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s-pure extensions of locally compact abelian groups

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Abstract

A subgroup H of a locally compact abelian (LCA) group G is called s-pure if $\overline{H \bigcap nG} = H$ for every positive integer n. A proper short exact sequence $0 \to A \xrightarrow{\phi} B \to C \to 0$ in the category of LCA groups is said to be *s-pure* if $\phi(A)$ is an s-pure subgroup of G. We establish conditions under which the s-pure exact sequences split and determine those LCA groups which are s-pure injective. We also gives a necessary condition for an LCA group to be s-pure projective in $\pounds$.

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Intr

All groups considered in this paper are Hausdorff topological abelian groups and they will be written additively. For a group G and a positive integer n, we denote by nG, the subgroup of G defined by $nG = \{nx : x \in G\}$ and $G[n]$, the subgroup of G defined by $G[n] = \{x \in G; nx = 0\}$. In a multiplicative group, we will use G^n instead of nG and define $G^n = \{x^n : x \in G\}$. Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called proper if it is open onto its image, and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be an extension of A by C if ϕ and ψ are proper morphism. We let $Ext(C, A)$ denote the group of extensions of A by C [6]. Let $\overline{S}$ denotes the closure of $S \subseteq G$. We say that a closed subgroup H of an LCA group G is s-pure if $\overline{H \bigcap nG} = H$ for every positive integer n. A subgroup H of a group G is said to be pure if $H \bigcap nG = nH$ for every positive integer n [3]. A pure subgroup need not be s-pure and vice versa (Example 1.9). In Section 1, we show that an s-pure subgroup is pure if and only if it is densely divisible (Lemma 1.10). An LCA group G is said to be pure simple if G contains no nontrivial closed pure subgroup [1]. Armacost [1] has determined the pure simple LCA group G. Also, Armacost has determined the LCA group G such that every closed subgroup of G is pure [1]. We say that an LCA group G is s-pure simple if G contains no nonzero s-pure

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subgroup. We say that a LCA group G is s-pure full if every closed subgroup of G is s-pure. We show that a LCA group G is s-pure full if and only if it is divisible (Theorem 1.15). Also, we show that a compact group G is s-pure simple if and only if it is totally disconnected(Theorem 1.16). A proper short exact sequence $0 \to A \xrightarrow{\phi} B \to C \to 0$ in $\pounds$ is said to be s-pure if $\phi(A)$ is s-pure in B. In section 2, we study s-pure exact sequence in $\pounds$. In [4], Fulp studied pure injective and pure projective in $\pounds$. In section 3, we study s-pure injective and s-pure projective in $\pounds$. An LCA group G is an s-pure injective group in $\pounds$ if and only if $G \cong R^n \bigoplus (R/Z)^\sigma$ (Theorem 3.2). If G is an s-pure projective group in $\pounds$ then $G \cong R^n \bigoplus G'$ where G' is a discrete torsion-free, non divisible group (Theorem 3.4).

The additive topological group of real numbers is denoted by R, Q is the group of rationals with the discrete topology and Z is the group of integers. Also, $Z(n)$ is the cyclic group of order n and $Z(p^\infty)$ denotes the quasicyclic group. For any group G, G_0 is the identity component of G, tG is the maximal torsion subgroup of G and 1_G is the identity map $G \to G$. An element $g \in G$ is called compact if the smallest closed subgroup which its contains is compact [8, Definition 9.9]. We denote by bG, the subgroup of all compact elements of G. If $\{G_i\}_{i\in I}$ is a family of groups in $\pounds$, then we denote their direct product by $\prod_{i\in I} G_i$. If all the G_i are equal, we will write G^I instead of $\prod_{i\in I} G_i$. For any group G and H, $Hom(G,H)$ is the group of all continuous homomorphisms from G to H, endowed with the compact-open topology. The dual group of G is $\hat{G} = Hom(G, R/Z)$ and $(\hat{G}, S)$ denotes the annihilator of $S \subseteq G$ in $\hat{G}$. For a group G, we define $G^{(1)} = \bigcap_{n=1}^{\infty} nG$.

1. s-pure subgroups

Let $G \in \pounds$. In this section, we introduce the concept and study some properties of an s-pure subgroup of G.

