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INJECTIVE MODULES WITH RESPECT TO MODULES OF PROJECTIVE DIMENSION AT MOST ONE

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Abstract. Several authors have been interested in cotorsion theories. Among these theories we figure the pairs $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$, where $\mathcal{P}_n$ designates the set of modules of projective dimension at most a given integer $n \geq 1$ over a ring R. In this paper, we shall focus on homological properties of the class $\mathcal{P}_1^{\perp}$ that we term the class of $\mathcal{P}_1$-injective modules. Numerous nice characterizations of rings as well as of their homological dimensions arise from this study. In particular, it is shown that a ring R is left hereditary if and only if any $\mathcal{P}_1$-injective module is injective and that R is left semi-hereditary if and only if any $\mathcal{P}_1$-injective module is FP-injective. Moreover, we prove that the global dimensions of R might be computed in terms of $\mathcal{P}_1$-injective modules, namely the formula for the global dimension and the weak global dimension turn out to be as follows

$$\text{wgl-dim}(R) = \sup\{\text{fd}_R(M) : M \text{ is a } \mathcal{P}_1\text{-injective left } R\text{-module}\}$$

and

$$\text{l-gl-dim}(R) = \sup\{\text{pd}_R(M) : M \text{ is a } \mathcal{P}_1\text{-injective left } R\text{-module}\}.$$

We close the paper by proving that, given a Matlis domain R and an R-module $M \in \mathcal{P}_1$, $\text{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective module N if and only if M is strongly flat.

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1. Introduction

Throughout this paper, R denotes an associative ring with unit element and the R-modules are supposed to be unital. Given an R-module M, M^+ denotes the character R-module of M, that is, $M^+ := \text{Hom}_{\mathbb{Z}}\left(M, \frac{\mathbb{Q}}{\mathbb{Z}}\right)$, $\text{pd}_R(M)$ denotes the projective dimension of M, $\text{id}_R(M)$ the injective dimension of M and $\text{fd}_R(M)$ the flat dimension of M. As for the global dimensions, l-gl-dim(R) designates the left global dimension of R and wgl-dim(R) the weak global dimension of R.

Finally, $\mathrm{Mod}(R)$ stands for the class of left R-modules, $\mathcal{P}(R)$ stands for the class of projective left R-modules and $\mathcal{I}(R)$ the class of injective left R-modules. Any unreferenced material is standard as in [23,25,26,27].

The cotorsion theories were introduced by L. Salce in the category of abelian groups. Their role proves to be significant in the study of covers and envelopes and particularly in the proof of the flat cover conjecture. The most known and useful cotorsion pair is the flat cotorsion pair $(\mathcal{D}, \mathcal{E})$, where $\mathcal{D}$ is the class of flat modules and $\mathcal{E}$ is the class of cotorsion modules. Also, among the interesting examples of cotorsion theories we figure the pairs $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$, where $\mathcal{P}_n$ designates the set of modules of projective dimension at most a given integer $n \geq 1$ over a ring R. These pairs have been proved to be complete with enough projective and injective modules and they play an important role in generalizing many classical results of Fuchs and Salce in the context of Prüfer domains by means of the approximation theory.

