International Electronic Journal of Algebra
Volume 25 (2019) 212-223
DOI: 10.24330/ieja.504155

ON 2-ABSORBING MODULES OVER NONCOMMUTATIVE RINGS

N. J. Groenewald and Bac T. Nguyen

Received: 27 July 2018; Revised: 27 October 2018; Accepted: 31 October 2018

Communicated by Sait Halıcıoğlu

Abstract. Let R be a noncommutative ring with identity. We define the notion of a 2-absorbing submodule and show that if the ring is commutative then the notion is the same as the original definition of that of A. Darani and F. Soheilnia. We give an example to show that in general these two notions are different. Many properties of 2-absorbing submodules are proved which are similar to the results for commutative rings.

Mathematics Subject Classification (2010): 16N60

Keywords: 2-Absorbing submodule, strong 2-absorbing submodule, prime submodule, completely prime submodule

1. Introduction

In 2007, Badawi [1] introduced the concept of 2-absorbing ideals of commutative rings with identity, which is a generalization of prime ideals, and investigated some properties. He defined a 2-absorbing ideal P of a commutative ring R with identity to be a proper ideal of R and if whenever $a, b, c \in R$ with $abc \in P$, then $ab \in P$ or $bc \in P$ or $ac \in P$. In 2011, Darani and Soheilnia [3] introduced the concept of 2-absorbing submodules of modules over commutative rings with identities. A proper submodule P of a module M over a commutative ring R with identity is said to be a 2-absorbing submodule of M if whenever $a, b \in R$ and $m \in M$ with $abm \in P$, then $abM \subseteq P$ or $am \in P$ or $bm \in P$. One can see that 2-absorbing submodules are generalization of prime submodules. Moreover, it is obvious that 2-absorbing ideals are special cases of 2-absorbing submodules.

Throughout all rings (not necessarily commutative rings) have identities and all modules are unital left modules. In recent years the study of the absorbing property of rings, modules and related notions have been some of the topics of interest in the development of the ring and module theory. In this paper we study the notion of 2-absorbing modules over noncommutative rings. We prove basic properties of 2-absorbing submodules analogous to properties studied by Payrovi and Babaei in

[6] and [7] for 2-absorbing submodules over commutative rings. We also introduce the notion of strong 2-absorbing submodules and show that in general if R is not a commutative ring then the notions of 2-absorbing and strong 2-absorbing submodules are not the same. If R is a commutative ring then the notions of 2-absorbing and strong 2-absorbing submodule coincide with that of the original definition introduced by Darani and Soheilnia in [3].

2. 2-Absorbing submodules and ideals

Prime submodules of modules over associative rings were introduced by Dauns [4]. A proper submodule N of M is called prime, if $rRm \subseteq N$ implies $rM \subseteq N$ or $m \in N$ for all $r \in R$ and $m \in M$. Recently the concept of 2-absorbing submodules was introduced by Darani and Soheilnia as a generalization of 2-absorbing ideals in [3].

Definition 2.1. Let P be a proper ideal of a ring R. Then P is a 2-*absorbing ideal* of R if $aRbRc \subseteq P$ implies $ab \in P$ or $bc \in P$ or $ac \in P$ for all $a, b, c \in R$.

Definition 2.2. Let R be a ring and N be a proper submodule of an R-module M. Then N is 2-*absorbing submodule* of M if $aRbRm \subseteq N$ implies $abM \subseteq N$ or $am \in N$ or $bm \in N$ for all $a, b \in R$ and $m \in M$.

Remark 2.3. *If R is a commutative ring then this notion of a 2-absorbing submodule coincides with that of Darani and Soheilnia.*

Definition 2.4. Let R be a noncommutative ring and M an R-module. We define a proper submodule P of M to be a *strong* 2-*absorbing submodule* if whenever $a, b \in R$ and $m \in M$ with $abm \in P$, then $abM \subseteq P$ or $am \in P$ or $bm \in P$.

Proposition 2.5. *Let R be a ring and N be a prime submodule of an R-module M. If $aRbRm \subseteq N$ and $am \notin N$, then $bM \subseteq N$ for all $a, b \in R$ and $m \in M$.*

Proof. Let $a, b \in R$ and $m \in M$. Assume that $aRbRm \subseteq N$ and $am \notin N$. First, we show that $bRm \subseteq N$. Let r be any element of the ring R. Then $aR(brm) \subseteq aR(bRm) \subseteq N$. Since N is a prime submodule, $aM \subseteq N$ or $brm \in N$. Then $brm \in N$ because $am \notin N$. That is $bRm \subseteq N$. Since N is a prime submodule and $am \notin N$, it follows that $m \notin N$ so that $bM \subseteq N$. □

From [5], a proper submodule P of an R-module M is called completely prime, if $am \in P$ implies $m \in P$ or $aM \subseteq P$, for each $a \in R$ and $m \in M$. An R-module M is completely prime if the zero submodule of M is a completely prime submodule of M. In general, an R-module M/P is a completely prime module if and only if P is a completely prime submodule of M.