1.1. Definition. *A closed subgroup H of a group G is called s-pure if $\overline{H \bigcap nG} = H$ for every positive integer n.*

1.2. Note.

(a) A closed divisible subgroup of a group is s-pure.
(b) A closed subgroup of a divisible group is s-pure.

1.3. Remark. *Let $G \in \pounds$. Then G has two trivial subgroups,$\{0\}$ and G. Clearly, $\{0\}$ is s-pure. But G need not be an s-pure in itself.*

Recall that a group G is said to be densely divisible if it has a dense divisible subgroup.

1.4. Lemma. *A group G is densely divisible if and only if $\overline{nG} = G$ for every positive integer n.*

Proof. *See [2, 4.16(a)].*

1.5. Corollary. *Let $G \in \pounds$. Then, G is s-pure in itself if and only if G is densely divisible.*

Proof. *It is clear by Lemma 1.4.*

1.6. Lemma. *Let $G \in \pounds$. Then, $\overline{G^{(1)}}$ is an s-pure subgroup of G.*

Proof. It is clear that $G^{(1)} \subseteq \overline{G^{(1)}} \bigcap mG$ for every positive integer m. So, $\overline{G^{(1)}} \subseteq \overline{\overline{G^{(1)}} \bigcap mG} \subseteq \overline{G^{(1)}}$ for all m. Hence, $\overline{G^{(1)}}$ is an s-pure subgroup.

1.7. Remark. *Let $G \in \pounds$ and H be an s-pure subgroup of G. Then, $H \subseteq \overline{nG}$ for all positive integers n. Hence, $H \subseteq \bigcap_{n=1}^{\infty} \overline{nG} = (G, t\hat{G})$ [8].*

Now, we present an example of a LCA group G and a closed subgroup H of G such that $H \subseteq (G, t\hat{G})$, but H is not an s-pure subgroup.

1.8. Example. *Let S^1 be the (multiplicative) circle group of unitary complex numbers and σ any infinite cardinal number. Let G be the subgroup of $(S^1)^{\sigma}$ consisting of all (x_ι) such that $x_\iota = \pm 1$ for all but a finite number of ι. Let K be the subgroup of G consisting of all (x_ι) such that $x_\iota = 1$ for all but a finite number of ι. By [8, section 24.44(a)], G is a locally compact abelian group, and $\hat{G}$ is torsion-free. Let $H = \{(x)_\iota, (y)_\iota\}$ where $x_\iota = 1$ and $y_\iota = -1$ for $\iota \neq \iota_1, ..., \iota_m$ and $x_\iota = y_\iota = 0$ for $\iota = \iota_1, ..., \iota_m$. Then, H is a closed subgroup of G, and $H \subseteq G = (G, t\hat{G})$. Now, suppose that n is even. Then, $\overline{H \bigcap G^n} = \overline{H \bigcap K} = \{(x)_\iota\}$. Hence, H is not s-pure.*

Recall that a subgroup H of a group G is called pure if $nH = H \bigcap nG$ for every positive integer n[3]. A pure subgroup need not be s-pure, and an s-pure subgroup need not be pure.

1.9. Example. *Since R is divisible, so the subgroup Z of R is s-pure. But it is not a pure subgroup. Let p be a prime and $G = \prod_{n=1}^{\infty} Z(p^n)$, with discrete topology. Then, tG is a pure subgroup of G. Since $(1,0,0,...) \in tG$ and $(1,0,0,...) \notin p(tG)$, so it is not s-pure.*

1.10. Lemma. *A pure subgroup is s-pure if and only if it is densely divisible.*

Proof. Let H be a pure subgroup of G. If H is an s-pure subgroup, then $\overline{nH} = H$ for every positive integer n. So, by Lemma 1.4, H is densely divisible. Conversely, let H be a densely divisible, pure subgroup of G. Then, $\overline{H \bigcap nG} = \overline{nH}$ for every positive integer n. By Lemma 1.4, $\overline{nH} = H$ for all n. So, $\overline{H \bigcap nG} = H$ for all n. Hence, H is an s-pure subgroup in G.

Let G be a group in $\pounds$. Then G is called s-pure simple if G contains no nonzero s-pure subgroups. Similarly, G is called s-pure full if every closed subgroup of G is s-pure.