Our main purpose in this paper is to study the homological properties of the class $\mathcal{P}_1^{\perp}$ that we term the class of $\mathcal{P}_1$-injective modules over an arbitrary ring R. Let us denote by $\mathcal{P}_1\mathcal{I}(R) := \mathcal{P}_1^{\perp}$ this class of $\mathcal{P}_1$-injective R-modules. Note that in the context of integral domains, by [4, Theorem 7.2 and Corollary 8.2], $\mathcal{P}_1\mathcal{I}(R)$ coincides with the class $\mathcal{DI}(R)$ of divisible R-modules. This means that the $\mathcal{P}_1$-injective module notion extends in some sense the divisible module notion from integral domains to arbitrary rings, and thus our study will permit, in particular, to shed light on homological properties of divisible modules in the case when R is a domain. It is worth reminding the reader at this point that several generalizations of injective R-modules were studied in the literature. Recently, S.B. Lee introduced the notion of weak-injective modules which are injective with respect to $\mathcal{F}_1$ the class of modules of flat dimension at most one in the case of integral domains R [18]. It is to be noted that this concept of weak-injective module is different from the notion of weak injective module introduced by Gao and Wang in [16]. Observe that $(\mathcal{F}_1, \mathcal{W})$ is a cotorsion theory, where $\mathcal{W}$ denotes the class of weak-injective modules over R, and that it is proved in [12] that the pairs $(\mathcal{F}_1, \mathcal{W})$ and $(\mathcal{P}_1, \mathcal{P}_1\mathcal{I}(R))$ coincide over an integral domain R if and only if R is almost perfect in the sense of Bazzoni and Salce [5]. Lee provided many characterizations of Prüfer domains and semi-Dedekind domains in terms of weak-injective modules. Subsequently, in [13], Fuchs and Lee studied the weak-injective envelopes of modules over an integral domain R and their relations to flat covers. They proved that any R-module admits a weak-injective envelope. Further, in [19], Lee extends weak-injectivity from modules over a domain to modules over a commutative ring R. The author discusses many properties of weak-injective modules over R, namely their relations to h-divisible, pure-injective and absolutely pure modules over R. For basics and later investigations on weak-injective modules the reader is kindly referred to [11,12,13,14,15,18,19]. It arises from the present study of $\mathcal{P}_1$-injective modules over an arbitrary ring R numerous nice characterizations of specific rings as well as of their homological dimensions. For instance, it is shown that R is left hereditary if and only if $\mathcal{P}_1\mathcal{I}(R) = \mathcal{I}(R)$ and that R is left semi-hereditary if and only if $\mathcal{P}_1\mathcal{I}(R) \subseteq \text{FP-}\mathcal{I}(R)$, where $\text{FP-}\mathcal{I}(R)$ stands for the class of FP-injective R-modules. Also, we characterize those rings R over which any R-module is $\mathcal{P}_1$-injective. In this aspect, we prove that $\mathcal{P}_1\mathcal{I}(R) = \text{Mod}(R)$ if and only if $\text{FPD}(R) = 0$, where $\text{FPD}(R)$ denotes the finitistic projective dimension of R. Moreover, if R is commutative, it turns out that $\mathcal{P}_1\mathcal{I}(R) = \text{Mod}(R)$ if and only if R is a perfect ring. On the other hand, we discuss the homological dimensions of R in terms of $\mathcal{P}_1$-injective modules. In particular, we prove the following formula for the weak global dimension and the global dimension of a ring R:

$$\text{wgl-dim}(R) = \sup\{\text{fd}_R(M) : M \text{ is a } \mathcal{P}_1\text{-injective left } R\text{-module}\}$$

and

$$\text{l-gl-dim}(R) = \sup\{\text{pd}_R(M) : M \text{ is a } \mathcal{P}_1\text{-injective left } R\text{-module}\}.$$

In the last section, we characterize the modules M such that $\text{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective module N. It is worthwhile pointing out, in this case, that Fuchs and Lee proved that given an integral domain R and an R-module M, then $\text{Hom}_R(M, N)$ is weak-injective for each weak-injective module N if and only if M is flat [12, Theorem 4.3]. We prove that, given a commutative ring R and an R-module M, $M \in \mathcal{P}_1$ and $\text{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective module N if and only if $\text{Tor}_1^R(E, M) = 0$ and $E \otimes_R M \in \mathcal{P}_1$ for each $E \in \mathcal{P}_1$. It turns out, in the case where R is a Matlis domain and for an R-module $M \in \mathcal{P}_1$, that $\text{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective module N if and only if M is strongly flat.

2. $\mathcal{P}_1$-injective modules and homological dimensions

This section discusses homological properties of $\mathcal{P}_1$-injective modules especially those related to the different homological dimensions. Let $\mathcal{P}_1 := \{X \in \text{Mod}(R) : \text{pd}_R(X) \leq 1\}$. Then it is easy to check that $\mathcal{P}_1$ satisfies the following statements:

1) $\mathcal{P}_1$ is contains all projective left R-modules.

2) $\mathcal{P}_1$ is closed under extensions.

3) $\mathcal{P}_1$ is closed under kernels of epimorphisms.

$\mathcal{P}_1$ is thus a projectively resolving class of R-modules. Moreover, if R is left hereditary, then $\mathcal{P}_1 = \mathrm{Mod}(R)$.

Definition 2.1. A left R-module M is said to be $\mathcal{P}_1$-injective if $\mathrm{Ext}^1_R(H, M) = 0$ for all $H \in \mathcal{P}_1$ and R is said to be a self $\mathcal{P}_1$-injective ring if it is a $\mathcal{P}_1$-injective left R-module.

Let ζ denotes the set of exact sequences of the form $0 \longrightarrow K \longrightarrow N \longrightarrow H \longrightarrow 0$ such that $H \in \mathcal{P}_1$. It is easy to see that a module M is $\mathcal{P}_1$-injective if M is ζ-injective, that is, the functor $\mathrm{Hom}_R(-, M)$ leaves exact all short exact sequences of ζ.

Next, we list some properties of $\mathcal{P}_1$-injective R-modules.