Proposition 2.6. *If N is a prime (completely prime) submodule of an R-module M, then N is a 2-absorbing (strong 2-absorbing) submodule of M.*

Proof. Assume that N is a prime (completely prime) submodule of an R-module M. Let $a, b \in R$, $m \in M$ and assume N is a prime submodule such that $aRbRm \subseteq N$ but $am \notin N$. Thus $bM \subseteq N$ by Proposition 2.5. Then $bm \in N$ and $abM \subseteq aN \subseteq N$. Hence N is a 2-absorbing submodule of M. Now, assume N is a completely prime submodule and that $abm \in N$ but $bm \notin N$. Since N is completely prime and $bm \notin N$, we have $aM \subseteq N$. Hence $abM \subseteq aM \subseteq N$ and we have N is a strong 2-absorbing submodule of M. □

In general if R is not a commutative ring then the notions of 2-absorbing and strong 2-absorbing submodules are not the same.

Example 2.7. *Let* $M = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ *where entries of matrices in M are from $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and $R = M_2(\mathbb{Z})$. ${}_RM$ is a simple module and hence also a prime module and as such 2-absorbing. However, M is not a strong 2-absorbing module.*

Proof. It is clear that ${}_RM$ is a simple module and hence also a prime module.

Take $a = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} \in R$ and $m = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \in M$.

$abm = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$ but

$am = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$,

$bm = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$

and $abM \neq \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$ since

$ab \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$. □

In [2] a module is defined to be a semi-commutative module if whenever $am = 0$ for $a \in R$ and $m \in M$, we have $aRm = 0$. A submodule N of an R-module M is a semi-commutative submodule if whenever $am \in N$ for $a \in R$ and $m \in M$, we have $aRm \subseteq N$.

Proposition 2.8. *Let M be a left R module. If N is a 2-absorbing submodule which is also a semi-commutative submodule, then N is a strong 2-absorbing submodule.*

Proof. Suppose N is 2-absorbing. Let $a, b \in R$ and $m \in M$ such that $abm \in N$. Since N is a semi-commutative submodule, we have $aRbRm \subseteq N$. Now N 2-absorbing implies that $am \in N$ or $bm \in N$ or $abM \subseteq N$. Hence N is a strong 2-absorbing submodule of M. □

Compare the next Theorem with Theorem 2.3 in [6].

Theorem 2.9. *Let N be a proper submodule of an R-module M. If N is a 2-absorbing submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a 2-absorbing ideal of R.*

Proof. Let $a, b, c \in R$ such that $aRbRc \subseteq (N :_R M)$ and suppose $ac \notin (N :_R M)$ and $bc \notin (N :_R M)$. We show that $ab \in (N :_R M)$. Since $ac, bc \notin (N :_R M)$, there exists $x_1, x_2 \in M$ such that $acx_1 \notin N$ and $bcx_2 \notin N$. Now, $aRbRc(x_1 + x_2) \subseteq N$. Since N is a 2-absorbing submodule of M we have $ab \in (N :_R M)$ or $ac(x_1+x_2) \in N$ or $bc(x_1 + x_2) \in N$. If $ac(x_1 + x_2) \in N$, then $acx_2 \notin N$ since $acx_1 \notin N$. Since $aRbRcx_2 \subseteq N$ and $bcx_2 \notin N$ and $acx_2 \notin N$ we have $ab \in (N :_R M)$. Similar to the case $bc(x_1 + x_2) \in N$, we get $ab \in (N :_R M)$. Hence $(N :_R M)$ is a 2-absorbing ideal of R. □

Corollary 2.10. [6, Theorem 2.3] *If R is a commutative ring and N is a 2-absorbing submodule of M, then $(N :_R M)$ is a 2-absorbing ideal of R.*

Remark 2.11. *The converse of the above theorem is not true in general.*

Proof. Let p be a fixed prime integer. Then $\mathbb{Z}(p^\infty) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = r/p^n + \mathbb{Z}$ for some $r \in \mathbb{Z}$ and $n \geq 0\}$ is a non-zero submodule of $\mathbb{Q}/\mathbb{Z}$. Let $G_t = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = r/p^t + \mathbb{Z}$ for some $r \in \mathbb{Z}\}$ for all $t \geq 0$. It is well known that each proper submodule of $\mathbb{Z}(p^\infty)$ is equal to G_t for some $t \geq 0$. By [6, page 914], G_t is not a 2-absorbing submodule of $\mathbb{Z}(p^\infty)$. We can see that $(G_t :_{\mathbb{Z}} \mathbb{Z}(p^\infty)) = 0$ is a 2-absorbing ideal of $\mathbb{Z}$ for all $t \geq 0$. □

Compare the next Proposition with Theorem 2.2 (i) in [6].

