];\; \ldots;\; [b_s:1]; \\ \left[\quad - \quad\right]: \;-;\; [\alpha:1];\; [c_2:1];\; \ldots;\; [c_s:1]; \end{array} \; (-\tfrac{u_1 t}{1-t}), \left(\tfrac{u_1 x t}{(1-t)(1-(1-x)t)}\right), u_2,\ldots,u_s \right),$$

where the coefficients $\psi^{(n)}$ are given by $\psi^{(1)} = \psi^{(2)} = \ldots = \psi^{(s+1)} = 1.$

II. By letting

$$\Omega\left(f(m_1,m_2,\ldots,m_s),m_2,\ldots,m_s\right) = \frac{(a)_{m_1+\ldots+m_s}(b_2)_{m_2}\ldots(b_s)_{m_s}}{(a)_{m_1+\ldots+m_s}}$$

and

$$\phi = \delta = 0,$$

in Theorem 5, we easily get the following result.

Corollary 5. *The following bilateral generating function applies:*

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\beta)}(x) F_D^{(s)}\left[a,-n,b_2,\ldots,b_s;a;u_1,u_2,\ldots,u_s\right]t^n$$

$$= (1-t)^{\beta-\alpha}\left[1-(1-x)t\right]^{-\beta} F^{1:0;1;1;\ldots;1}_{0:0;1;1;\ldots;1}$$

$$\left(\begin{array}{l} \left[(\alpha):\; \theta^{(1)},\ldots,\theta^{(s+1)}\right]: \;-;\; [\beta:1];\; [b_2:1];\; \ldots;\; [b_s:1]; \\ \left[\quad - \quad\right]: \;-;\; [\alpha:1];\; -;\; \ldots;\; -; \end{array} \; (-\tfrac{u_1 t}{1-t}), \left(\tfrac{u_1 x t}{(1-t)(1-(1-x)t)}\right), u_2,\ldots,u_s \right)$$

where the coefficients $\theta^{(\eta)}$ *are given by*

$$\theta^{(\eta)} = \begin{cases} 1, & (1 \le \eta \le 2) \\ 0, & (2 < \eta \le s+1) \end{cases}.$$

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Chen, K.Y., Srivastava, H.M, “A New Result For Hypergeometric Polynomials”, Proceeding of The American Mathematical Society, 133, 3295-3302, (2005).

[2] Srivastava, H.M., Manocha, H.L., “A Treatise on Generating Functions”, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, (1984).

[3] Lauricella, G., “Sulle funzioni ipergeometriche a più variabili”, Rend. Circ. Mat. Palermo 7, 111-158, (1893).

[4] Srivastava, H.M., Daoust, M.C., “Certain generalized Neumann expansions associated with the Kampé de Fériet function”, Nederl. akad. Westensch. Indag. Math. 31, 449-457, (1969).

[5] Exton, H., “Multiple Hypergeometric Functions and Applicaions”, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, (1976).

[6] Qureshi, M.I. , Khan, M.S., Pathan, M.A., “Some multiple Gaussian hypergeometric generalizations of Buschman-Srivastava theorem”, Internal J. Math. Math. Sci. 2005, 143-153, (2005).

[7] Srivastava, H.M., Karlsson, P.W. , “Multiple Gaussian Hypergeometric Series”, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, (1985).

[8] Liu, S.-J., Lin, S.-D., Srivastava, H.M., Wong, M.-M., “Bilateral generating functions for the Erkus-Srivastava polynomials and the generalized Lauricella functions”, App. Mathematcis and Comp., 218, 7685-7693, (2012).

[9] Aktaş, R. Şahin, R., Altın, A., “On a multivariable extension of the Humbert polynomials”, Applied Mathematics and Computation, 218, 662-666, (2011).

[10] Aktaş, R., Taşdelen, F., Yavuz, N., “Bilateral and Bilinear generating functions for the Generalized Zernike or disc polynomials”, Ars Combinatoria, 108, 389-400, (2013).

[11] Altın, A., Erkuş, E., Taşdelen, F., “The q-Lagrange polynomials in several variables” Taiwanese J. Math., 10(5): 1131-1137, (2006).

[12] Srivastava, H.M., Taşdelen, F., Şekeroğlu, B., ‘‘Some families of generating functions for the q-Konhauser polynomials’’, Taiwanese J. Math., 12(3): 841-850, (2008).

[13] Erdélyi, A., Magnus, W., Oberhettinger F., Tricomi, F.G., “Higher Transcendental Functions”, Vol. II, McGraw-Hill Book Company, New York, (1955).

[14] Özmen, N., Erkuş-Duman, E., “On the Poisson-Charlier polynomials”, Serdica Math. J. 41(4): 457-470, (2015).

[15] Özmen, N., Erkuş-Duman, E., “Some families of generating functions for the generalized Cesáro polynomials”, J. Comput. Anal. Appl., 25(4): 670-683, (2018).