Proposition 2.2. *Let R be a ring. Then the following conditions hold:*

(1) *Any injective R-module is $\mathcal{P}_1$-injective.*
(2) *Any quotient of a $\mathcal{P}_1$-injective module is $\mathcal{P}_1$-injective.*
(3) *The class $\mathcal{P}_1\mathcal{I}(R)$ of all $\mathcal{P}_1$-injective R-modules is closed under extensions.*
(4) *Let $(M_i)_{i\in I}$ be a family of R-modules. Then $\prod_{i\in I} M_i$ is $\mathcal{P}_1$-injective if and only if each M_i is $\mathcal{P}_1$-injective.*
(5) *Any finite direct sum of $\mathcal{P}_1$-injective R-modules is $\mathcal{P}_1$-injective.*
(6) *A direct summand of a $\mathcal{P}_1$-injective R-module is $\mathcal{P}_1$-injective.*

Proof. (1) It is clear.

(2) Let M be a $\mathcal{P}_1$-injective left R-module and let N be a submodule of M. Consider the short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow \dfrac{M}{N} \longrightarrow 0$. Let $K \in \mathcal{P}_1$. Applying the functor $\mathrm{Hom}_R(K, -)$ to the considered sequence, we get the following exact sequence

$$\mathrm{Ext}^1_R(K, N) \longrightarrow \mathrm{Ext}^1_R(K, M) = 0 \longrightarrow \mathrm{Ext}^1_R\left(K, \frac{M}{N}\right) \longrightarrow \mathrm{Ext}^2_R(K, N).$$

As $K \in \mathcal{P}_1$, $\mathrm{Ext}^2_R(K, N) = 0$. Hence $\mathrm{Ext}^1_R\left(K, \dfrac{M}{N}\right) = 0$. It follows that $\dfrac{M}{N}$ is $\mathcal{P}_1$-injective, as desired.

The remaining assertions (3), (4), (5) and (6) are straightforward completing the proof. □

Let $\mathcal{L}$ be a class of R-modules. Consider the following two associated classes:

$\mathcal{L}^{\perp} = \{X \in \mathrm{Mod}(R) : \mathrm{Ext}^1_R(L, X) = 0, \forall L \in \mathcal{L}\}$ and ${}^{\perp}\mathcal{L} = \{X \in \mathrm{Mod}(R) : \mathrm{Ext}^1_R(X, L) = 0, \forall L \in \mathcal{L}\}$.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a cotorsion theory provided that ${}^{\perp}\mathcal{C} = \mathcal{F}$ and $\mathcal{F}^{\perp} = \mathcal{C}$ [10]. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called complete if every R-module has a special $\mathcal{C}$-preenvelope and a special $\mathcal{F}$-precover [25].

Lemma 2.3. *The pair $(\mathcal{P}_1, \mathcal{P}_1\mathcal{I}(R))$ is a complete cotorsion theory with enough injectives and projectives.*

Proof. This follows from [10, Theorem 7.4.6]. □

Taking into account [19], we call a module M weak-injective over an arbitrary ring R if $\mathrm{Ext}^1_R(H, M) = 0$ for each left R-module $H \in \mathcal{F}_1$. The next results compare the two classes $\mathcal{P}_1\mathcal{I}(R)$ of $\mathcal{P}_1$-injective modules and $\mathcal{W}$ of weak-injective modules.

Lemma 2.4. *Let R be a ring. Then any weak-injective module is $\mathcal{P}_1$-injective.*

Proof. It is straightforward as $\mathcal{P}_1 \subseteq \mathcal{F}_1$. □

Proposition 2.5. *Let R be a ring. Then the following assertions are equivalent:*

(1) $\mathcal{P}_1\mathcal{I}(R) = \mathcal{W}$;
(2) $\mathcal{P}_1 = \mathcal{F}_1$;
(3) *Any flat submodule of a free R-module is projective.*

Proof. (1) $\Leftrightarrow$ (2) It is straightforward as $(\mathcal{P}_1, \mathcal{P}\mathcal{I}(R))$ and $(\mathcal{F}_1, \mathcal{W})$ are cotorsion theories.
(2) $\Rightarrow$ (3) Let $0 \longrightarrow F \longrightarrow L \longrightarrow K = \dfrac{L}{F} \longrightarrow 0$ be an exact sequence such that F is flat and L is a free module. Then $K \in \mathcal{F}_1 = \mathcal{P}_1$. If K is projective, then the sequence splits and thus F is projective. If $\mathrm{pd}_R(K) = 1$, then F is projective, as desired.
(3) $\Rightarrow$ (2) Let $M \in \mathcal{F}_1$ and let $0 \longrightarrow F \longrightarrow L \longrightarrow M \longrightarrow 0$ be an exact sequence such that L is a free module and F is flat. By assumption, F, being a flat submodule of the free module L, is projective. Thus $M \in \mathcal{P}_1$ completing the proof. □

Corollary 2.6. *Let R be a left perfect ring. Then a module M is $\mathcal{P}_1$-injective if and only if it is weak-injective.*

Proof. It is follows easily from Proposition 2.5 as if R is left perfect, then $\mathcal{P}_1 = \mathcal{F}_1$. □

Corollary 2.7. *Let R be a domain. Then the following assertions are equivalent.*