Proposition 2.12. *The intersection of each pair of prime (completely prime) submodules of an R-module M is a 2-absorbing (strong 2-absorbing) submodule of M.*

Proof. Let N and K be two prime (completely prime) submodules of M. If $N = K$, then $N \cap K$ is a prime (completely prime) submodule of M so that $N \cap K$ is a 2-absorbing (strong 2-absorbing) submodule of M. Assume that N and K are distinct. Since N and K are proper submodules of M, it follows that $N \cap K$ is a proper submodule of M. Next, let $a, b \in R$ and $m \in M$ be such that $aRbRm \subseteq N \cap K$ ($abm \in N \cap K$) but $am \notin N \cap K$ and $abM \nsubseteq N \cap K$. Then, we can conclude that

(a): $am \notin N$ or $am \notin K$, and
(b): $abM \nsubseteq N$ or $abM \nsubseteq K$.

These two conditions give 4 cases:

(1): $am \notin N$ and $abM \nsubseteq N$;
(2): $am \notin N$ and $abM \nsubseteq K$;
(3): $am \notin K$ and $abM \nsubseteq N$;
(4): $am \notin K$ and $abM \nsubseteq K$.

Let N and K be two prime submodules. We first consider Case(1). Since $aRbRm \subseteq N \cap K \subseteq N$ and $am \notin N$, it follows from Proposition 2.5 that $bM \subseteq N$. This is a contradiction because $abM \nsubseteq N$. Hence Case(1) does not occur. Similarly, Case(4) is not possible.

Next, Case(2) is considered. Again, we obtain that $bM \subseteq N$ and then $bm \in N$. Let $r \in R$. Since $aRbRm \subseteq N \cap K \subseteq K$, it follows that $aR(brm) \subseteq aR(bRm) \subseteq K$. Hence $aM \subseteq K$ or $brm \in K$ because K is a prime submodule of M. If $aM \subseteq K$, then $abM \subseteq aM \subseteq K$ contradicts $abM \nsubseteq K$. Thus $brm \in K$. That is $bRm \subseteq K$. Since K is a prime submodule, $bM \subseteq K$ or $m \in K$. If $bM \subseteq K$, then $abM \subseteq K$ leading to the same contradiction. Therefore, $m \in K$ and then $bm \in K$. Hence $bm \in N \cap K$. The proof of Case(3) is similar to that of Case(2).

Now, let N and K be completely prime submodules of M and $abm \in N \cap K \subseteq N$. We consider Case(1): Since $abm \in N \cap K \subseteq N$ and N completely prime we have $aM \subseteq N$ or $bm \in N$. If $aM \subseteq N$, then $abM \subseteq aM \subseteq N$ which is not possible. So, suppose $bm \in N$. Now $bM \subseteq N$ or $m \in N$. This is not possible and therefor Case (1) does not occur. Similarly, Case(4) is not possible. Next, Case(2) is considered. We have $abm \in N \cap K \subseteq K$ and since K is completely prime it follows that $aM \subseteq K$ or $bm \in K$. If $aM \subseteq K$ then $abM \subseteq aM \subseteq K$ which contradicts $abM \nsubseteq K$ thus $bm \in K$. From $abm \in N \cap K \subseteq N$ we have $aM \subseteq N$ or $bm \in N$. Since $am \notin N$, $aM \subseteq N$ is not possible. Hence $bm \in N \cap K$. The proof of Case(3) is similar to that of Case(2). $\square$

Corollary 2.13. [6, Theorem 2.2 (i)] *If R is a commutative ring and N is a prime submodule of M or is an intersection of two prime submodules of M, then N is 2-absorbing.*

Proposition 2.6 guarantees that every prime (completely prime) submodule is a 2-absorbing (strong 2-absorbing) submodule. The converse does not necessarily hold.

Example 2.14. *Consider $\mathbb{Z}_6$ as $\mathbb{Z}$-module, $(\bar{0})$ is not a prime (completely prime) submodule of $\mathbb{Z}_6$ since $\bar{2}.\bar{3} \in (\bar{0})$ but $\bar{3} \notin (\bar{0})$ and $2 \notin ((\bar{0})_{\mathbb{Z}} : \mathbb{Z}_6) = 6\mathbb{Z}$. By Proposition 2.12 $(\bar{0}) = (\bar{2}) \cap (\bar{3})$ is a 2-absorbing (strong 2-absorbing) submodule of $\mathbb{Z}_6$ as $\mathbb{Z}$-module.*

Proposition 2.15. *Let N, W be two submodules of an R-module M and $N \subseteq W$. If N is a 2-absorbing (strong 2-absorbing) submodule of M, then N is a 2-absorbing (strong 2-absorbing) submodule of W.*