1.11. Lemma. *Let G_1 and G_2 be two groups in $\pounds$. If $G_1 \times G_2$ is s-pure full, then G_1 and G_2 are s-pure full.*

Proof. Let G_1,$G_2 \in \pounds$ and H be a closed subgroup of G_1. Then, $H \times G_2$ is a closed subgroup of $G_1 \times G_2$. So, $\overline{(H \times G_2) \bigcap (nG_1 \times nG_2)} = H \times G_2$ for any positive integer n. Hence, $\overline{(H \bigcap nG_1) \times (G_2 \bigcap nG_2)} = H \times G_2$. Therefore, $\pi_1(\overline{(H \bigcap nG_1) \times (nG_2)}) = \pi_1(H \times G_2)$ where π_1 is the first projection map of $G_1 \times G_2$ onto G_1. Consequently, $\overline{H \bigcap nG_1} = H$. Similarly, it can be show that G_2 is s-pure full.

1.12. Remark. *Recall that a discrete group is densely divisible if and only if it is divisible.*

1.13. Remark. *Let G be a densely divisible group and H a closed subgroup of G. Since $(\hat{G}, H)$ is a subgroup of $\hat{G}$ and $\hat{G}$ is torsion-free, so G/H is densely divisible.*

1.14. Remark. *Let G be a densely divisible group and H an open, pure subgroup of G. An easy calculation shows that H is divisible.*

1.15. Theorem. *Let $G \in \pounds$. Then, G is s-pure full if and only if G is divisible.*

Proof. *Let G be an s-pure full group in $\pounds$. By [8, Theorem 24.30], $G \cong R^n \bigoplus G'$, where G' is an LCA group which contains a compact open subgroup. By Lemma 1.11, G' is s-pure full. So, by Corollary 1.5, G' is densely divisible. By Remark 1.13 , G'/bG' is densely divisible. On the other hand, G'/bG' is discrete and torsion-free (see the proof of Theorem 2.7 [9]). Hence, by Remark 1.12, G'/bG' is divisible. By Remark 1.14, bG' is divisible. Consequently, the short exact sequence $0 \to bG' \to G' \to G'/bG' \to 0$ splits. Hence, $G' \cong bG' \bigoplus G'/bG'$ and G' is divisible. Therefore, G is divisible. The converse is clear by Note 1.2.b.*

1.16. Theorem. *A compact group G is an s-pure simple group if and only if it is totally disconnected.*

Proof. *Let G be a compact group. If G is an s-pure simple group, then by Note 1.2(a), $G_0 = 0$ because G_0 is a closed divisible subgroup of G. So G is totally disconnected. Conversely, Let G be a compact, totally disconnected group and H an s-pure subgroup of G. By Remark 1.7, $H \subseteq (G, t\hat{G})$. Since $\hat{G}$ is a discrete and a torsion group, so $t\hat{G} = \hat{G}$. Hence, $H = 0$.*

2. s-pure exact sequence

In this section, we introduce the concept and study some properties of s-pure extensions in $\pounds$.

2.1. Definition. *An extension $0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$ in $\pounds$ is called s-pure if $\phi(A)$ is s-pure in B.*

2.2. Remark. *Let A be a divisible group in $\pounds$ and $E : 0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$ an extension in $\pounds$. Then $\phi(A)$ is a closed divisible subgroup of B. So, by Note 1.2(a),E is an s-pure extension.*

2.3. Lemma. *Let A, C be groups in $\pounds$. Then the extension $0 \to A \to A \bigoplus C \to C \to 0$ is an s-pure extension if and only if A is densely divisible.*

Proof. *The extension $0 \to A \to A \bigoplus C \to C \to 0$ is pure. Hence, by Lemma 1.10, it is s-pure if and only if A is densely divisible.*

2.4. Remark. *Lemma 2.3 shows that the set of all s-pure extensions of A by C need not be a subgroup of $Ext(C, A)$.*

The dual of an extension $E : 0 \to A \to B \to C \to 0$ is defined by $\hat{E} : 0 \to \hat{C} \to \hat{B} \to \hat{A} \to 0$. The following example shows that the dual of an s-pure extension need not be s-pure.