(1) $\mathcal{P}_1\mathcal{I}(R) = \mathcal{W}$;
(2) *R is almost perfect.*

Proof. It is straightforward using [19, Corollary 6.4] and Proposition 2.5. □

Remark 2.8. It is clear from Proposition 2.2 that any quotient module of an injective R-module is $\mathcal{P}_1$-injective. Also, it is notable that $\mathcal{P}_1\mathcal{I}(R)$ is not stable under direct limit and arbitrary direct sum. In fact, take any left hereditary ring R which is not left Noetherian. For instance, let R be the ring

$$R := \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Then R is a hereditary ring which is not Noetherian (see [1, Example 28.12]). Observe that $\mathcal{P}_1\mathcal{I}(R) = \mathcal{I}(R)$ and, as R is not Noetherian, $\mathcal{I}(R)$ is not stable under direct limit and arbitrary direct sum [23, Theorem 4.10]. Thus $\mathcal{P}_1\mathcal{I}(R)$ is not stable under direct limit and arbitrary direct sum, as desired. An immediate consequence of this is that the two classes of $\mathcal{P}_1$-injective modules and divisible modules are different in general as this latter is stable under arbitrary direct sums.

The following result discusses possible connections between the stability of $\mathcal{P}_1\mathcal{I}(R)$ under direct limit and under arbitrary direct sum.

Proposition 2.9. *Let R be a ring. Then the following assertions are equivalent.*

(1) $\mathcal{P}_1\mathcal{I}(R)$ *is stable under direct limits;*
(2) $\mathcal{P}_1\mathcal{I}(R)$ *is stable under direct sums.*
(3) $\mathcal{P}_1\mathcal{I}(R)$ *is stable under direct unions.*
(4) $\mathcal{P}_1\mathcal{I}(R)$ *is stable under increasing unions.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2) It holds as direct unions are direct limits, increasing unions are direct unions and direct sums are increasing unions.
(2) $\Rightarrow$ (1) Observe that a direct limit of a direct system $(M_i)_{i\in\Lambda}$ of $\mathcal{P}_1$-injective modules is a quotient of the direct sum $\bigoplus\limits_i M_i$. As, by Proposition 2.2, any quotient of $\bigoplus\limits_i M_i$ is a $\mathcal{P}_1$-injective, we get $\varinjlim M_i$ is $\mathcal{P}_1$-injective, as desired. ☐

Next, we present a large class of $\mathcal{P}_1$-injective modules. First, we begin by recalling some notions from Gorenstein homological theory. In fact, the Gorenstein projective modules, Gorenstein injective modules and Gorenstein flat modules stem from the classical notions of projective modules, injective modules and flat modules, respectively, by standing as images and kernels of the differentials of complete resolutions of projective modules, injective modules and flat modules. Effectively, a module M is said to be Gorenstein projective if there exists an exact sequence of projective modules, called a complete projective resolution,

$$\mathbf{P} := \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

such that $\mathbf{P}$ remains exact after applying the functor $\mathrm{Hom}_R(-, P)$ for each projective module P and $M := \mathrm{Im}(P_0 \to P_{-1})$. The Gorenstein injective modules are defined dually. These new concepts allows Enochs and Jenda [8,9] to introduce new (Gorenstein homological) dimensions in order to extend the G-dimension defined by Auslander and Bridger in [2]. It turns out, in particular, that these Gorenstein homological dimensions are refinements of the classical dimensions of a module M, in the sense that $\mathrm{Gpd}_R(M) \le \mathrm{pd}_R(M)$ and $\mathrm{Gid}_R(M) \le \mathrm{id}_R(M)$ with equality each time the corresponding classical homological dimension is finite. Moreover the Gorenstein global dimension is defined to be $\text{G-gldim}(R) := \max\{\mathrm{Gpd}_R(M) : M \in \mathrm{Mod}(R)\}$. A ring R is called Gorenstein semi-simple if $\text{G-gldim}(R) = 0$.

Note that any Gorenstein injective module over a ring R, being a quotient of an injective module, is $\mathcal{P}_1$-injective. Then

$$\mathcal{I}(R) \subseteq \mathcal{GI}(R) \subseteq \mathcal{P}_1\mathcal{I}(R)$$

where $\mathcal{GI}(R)$ denotes the class of Gorenstein injective modules.

Proposition 2.10. *Let R be a ring. Then $(\mathcal{P}_1\mathcal{I}(R))^\perp = \mathcal{I}(R)$.*

Proof. We only need to prove that if $M \in (\mathcal{P}_1\mathcal{I}(R))^\perp$, then M is injective. In fact, let $M \in (\mathcal{P}_1\mathcal{I}(R))^\perp$. There exists a short exact sequence of left R-modules $0 \longrightarrow M \longrightarrow I \longrightarrow G \longrightarrow 0$ with I injective. Then G is $\mathcal{P}_1$-injective, by Proposition 2.2. Hence, $\mathrm{Ext}^1_R(G, M) = 0$, and thus the sequence $0 \longrightarrow M \longrightarrow I \longrightarrow G \longrightarrow 0$ splits. It follows that M is injective, as desired. □

The next results show that the homological dimensions of a ring R might be characterized by the vanishing of the functors Ext and Tor with respect to the class $\mathcal{P}_1\mathcal{I}(R)$ of $\mathcal{P}_1$-injective modules.