Proof. If $W = M$, then there is nothing to prove. Let $aRbRx \subseteq N$ ($abx \in N$), where $a, b \in R$, $x \in W$. N is a 2-absorbing (strong 2-absorbing) submodule of M, so either $ax \in N$ or $bx \in N$ or $ab \in (N :_R M)$. Since $N \subseteq W$ implies $(N :_R M) \subseteq (N :_R W)$, then either $ax \in N$ or $bx \in N$ or $ab \in (N :_R W)$. Hence N is 2-absorbing (strong 2-absorbing) in W. $\square$

Lemma 2.16. *Let N be a proper submodule of an R-module M. Then N is a 2-absorbing (strong 2-absorbing) submodule of M if and only if $aRbK \subseteq N$ ($abK \subseteq N$) implies $ab \in (N : M)$, or $aK \subseteq N$ or $bK \subseteq N$ for each $a, b \in R$ and submodule K of M.*

Proof. Suppose that $ab \notin (N : M)$ and $aK \nsubseteq N$ and $bK \nsubseteq N$. Then there exist m_1, m_2 in K such that $am_1 \notin N$ and $bm_2 \notin N$. Since $aRbRm_1 \subseteq aRbK \subseteq N$ ($abm_1 \in abK \subseteq N$) and $ab \notin (N : M)$, $am_1 \notin N$, we get $bm_1 \in N$. Also, since $aRbRm_2 \subseteq aRbK \subseteq N$ ($abm_2 \in abK \subseteq N$) and $ab \notin (N : M)$, $bm_2 \notin N$, we get $am_2 \in N$. Now, since $aRbR(m_1 + m_2) \subseteq aRbK \subseteq N$ ($ab(m_1 + m_2) \in abK \subseteq N$) and $ab \notin (N : M)$ we have $a(m_1 + m_2) \in N$ or $b(m_1 + m_2) \in N$. If $a(m_1 + m_2) \in N$, i.e. $(am_1 + am_2) \in N$ then since $am_2 \in N$ we get $am_1 \in N$ which is contradiction. If $b(m_1 + m_2) \in N$, i.e. $(bm_1 + bm_2) \in N$, then since $bm_1 \in N$ we get $bm_2 \in N$ which is a contradiction. Thus either $ab \in (N : M)$ or $aK \subseteq N$ or $bK \subseteq N$. The converse is clear. $\square$

As for the commutative case [6, Theorem 2.4] we have.

Proposition 2.17. *If N is a 2-absorbing submodule of the R-module M then $(N : Rm)$ is a 2-absorbing ideal of R for every $m \in M \backslash N$.*

Proof. Let $a, b, c \in R$ and $m \in M \backslash N$ such that $aRbRc \subseteq (N : Rm)$. Hence $aRb(RcR)m \subseteq N$. Since RcR is an ideal of R, we have $(RcR)m$ is a submodule of M. It now follows from Lemma 2.16 that $a(RcR)m \subseteq N$ or $b(RcR)m \subseteq N$ or $abM \subseteq N$. Hence $acRm \subseteq N$ or $bcRm \subseteq N$ or $abM \subseteq N$. Thus $ac \in (N : Rm)$ or $bc \in (N : Rm)$ or $abRm \subseteq abM \subseteq N$ i.e. $ab \in (N : Rm)$ and we are done. □

Compare the next Theorem with [7, Theorem 2.3].

Theorem 2.18. *Let N be a proper submodule of the R module M. If N is 2-absorbing (strong 2-absorbing) submodule of M and if I and J are ideals of R and K a submodule of M such that $IJK \subseteq N$, then $IK \subseteq N$ or $JK \subseteq N$ or $IJ \subseteq (N :_R M)$. The converse holds for 2-absorbing submodules.*

Proof. Suppose $IJK \subseteq N$ and $IJ \nsubseteq (N :_R M)$. We show that $IK \subseteq N$ or $JK \subseteq N$. Suppose $IK \nsubseteq N$ and $JK \nsubseteq N$. There exists $a_1 \in I$ and $a_2 \in J$ such that $a_1K \nsubseteq N$ and $a_2K \nsubseteq N$. But $a_1Ra_2K \subseteq IJK \subseteq N$ ($a_1a_2K \subseteq IJK \subseteq N$). Since N is a 2-absorbing (strong 2-absorbing) submodule of M it follows from Lemma 2.16 that $a_1a_2 \in (N :_R M)$. Since $IJ \nsubseteq (N :_R M)$. there exists $b_1 \in I$ and $b_2 \in J$ such that $b_1b_2M \nsubseteq N$. Now, since N is 2-absorbing (strong 2-absorbing) and $b_1Rb_2K \subseteq IJK \subseteq N$ ($b_1b_2K \subseteq IJK \subseteq N$) and also $b_1b_2M \nsubseteq N$ it follows from Lemma 2.16 that $b_1K \subseteq N$ or $b_2K \subseteq N$. We have the following cases:

Case (1) $b_1K \subseteq N$ and $b_2K \nsubseteq N$

Since $a_1Rb_2K \subseteq IJK \subseteq N$ ($a_1b_2K \subseteq IJK \subseteq N$) and $a_1K \nsubseteq N$ and $b_2K \nsubseteq N$ it follows from Lemma 2.16 that $a_1b_2 \in (N :_R M)$. Since $b_1K \subseteq N$ and $a_1K \nsubseteq N$, we conclude $(a_1 + b_1)K \nsubseteq N$. On the other hand, $(a_1 + b_1)Rb_2K \subseteq N$ ($(a_1 + b_1)b_2K \subseteq N$) and neither $(a_1 + b_1)K \subseteq N$ nor $b_2K \subseteq N$, we get that $(a_1 + b_1)b_2 \in (N : M)$ by Lemma 2.16. But since $(a_1 + b_1)b_2 = (a_1b_2 + b_1b_2) \in (N : M)$ and $a_1b_2 \in (N : M)$, we get $b_1b_2 \in (N : M)$ which is a contradiction.

Case (2) $b_2K \subseteq N$ and $b_1K \nsubseteq N$

By a similar argument to case (1) we get a contradiction.

Case (3) $b_1K \subseteq N$ and $b_2K \subseteq N$

$b_2K \subseteq N$ and $a_2K \nsubseteq N$ gives $(a_2 + b_2)K \nsubseteq N$. But $a_1R(a_2 + b_2)K \subseteq N$ ($a_1(a_2 + b_2)K \subseteq N$) and neither $a_1K \subseteq N$ nor $(a_2 + b_2)K \subseteq N$, hence $a_1(a_2 + b_2) \in (N : M)$ by Lemma 2.16. Since $a_1a_2 \in (N : M)$ and $(a_1a_2 + a_1b_2) \in (N : M)$, we have $a_1b_2 \in (N : M)$. Since $(a_1 + b_1)Ra_2K \subseteq N$ ($(a_1 + b_1)a_2K \subseteq N$) and neither $a_2K \subseteq N$ nor $(a_1 + b_1)K \subseteq N$, we conclude $(a_1 + b_1)a_2 \in (N : M)$ by

Lemma 2.16. But $(a_1+b_1)a_2 = a_1a_2 + b_1a_2$, so $(a_1a_2+b_1a_2) \in (N : M)$ and since $a_1a_2 \in (N : M)$, we get $b_1a_2 \in (N : M)$. Now, since $(a_1+b_1)R(a_2+b_2)K \subseteq N$ $((a_1+b_1)(a_2+b_2)K \subseteq N)$ and neither $(a_1+b_1)K \subseteq N$ nor $(a_2+b_2)K \subseteq N$, we have $(a_1+b_1)(a_2+b_2) = (a_1a_2+a_1b_2+b_1a_2+b_1b_2) \in (N : M)$ by Lemma 2.16. But $a_1a_2, a_1b_2, b_1a_2 \in (N : M)$, so $b_1b_2 \in (N : M)$ which is a contradiction. Consequently $IK \subseteq N$ or $JK \subseteq N$.

The converse is clear for the 2-absorbing case since if $aRbRm \subseteq N$, then $(RaR)(RbR)Rm \subseteq N$ and we have $am \in (RaR)Rm \subseteq N$ or $bm \in (RbR)Rm \subseteq N$ or $abM \subseteq (RaR)(RbR)M \subseteq N$. □

Corollary 2.19. *Let I and J be two ideals of R and P a 2-absorbing submodule of M. If $m \in M$ such that $IJm \subseteq P$, then $Im \subseteq P$ or $Jm \subseteq P$ or $IJ \subseteq (N :_R M)$.*

Proof. If $IJm \subseteq P$, then $IJRm \subseteq P$ and consequently $Im \subseteq IRm \subseteq P$ or $Jm \subseteq JRm \subseteq P$ or $IJ \subseteq (P :_R M)$. □

Lemma 2.20. *Let I be an ideal of R and N be a 2-absorbing (strong 2-absorbing) submodule of M. If $a \in R$, $m \in M$ and $IRaRm \subseteq N$ $(Iam \subseteq N)$, then $am \in N$ or $Im \subseteq N$ or $Ia \subseteq (N_R : M)$.*