2.5. Example *There exists a non splitting extension*

$$E : 0 \to Z(p^{\infty}) \to B \to C \to 0$$

of $Z(p^{\infty})$ with compact group C which is not torsion-free [2, Example 6.4]. By Note 1.2(a), E is s-pure. Since $\widehat{Z(p^{\infty})}$ is torsion-free, so $\hat{E}$ is pure. By Lemma 1.10, $\hat{E}$ is s-pure if and only if $\hat{C}$ is densely divisible. But C is compact. So, $\hat{C}$ is discrete. Hence, $\hat{E}$ is s-pure if and only if $\hat{C}$ is a discrete divisible group. Consequently, $\hat{E}$ is s-pure if and only if C is a compact torsion-free group. Since C is not torsion-free, it follows that $\hat{E}$ is not s-pure.

Recall that two extensions $0 \to A \xrightarrow{\phi_1} B \xrightarrow{\psi_1} C \to 0$ and $0 \to A \xrightarrow{\phi_2} X \xrightarrow{\psi_2} C \to 0$ is said to be equivalent if there is a topological isomorphism $\beta : B \to X$ such that the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\phi_1} & B & \xrightarrow{\psi_1} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\
0 & \longrightarrow & A & \xrightarrow{\phi_2} & X & \xrightarrow{\psi_2} & C & \longrightarrow & 0
\end{array}$$

is commutative.

2.6. Lemma *An extension equivalent to an s-pure extension is s-pure.*

Proof. Suppose that

$$E_1 : 0 \to A \xrightarrow{\phi_1} B \to C \to 0, E_2 : 0 \to A \xrightarrow{\phi_2} X \to C \to 0$$

be two equivalent extension such that E_1 is s-pure. Then there is a topological isomorphism $\beta : B \to X$ such that $\beta\phi_1 = \phi_2$. Since E_1 is s-pure, $\phi_1(A) = \overline{\phi_1(A) \bigcap nB}$. Then $\beta\phi_1(A) = \beta(\overline{\phi_1(A) \bigcap nB})$. So, $\phi_2(A) = \overline{\phi_2(A) \bigcap nX}$. Hence, E_2 is s-pure.

2.7. Corollary. *If the s-pure extension $0 \to A \to B \to C \to 0$ splits, Then A is densely divisible.*

Proof. Let $0 \to A \to B \to C \to 0$ be a split, s-pure extension. Then, it is equivalent to $0 \to A \to A \bigoplus C \to C \to 0$. So, $0 \to A \to A \bigoplus C \to C \to 0$ is s-pure. Hence, by Lemma 2.3, A is densely divisible.

2.8. Remark. *The converse of Corollary 2.7 may not hold. Consider Example 2.5.*

We will now show that a pullback or pushout of an s-pure extension need not be s-pure. For more on a pullback and a pushout of an extension in £, see [6].

2.9. Example *Let α be the map $\alpha : Z \to Z : n \longmapsto 2n$. Consider the s-pure extension $E : 0 \to Z_2 \to R/Z \to R/Z \to 0$ which is the dual of $0 \to Z \xrightarrow{\alpha} Z \to Z_2 \to 0$. Let $f : Q \to R/Z$ be any continuous homomorphism. Since Q is torsion-free, so the standard pullback of E is pure, but not s-pure by Lemma 1.10 because Z_2 is not densely divisible. Now consider the s-pure extension $E' : 0 \to Z \to Q \to Q/Z \to 0$. Then the map α*

induces a pushout diagram

$$\begin{array}{ccccccccc} E' : 0 & \longrightarrow & Z & \longrightarrow & Q & \longrightarrow & Q/Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle 1_{Q/Z}} & & \\ \alpha E' : 0 & \longrightarrow & Z & \xrightarrow{\mu} & (Z \bigoplus Q)/H & \longrightarrow & Q/Z & \longrightarrow & 0 \end{array}$$

Where $H = \{(2n, -n); n \in Z\}$ *and* $\mu : n \longmapsto (n, 0) + H$. *If* $\alpha E'$ *is s-pure, then* $\mu(Z) \subseteq 2((Z \bigoplus Q)/H)$ *which is a contradiction.*

3. s-pure injectives and s-pure projectives

In this section, we define the concept of s-pure injective and s-pure projective in £ and express some of their properties .

3.1. Definition *Let G be a group in £. We call G an s-pure injective group in £ if for every s-pure exact sequence*

$$0 \to A \xrightarrow{\phi} B \to C \to 0$$

and continuous homomorphism $f : A \to G$, there is a continuous homomorphism $\bar{f} : B \longrightarrow G$ such that $\bar{f}\phi = f$. Similarly, we call G an s-pure projective group in £ if for every s-pure exact sequence

$$0 \to A \to B \xrightarrow{\psi} C \to 0$$

and continuous homomorphism $f : G \to C$, there is a continuous homomorphism $\bar{f} : G \to B$ such that $\psi \bar{f} = f$.