Proposition 2.11. *Let R be a ring. Let M be a left R-module and $n \ge 1$ an integer. Then the following statements are equivalent:*

(1) $\mathrm{pd}_R(M) \le n$;
(2) $\mathrm{Ext}^n_R(M, N) = 0$ *for any $\mathcal{P}_1$-injective left R-module N.*

Proof. Our argument uses induction on n. The equivalence holds for $n = 1$ as, by Lemma 2.3, the pair $(\mathcal{P}_1, \mathcal{P}_1\mathcal{I}(R))$ is a cotorsion theory. Assume that $n \ge 2$. Then $\mathrm{pd}_R(M) \le n$ if and only if there exists a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P is a projective module and $\mathrm{pd}_R(K) \le n - 1$ if and only if there exists a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P is a projective module and $\mathrm{Ext}^{n-1}_R(K, N) = 0$ for any $N \in \mathcal{P}_1\mathcal{I}(R)$ (by induction) if and only if $\mathrm{Ext}^n_R(M, N) = 0$ for any $N \in \mathcal{P}_1\mathcal{I}(R)$, as desired. □

Proposition 2.12. *Let R be a ring. Let M be a right R-module and n be a positive integer. Then the following statements are equivalent:*

(1) $\mathrm{fd}_R(M) \leq n$;
(2) $\mathrm{Tor}^R_{n+1}(M, N) = 0$ *for any $\mathcal{P}_1$-injective left R-module N.*

First, we establish the following lemma.

Lemma 2.13. *Let R be a ring and M a right R-module. Then the following assertions are equivalent:*

(1) *M is a flat right R-module;*
(2) $\mathrm{Tor}^R_1(M, N) = 0$ *for any $\mathcal{P}_1$-injective left R-module N.*

Proof. We only need to prove that (2) $\Rightarrow$ (1). Assume that $\mathrm{Tor}^R_1(M, N) = 0$ for every $\mathcal{P}_1$-injective left R-module N. Consider the short exact sequence of left R-modules $0 \longrightarrow M^+ \longrightarrow E \longrightarrow G \longrightarrow 0$ with E an injective left module. Then E is $\mathcal{P}_1$-injective and thus G is $\mathcal{P}_1$-injective by Proposition 2.2. Hence, $\mathrm{Ext}^1_R(G, M^+) \cong \mathrm{Tor}^R_1(M, G)^+ = 0$. Therefore, the considered exact sequence $0 \longrightarrow M^+ \longrightarrow E \longrightarrow G \longrightarrow 0$ splits, and thus M^+ is an injective left R-module. Hence, M is a flat right R-module completing the proof. □

Proof of Proposition 2.12. It suffices to prove that (2) $\Rightarrow$ (1). Assume that (2) holds. Let F_{n-1} be the $(n-1)$th yoke of a flat resolution of M and let N be any $\mathcal{P}_1$-injective left R-module. By [23, Corollary 8.8], $\mathrm{Tor}^R_{n+1}(M, N) \cong \mathrm{Tor}^R_1(F_{n-1}, N)$. Then, using (2), we get $\mathrm{Tor}^R_1(F_{n-1}, N) = 0$, and thus by Lemma 2.13, F_{n-1} is flat. Hence $\mathrm{fd}_R(M) \leq n$, as desired. □

Corollary 2.14. *Let R be a ring. Then*

$$\text{wgl-dim}(R) = \sup\{\mathrm{fd}_R(M) : M \in \mathcal{P}_1\mathcal{I}(R)\}.$$

Proof. If $\sup\{\mathrm{fd}_R(M) : M$ is a $\mathcal{P}_1$-injective left R-module$\} = +\infty$, then we are done. Assume that there exists a positive integer n such that $\mathrm{fd}_R(M) \leq n$ for any $\mathcal{P}_1$-injective left module M. Then $\mathrm{Tor}^R_{n+1}(A, M) = 0$ for any right R-module A. Then, by Proposition 2.12, $\mathrm{fd}_R(A) \leq n$ for each right R-module A. Therefore $\text{wgl-dim}(R) \leq n$. This establishes the desired equality. □

As $\mathcal{P}_1\mathcal{I}(R) = \mathcal{DI}(R)$ when R is a domain, we have the following consequence.