Proof. Let $am \notin N$ and $Ia \nsubseteq (N_R : M)$. Then there exists $b \in I$ such that $ba \notin (N_R : M)$. Now, $bRaRm \subseteq N$ $(bam \in N)$, implies that $bm \in N$, since N is a 2-absorbing (strong 2-absorbing) submodule of M. We have to show that $Im \subseteq N$. Let c be an arbitrary element of I. Thus $(b+c)RaRm \subseteq IRaRm \subseteq N$ $(b+c)am \in Iam \subseteq N)$. Hence, either $(b+c)m \in N$ or $(b+c)a \in (N_R : M)$. If $(b+c)m \in N$, then by $bm \in N$ it follows that $cm \in N$. If $(b+c)a \in (N_R : M)$, then $ca \notin (N_R : M)$, but $cRaRm \subseteq N$ $(cam \in N)$. Thus $cm \in N$. Hence, we conclude that $Im \subseteq N$. □

Corollary 2.21. *Let N be a 2-absorbing (strong 2-absorbing) submodule of the R-module M. Then $(N :_M I) = \{m \in M : Im \subseteq N\}$ is a 2-absorbing (strong 2-absorbing) submodule of M for every ideal I of R.*

Proof. Let I be an ideal of R and $a, b \in R$ and $m \in M$ such that $aRbRm \subseteq (N :_M I)(abm \subseteq (N :_M I))$. Thus $IaRbRm \subseteq N(Iabm \subseteq N)$, then $(IaR)RbRm \subseteq IaRbRm \subseteq N$. Hence from Lemma 2.20 we have $(IaR)m \subseteq N$ or $(IaRb) \subseteq (N_R : M)$ or $bm \in N$. If $bm \in N$, then $Ibm \subseteq N$ and consequently $bm \in (N :_M I)$ and we are done. If $IaRb \subseteq (N_R : M)$, then $ab \in aRb \subseteq ((N :_R M) :_R I) = ((N :_M I) :_R M)$. If $(IaR)m \subseteq N$, then $am \in aRm \subseteq (N :_M I)$. Thus $bm \in (N :_M I)$

or $am \in (N :_M I)$ or $ab \in ((N :_M I) :_R M)$ which complete the proof for 2-absorbing. For $Iabm \subseteq N$ and N strong 2-absorbing it follows from Lemma 2.20 that $abm \in N$ or $Im \subseteq N$ or $Iab \subseteq (N_R : M)$. If $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Hence for $am \in N$ it follows that $Iam \subseteq IN \subseteq N$ and we have $am \in (N :_M I)$. For $bm \in N$ it follows that $Ibm \subseteq IN \subseteq N$ and we have $bm \in (N :_M I)$. For $ab \in (N :_R M)$, we have $ab \in ((N :_R M) :_R I) = ((N :_M I) :_R M)$. For $Im \subseteq N$, we have $m \in (N :_M I)$ and thus $am \in (N :_M I)$. For $Iab \subseteq (N_R : M)$, we have $ab \in ((N :_R M) :_R I) = ((N :_M I) :_R M)$ and $(N :_M I)$ is a strong 2-absorbing submodule of M. □

For the next Theorem compare also [7, Theorem 2.7 and Corollary 2.8].

Theorem 2.22. *Let N be a 2-absorbing submodule of M. Then $(N :_R M)$ is a prime ideal of R if and only if $(N :_R P)$ is a prime ideal for every submodule P of M containing N.*

Proof. Let I and J be ideals of R and K a submodule of M such that $N \subseteq P$ and $IJ \subseteq (N :_R P)$. Hence $IJP \subseteq N$. Since N is a 2-absorbing submodule of M it follows from Theorem 2.18 that $IP \subseteq N$ or $JP \subseteq N$ or $IJ \subseteq (N :_R M)$. Hence $I \subseteq (N :_R P)$ or $J \subseteq (N :_R P)$. Also for $IJ \subseteq (N :_R M)$ by the assumption that $(N :_R M)$ is a prime ideal, we get $IP \subseteq IM \subseteq N$ or $JP \subseteq JM \subseteq N$. Hence also $I \subseteq (N :_R P)$ or $J \subseteq (N :_R P)$ and we have $(N :_R P)$ is a prime ideal.

This is clear, just take $P = M$. □

Proposition 2.23. *Let N and K be submodules of an R-module M with $K \nsubseteq N$. If N is a 2-absorbing (strong 2-absorbing) submodule of M, then $K \cap N$ is a 2-absorbing (strong 2-absorbing) submodule of K.*

Proof. Since N and K are submodules of M and $K \nsubseteq N$, $K \cap N$ is a proper submodule of K. Assume that N is a 2-absorbing (strong 2-absorbing) submodule of M. Let $a, b \in R$ and $x \in K$ be such that $aRbRx \subseteq N$ ($abx \in N$). Since K is a submodule of M, $abK \subseteq K$ and $ax; bx \in K$. Moreover, since $aRbRx \subseteq K \cap N \subseteq N$ ($abx \in K \cap N \subseteq N$) and N is a 2-absorbing (strong 2-absorbing) submodule of M, $abM \subseteq N$ or $ax \in N$ or $bx \in N$. Thus $abK \subseteq abK \cap abM \subseteq K \cap N$ or $ax \in K \cap N$ or $bx \in K \cap N$. Therefore, $K \cap N$ is a 2-absorbing (strong 2-absorbing) submodule of K. □