3.2. Theorem *Let $G \in$ £. The following statements are equivalent:*

(1) *G is an s-pure injective in £.*
(2) *$G \cong R^n \bigoplus (\frac{R}{Z})^\sigma$ where σ is a cardinal number.*

Proof. $1 \Longrightarrow 2$*: Let G be an s-pure injective in £. For a group X in £, consider the s-pure extension*

$$E : 0 \to G \xrightarrow{\phi} B \longrightarrow X \to 0$$

Then there is a continuous homomorphism $\bar{\phi} : B \to G$ such that $\bar{\phi}\phi = 1_G$. Consequently, E splits. In particular, the s-pure extension $0 \to G \longrightarrow G^* \to G^*/G \to 0$ splits where G^* is the minimal divisible extension of G. Hence, G is divisible. So, by Remark 2.2, every extension of G by X is an s-pure extension. On the other hand, every s-pure extension of G by X splits. Hence, $Ext(X, G) = 0$. By [10, Theorem 3.2], $G \cong R^n \bigoplus (R/Z)^\sigma$.

$2 \Longrightarrow 1$*: It is clear.*

Recall that a discrete group G is called reduced if it has no nontrivial divisible subgroup.

3.3. Lemma *Q is not an s-pure projective group.*

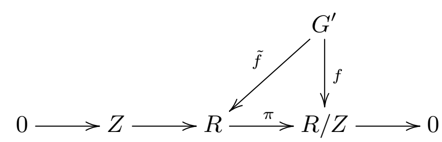
Proof. *Consider the s-pure exact sequence $0 \to Z \longrightarrow R \xrightarrow{\pi} R/Z \to 0$ where π is the natural mapping. Assume that Q is an s-pure projective group and $f \in Hom(Q, R/Z)$. Then, there is $\bar{f} \in Hom(Q, R)$ such that $\pi\bar{f} = f$. Hence, $\pi^* : Hom(Q, R) \to Hom(Q, R/Z)$ is surjective. Now consider the following exact sequence*

$$0 \to Hom(Q, Z) \to Hom(Q, R) \xrightarrow{\pi_*} Hom(Q, R/Z) \to Ext(Q, Z) \to Ext(Q, R)$$

Since Q is divisible and Z is reduced, so $Hom(Q,Z)=0$. Hence, π^* is one to one. This shows that π^* is an isomorphism. On the other hand, $Ext(Q,R)=0$. Consequently, $Ext(Q,Z)=0$ which is a contradiction.

3.4. Theorem *Let $G \in \pounds$. If G is an s-pure projective in $\pounds$, then $G \cong R^n \bigoplus G'$ where G' is a discrete torsion-free , reduced group.*

Proof. *It is known that an LCA group G can be written as $G \cong R^n \bigoplus G'$ where G' contains a compact open subgroup [8, Theorem 24.30]. An easy calculation shows that if G is an s-pure projective group, then G' is an s-pure projective in $\pounds$. Let $f \in Hom(G', \frac{R}{Z})$. Then there exists a continuous homomorphism $\tilde{f}: G' \to R$ such that the following diagram is commutative:*



Consider the following exact sequence

$$0 \to Hom(G',Z) \to Hom(G',R) \xrightarrow{\pi_*} Hom(G',R/Z) \to Ext(G',Z) \to 0$$

Since π_* is surjective, so $Ext(G',Z)=0$. Let K be a compact open subgroup of G'. Then the inclusion map $i: K \to G'$ induces the surjective homomorphism $i_*: Ext(G',Z) \to Ext(K,Z)$. So, $Ext(K,Z)=0$. Hence, $Ext(R/Z,\hat{K})=0$. By [7, Proposition 2.17], $\hat{K}=0$. So, $K=0$. Hence, G' is discrete. If G' contains a subgroup of the form $Z(n)$, then $Z(n)$ is a nontrivial compact open subgroup of G' which is a contradiction. So G' is torsion-free. Suppose G' has a nontrivial divisible subgroup. Then G' has a direct summand $H \cong Q$. But then H is s-pure projective, contradicting Lemma 3.3. Therefore, G' is reduced.

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