Corollary 2.15. *Let R be a domain. Then*

$$\text{wgl-dim}(R) = \sup\{\mathrm{fd}_R(M) : M \in \mathcal{DI}(R)\}.$$

Let R be an arbitrary ring. Recall the following cohomological invariants which are inherent to the ring R:

$$\text{l-sfli}(R) = \sup\{\text{fd}_R(M) : M \text{ is a left injective } R\text{-module}\}$$

and

$$\text{r-sfli}(R) = \sup\{\text{fd}_R(M) : M \text{ is a right injective } R\text{-module}\}.$$

It is known that $\text{r-sfli}(R) \neq \text{l-sfli}(R)$, in general. But still, it remains one of the open problems of homological algebra to know whether $\text{r-sfli}(R) = \text{l-sfli}(R)$ in the case of Noetherian rings R. A positive answer to this problem will have a positive impact on the resolution of the well known Gorenstein symmetry conjecture. Let us define the $\mathcal{P}_1$-analogs of these two entities. Denote by

$$\mathcal{P}_1\text{- l-sfli}(R) = \sup\{\text{fd}_R(M) : M \text{ is a left } \mathcal{P}_1\text{-injective } R\text{-module}\}$$

and

$$\mathcal{P}_1\text{- r-sfli}(R) = \sup\{\text{fd}_R(M) : M \text{ is a right } \mathcal{P}_1\text{-injective } R\text{-module}\}.$$

The next corollary states that the new cohomological invariants $\mathcal{P}_1$- l-sfli(R) and $\mathcal{P}_1$- r-sfli(R) do, in fact, coincide for any ring R.

Corollary 2.16. *Let R be a ring. Then*

$$\mathcal{P}_1\text{- l-sfli}(R) = \mathcal{P}_1\text{- r-sfli}(R).$$

Proof. It suffices to observe that $\mathcal{P}_1$- l-sfli(R) = l-wgl-dim(R) = r-wgl-dim(R) = $\mathcal{P}_1$- r-sfli(R). $\square$

In Section 3, we deduce from the above result that if R is left hereditary, then r-sfli(R)=l-sfli(R).

The next corollary records the fact that von Neumann regular rings can totally be characterized by flatness of $\mathcal{P}_1$-injective R-modules.

Corollary 2.17. *Let R be a ring. Then the following assertions are equivalent.*

(1) *R is a von Neumann regular ring.*
(2) *Any $\mathcal{P}_1$-injective left R-module M is flat.*

Proposition 2.18. *Let R be a ring. Let M be a left R-module and n a positive integer. Then the following statements are equivalent.*

(1) $\text{id}_R(M) \leq n$;
(2) $\text{Ext}_R^{n+1}(N, M) = 0$ *for each $\mathcal{P}_1$-injective left R-module N.*

The proof requires the following lemma.

Lemma 2.19. *Let R be a ring and M a left R-module. Then the following assertions are equivalent.*

(1) *M is injective;*
(2) $\mathrm{Ext}^1_R(N,M)=0$ *for each $\mathcal{P}_1$-injective left R-module N.*

Proof. It suffices to prove that (2) $\Rightarrow$ (1). Assume that $\mathrm{Ext}^1_R(N,M)=0$ for each $\mathcal{P}_1$-injective left R-module N. Then $M\in\mathcal{P}_1\mathcal{I}(R)^{\perp}$. By Proposition 2.10, $\mathcal{P}_1\mathcal{I}(R)^{\perp}=\mathcal{I}(R)$. Hence M is injective, as desired. □

Proof of Proposition 2.18. (1) $\Rightarrow$ (2) is straightforward.
(2) $\Rightarrow$ (1) Assume that $\mathrm{Ext}^{n+1}_R(N,M)=0$ for each $\mathcal{P}_1$-injective left R-module N. If $n=0$, then we are done, by Lemma 2.19. Assume that $n\geq 1$ and let $0\longrightarrow M\xrightarrow{\varepsilon}E_0\xrightarrow{d_0}E_1\xrightarrow{d_1}E_2\longrightarrow\cdots$ be an injective resolution of M. Let $L^0=\mathrm{Im}(\varepsilon)$ and $L^i=\mathrm{Im}(d_{i-1})$ for each integer $i\geq 1$. Then, by [23, Corollary 6.16], $\mathrm{Ext}^1_R(N,L^{n-1})\cong\mathrm{Ext}^{n+1}_R(N,M)=0$ for any $\mathcal{P}_1$-injective module N. Hence, by Lemma 2.19, L^{n-1} is injective. It follows that $\mathrm{id}_R(M)\leq n$. □

Proposition 2.20. *Let R be a ring. Then*

(1) $\text{l-gl-dim}(R)=\sup\{\mathrm{pd}_R(M):M\in\mathcal{P}_1\mathcal{I}(R)\}$.
(2) *Either R is semi-simple or*

$$\text{l-gl-dim}(R)=1+\sup\{\mathrm{id}_R(M):M\in\mathcal{P}_1\mathcal{I}(R)\}.$$