Proposition 2.24. *Let N and K be submodules of an R-module M with $K \subseteq N$. Then N is a 2-absorbing (strong 2-absorbing) submodule of M if and only if N/K is a 2-absorbing (strong 2-absorbing) submodule of M/K.*

Proof. First, assume that N is a 2-absorbing (strong 2-absorbing) submodule of M. Then N/K is a proper submodule of M/K. Let $a, b \in R$ and $m \in M$ be such that $aRbR(m+K) \subseteq N/K$ $(ab(m+K) \subseteq N/K)$. Let $s; t \in R$. Thus $asbtm + K = asbt(m+K) \subseteq aRbR(m+K) \subseteq N/K$. Then there exists $n \in N$ such that $asbtm + K = n + K$ so that $-n + asbtm \in K \subseteq N$ and then $asbtm \in N$. This shows that $aRbRm \subseteq N$. (For the strong 2-absorbing case, $abm + K = ab(m+K) \subseteq N/K$ and then there exists $n \in N$ such that $abm + K = n + K$ so that $-n + abm \in K \subseteq N$ and then $abm \in N$). As a result, $am \in N$ or $bm \in N$ or $abM \subseteq N$ because N is a 2-absorbing (strong 2-absorbing) submodule of M. Therefore, $a(m+K) \in N/K$ or $b(m+K) \in N/K$ or $ab(M/K) \subseteq N/K$. Hence N/K is a 2-absorbing (strong 2-absorbing) submodule of M/K.

Conversely, assume that N/K is a 2-absorbing (strong 2-absorbing) submodule of M/K. Then N is a proper submodule of M. Let $a, b \in R$ and $m \in M$ be such that $aRbRm \subseteq N (abm \in N)$. Then $aRbR(m+K) \subseteq N/K$ $(ab(m+K) \subseteq N/K)$. Since N/K is a 2-absorbing (strong 2-absorbing) submodule of M/K, we obtain that $a(m+K) \in N/K$ or $b(m+K) \in N/K$ or $ab(M/K) \subseteq N/K$. That is $am \in N$ or $bm \in N$ or $abM \subseteq N$. This implies that N is a 2-absorbing (strong 2-absorbing) submodule of M. □

Consider $R = R_1 \times R_2$ where each R_i is an associative ring with identity, M_i be an R_i-module where $i = 1, 2$, and $M = M_1 \times M_2$ be the R-module with $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ where $r_i \in R_i$, $m_i \in M_i, i = 1, 2$.

Theorem 2.25. *Let M_1 be an R_1-module, M_2 be an R_2-module, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then*

(1) *N_1 is a 2-absorbing R_1-submodule of M_1 if and only if $N_1 \times M_2$ is a 2-absorbing submodule of M; and*

(2) *N_2 is a 2-absorbing R_2-submodule of M_2 if and only if $M_1 \times N_2$ is a 2-absorbing submodule of M.*

Proof. It suffices to prove only part 1. First, assume that N_1 is a 2-absorbing R_1-submodule of M_1. Suppose that $(a, b)R(c, d)R(m_1, m_2) \subseteq N_1 \times M_2$ where $(a, b), (c, d) \in R$ and $(m_1, m_2) \in M$.
Then $(aR_1cR_1m_1, bR_2dR_2m_2) = (a, b)R(c, d)R(m_1, m_2) \subseteq N_1 \times M_2$, i.e., $aR_1cR_1m_1 \subseteq N_1$ and $bR_2dR_2m_2 \subseteq M_2$. Since N_1 is a 2-absorbing R_1-submodule of M_1, $acM_1 \subseteq N_1$ or $am_1 \in N_1$ or $cm_1 \in N_1$. That is $(a, b)(c, d)M = (acM_1, bdM_2) \subseteq N_1 \times M_2$ or $(a, b)(m_1, m_2) = (am_1, bm_2) \in N_1 \times M_2$ or $(c, d)(m_1, m_2) = (cm_1, dm_2) \in N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a 2-absorbing submodule of M. Conversely, assume

that $N_1 \times M_2$ is a 2-absorbing R-submodule of M. Let $a, b \in R_1$ and $m_1 \in M_1$. Assume that $aR_1bR_1m_1 \subseteq N_1$. Let $x, y \in R_2$ and $m_2 \in M_2$. Then $(a, x)R(b, y)R(m_1, m_2) = (aR_1bR_1m_1, xR_2yR_2m_2) \subseteq N_1 \times M_2$. Since $N_1 \times M_2$ is a 2-absorbing R-submodule of M, $(a, x)(b, y)M \subseteq N_1 \times M_2$ or $(a, x)(m_1, m_2) \in N_1 \times M_2$ or $(b, y)(m_1, m_2) \in N_1 \times M_2$. Then $(abM_1, xyM_2) = (a, x)(b, y)M \subseteq N_1 \times M_2$ or $(am_1, xm_2) = (a, x)(m_1, m_2) \in N_1 \times M_2$ or $(bm_1, ym_2) = (b, y)(m_1, m_2) \in N_1 \times M_2$, i.e., $abM_1 \subseteq N_1$ or $am_1 \in N_1$ or $bm_1 \in N_1$. Therefore, N_1 is a 2-absorbing R_1-submodule of M_1. □