Proof. (1) First, note that

$$\text{l-gl-dim}(R)\geq\sup\{\mathrm{pd}_R(M):M\text{ is a }\mathcal{P}_1\text{-injective left }R\text{-module}\}.$$

If $\sup\{\mathrm{pd}_R(M):M\text{ is a }\mathcal{P}_1\text{-injective left }R\text{-module}\}=+\infty$, then we are done. Now, assume that there exists an integer $n\geq 0$ such that $\mathrm{pd}_R(M)\leq n$ for any $\mathcal{P}_1$-injective R-module M. Then $\mathrm{Ext}^{n+1}_R(M,N)=0$ for any $\mathcal{P}_1$-injective R-module M and any R-module N. Hence, by Proposition 2.18, $\mathrm{id}_R(N)\leq n$ for any R-module N. It follows that $\text{l-gldim}(R)\leq n$ and thus the desired equality follows.
(2) Assume that R is not semi-simple. Then $\text{l-gl-dim}(R)\geq 1$. Let $n\geq 1$ be an integer.

Then $\text{l-gl-dim}(R)\leq n$ if and only if $\mathrm{pd}_R(M)\leq n$ for each R-module M if and only if $\mathrm{Ext}^n_R(M,N)=0$ for each R-module M and each $\mathcal{P}_1$-injective module N (by Proposition 2.11) if and only if $\mathrm{id}_R(N)\leq n-1$ for each $\mathcal{P}_1$-injective module N if and only if $1+\sup\{\mathrm{id}_R(M):M\text{ is a }\mathcal{P}_1\text{-injective left }R\text{-module}\}\leq n$. It follows that

$$\text{l-gl-dim}(R)=1+\sup\{\mathrm{id}_R(M):M\text{ is a }\mathcal{P}_1\text{-injective left }R\text{-module}\}.$$

□

We get the following result allowing to compute the global dimension of a domain R in terms of divisible modules.

Corollary 2.21. *Let R be a domain. Then*

(1) $\text{l-gl-dim}(R) = \sup\{\text{pd}_R(M) : M \in \mathcal{DI}(R)\}$.

(2) *Either R is a field or*

$$\text{l-gl-dim}(R) = 1 + \sup\{\text{id}_R(M) : M \in \mathcal{DI}(R)\}.$$

We close this section by the following characterization of semi-simple rings in terms of $\mathcal{P}_1$-injective modules.

Proposition 2.22. *Let R be a ring. Then the following assertions are equivalent.*

(1) *R is semi-simple;*

(2) *Any $\mathcal{P}_1$-injective left R-module is projective;*

(3) *Any $\mathcal{P}_1$-injective left R-module is semi-simple.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.
(2) $\Rightarrow$ (1) Assume that any $\mathcal{P}_1$-injective left R-module is projective. Then, by Proposition 2.20, $\text{l-gl-dim}(R) = 0$. Hence R is semi-simple.
(3) $\Rightarrow$ (1) Let M be any left R-module. As M is a submodule of an injective R-module I and I is semi-simple, by our assumption. Then M is semi-simple. It follows that R is semi-simple, as desired. $\square$

3. $\mathcal{P}_1$-injective modules and specific rings

The main goal of this section is to characterize specific rings such as hereditary rings, semi hereditary rings and Noetherian rings in terms of inherent properties of $\mathcal{P}_1$-injective R-modules.

Let $\mathcal{A}$ be nonempty collection of left ideals of a ring R. A left R-module Q is said to be $\mathcal{A}$-injective provided that each R-homomorphism $f : A \longrightarrow Q$ with $A \in \mathcal{A}$ extends to R (see [24]), equivalently, $\text{Ext}^1_R\left(\frac{R}{A}, Q\right) = 0$ for each $A \in \mathcal{A}$. Let us denote the class of $\mathcal{A}$-injective modules by $\mathcal{AI}(R)$. In particular, an R-module M is called P-injective if M is $\mathcal{A}$-injective with $\mathcal{A} = \{\text{principal left ideals of } R\}$, M is said to be max-injective if M is $\mathcal{A}$-injective with $\mathcal{A} = \{\text{maximal left ideals of } R\}$ and M is said to be coflat if M is $\mathcal{A}$-injective with $\mathcal{A} = \{\text{finitely generated left ideals of } R\}$.