Theorem 2.26. *Let M_1 be an R_1-module, M_2 be an R_2-module, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then*

(1) *N_1 is a strong 2-absorbing R_1-submodule of M_1 if and only if $N_1 \times M_2$ is a strong 2-absorbing submodule of M; and*
(2) *N_2 is a strong 2-absorbing R_2-submodule of M_2 if and only if $M_1 \times N_2$ is a strong 2-absorbing submodule of M.*

Proof. It suffices to prove only part 1. First, assume that N_1 is a strong 2-absorbing R_1-submodule of M_1. Suppose that $(a, b)(c, d)(m_1, m_2) \in N_1 \times M_2$ where $(a, b), (c, d) \in R$ and $(m_1, m_2) \in M$.
Then $(acm_1, bdm_2) = (a, b)(c, d)(m_1, m_2) \in N_1 \times M_2$, i.e., $acm_1 \in N_1$ and $bdm_2 \in M_2$. Since N_1 is a strong 2-absorbing R_1-submodule of M_1, it follows that $acM_1 \subseteq N_1$ or am_1N_1 or $cm_1 \in N_1$. That is $(a, b)(c, d)M = (acM_1, bdM_2) \subseteq N_1 \times M_2$ or $(a, b)(m_1, m_2) = (am_1, bm_2) \in N_1 \times M_2$ or $(c, d)(m_1, m_2) = (cm_1, dm_2) \in N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a 2-absorbing submodule of M. Conversely, assume that $N_1 \times M_2$ is a strong 2-absorbing R-submodule of M. Let $a, b \in R_1$ and $m_1 \in M_1$. Assume that $abm_1 \in N_1$. Let $x, y \in R_2$ and $m_2 \in M_2$. Then $(a, x)(b, y)(m_1, m_2) = (abm_1, xym_2) \in N_1 \times M_2$. Since $N_1 \times M_2$ is a strong 2-absorbing R-submodule of M, it follows that $(a, x)(b, y)M \subseteq N_1 \times M_2$ or $(a, x)(m_1, m_2) \in N_1 \times M_2$ or $(b, y)(m_1, m_2) \in N_1 \times M_2$. Then $(abM_1, xyM_2) = (a, x)(b, y)M \subseteq N_1 \times M_2$ or $(am_1, xm_2) = (a, x)(m_1, m_2) \in N_1 \times M_2$ or $(bm_1, ym_2) = (b, y)(m_1, m_2) \in N_1 \times M_2$, i.e., $abM_1 \subseteq N_1$ or $am_1 \in N_1$ or $bm_1 \in N_1$. Therefore, N_1 is a 2-absorbing R_1-submodule of M_1. □

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

References

[1] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., 75(3) (2007), 417-429.

[2] A. M. Buhphang and M. B. Rege, *Semi-commutative modules and Armendariz modules*, Arab J. Math. Sci., 8(1) (2002), 53-65.

[3] A. Y. Darani and F. Soheilnia, 2-*Absorbing and weakly* 2-*absorbing submodules*, Thai J. Math., 9(3) (2011), 577-584.

[4] J. Dauns, *Prime modules*, J. Reine Angew. Math., 298 (1978), 156-181.

[5] N. J. Groenewald and D. Ssevviiri, *Completely prime submodules*, Int. Electron. J. Algebra, 13 (2013), 1-14.

[6] Sh. Payrovi and S. Babaei, *On* 2-*absorbing submodules*, Algebra Colloq., 19 (2012) 913-920.

[7] Sh. Payrovi and S. Babaei, *On the 2-absorbing submodules*, Iran. J. Math. Sci. Inform., 10(1) (2015), 131-137.

N. J. Groenewald (Corresponding Author)
Department of Mathematics
Faculty of Science
Nelson Mandela University
Port Elizabetyh
South Africa
e-mail: nico.groenewald@mandela.ac.za

Bac T. Nguyen
Nguyen Tat Thanh University
300 A Nguyen Tat Thanh street
Ho Chi Minh city, Vietnam
and Dept. of Basic Science
Thainguyen Univ. of Economics and Business Administration
Thainguyen
Vietnam
e-mail: bacnt2008@gmail.com