Proposition 3.1. *Let $\mathcal{A}$ be nonempty collection of left ideals of a ring R. Then $\mathcal{P}_1\mathcal{I}(R) \subseteq \mathcal{AI}(R)$ if and only if $\mathcal{A} \subseteq \mathcal{P}(R)$.*

Proof. Assume that $\mathcal{P}_1\mathcal{I}(R) \subseteq \mathcal{AI}(R)$. Let $A \in \mathcal{A}$. Then $\mathrm{Ext}^1_R\Big(\dfrac{R}{A}, M\Big) = 0$ for each $\mathcal{P}_1$-injective R-module M and thus $\dfrac{R}{A} \in {}^{\perp}(\mathcal{P}_1\mathcal{I}(R))$. By Lemma 2.3, we get $\mathrm{pd}_R\Big(\dfrac{R}{A}\Big) \leq 1$, and so A is projective. The inverse implication is obvious. □

Corollary 3.2. *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is left hereditary;*
(2) *$\mathcal{P}_1\mathcal{I}(R) = \mathcal{I}(R)$.*

Proof. Note that $\mathcal{I}(R) \subseteq \mathcal{P}_1\mathcal{I}(R)$ and that if $\mathcal{A} := \{\text{left ideals of } R\}$, then an R-module M is injective if and only if M is $\mathcal{A}$-injective. Hence, by Proposition 3.1, $\mathcal{P}_1\mathcal{I}(R) \subseteq \mathcal{I}(R)$ if and only if $\mathcal{P}_1\mathcal{I}(R) \subseteq \mathcal{AI}(R)$ if and only if $\mathcal{A} \subseteq \mathcal{P}(R)$ if and only if R is left hereditary. This completes the proof. □

Corollary 3.3. *Let R be a domain. Then the following conditions are equivalent.*

(1) *R is Dedekind domain;*
(2) *$\mathcal{DI}(R) = \mathcal{I}(R)$.*

As an immediate consequence of the above result, we deduce that l-sfli$(R) =$ r-sfli(R) in the context of a left hereditary ring R.

Corollary 3.4. *Let R be a left hereditary ring. Then*

$$\text{l-sfli}(R) = \text{r-sfli}(R).$$

Proof. It is straightforward using Corollary 2.16. □

It is known that, given a commutative ring R, any R-module is weak-injective (or $\mathcal{F}_1$-injective, where $\mathcal{F}_1 = \{M \in \mathrm{Mod}(R) : \mathrm{fd}(M) \leq 1\}$) if and only if R is perfect [19, Theorem 5.1]. Now, we present the following characterization of rings all modules over which are $\mathcal{P}_1$-injective via the vanishing of their finitistic projective dimension. This allows to give an alternate and simple proof of [19, Theorem 5.1].

Proposition 3.5. *Let R be a ring. Then the following conditions are equivalent.*

(1) *Any left R-module is $\mathcal{P}_1$-injective;*
(2) *R is self-$\mathcal{P}_1$-injective and $\mathcal{P}_1\mathcal{I}(R)$ is stable under direct limits;*
(3) *Any projective R-module is $\mathcal{P}_1$-injective;*
(4) *Every submodule of a $\mathcal{P}_1$-injective is $\mathcal{P}_1$-injective;*
(5) *$\mathcal{P}_1 = \mathcal{P}(R)$;*
(6) $\mathrm{FPD}(R) = 0$.

Proof. (1) $\Rightarrow$ (2) It is obvious.
(2) $\Rightarrow$ (3) It is easy as any projective module is a direct limit of finitely generated free modules which are $\mathcal{P}_1$-injective by hypotheses and as $\mathcal{P}_1\mathcal{I}(R)$ is stable under finite direct sums.
(3) $\Rightarrow$ (1) It follows from the fact any R-module M is a quotient of a projective module and thus it is $\mathcal{P}_1$-injective.
(1) $\Leftrightarrow$ (4) is clear as any R-module is a submodule of an injective module which is $\mathcal{P}_1$-injective.
(1) $\Rightarrow$ (5) Using (1), we get ${}^\perp\operatorname{Mod}(R) = {}^\perp(\mathcal{P}_1\mathcal{I}(R)) = \mathcal{P}_1$ as $(\mathcal{P}_1, \mathcal{P}_1\mathcal{I}(R))$ is a cotorsion theory. Since ${}^\perp\operatorname{Mod}(R) = \mathcal{P}(R)$, we get $\mathcal{P}_1 = \mathcal{P}(R)$.
(5) $\Rightarrow$ (6) It follows easily that $\operatorname{FPD}(R) = 0$.
(6) $\Rightarrow$ (1) $\operatorname{FPD}(R) = 0$ yields, in particular, that $\mathcal{P}_1 = \mathcal{P}(R)$ and thus $\mathcal{P}_1\mathcal{I}(R) = \mathcal{P}_1^\perp = \mathcal{P}(R)^\perp = \operatorname{Mod}(R)$, completing the proof. □

Corollary 3.6. *Let R be a ring. If any R-module is weak-injective, then* $\operatorname{FPD}(R) = 0$.

It turns out that in the commutative rings context the rings over which all modules are $\mathcal{P}_1$-injective are exactly the perfect rings. Moreover, the next result presents an alternate and simple proof of Lee's theorem [19, Theorem 5.1].

Corollary 3.7. *Let R be a commutative ring. Then the following conditions are equivalent.*

(1) *Any R-module is $\mathcal{P}_1$-injective;*
(2) *Any R-module is weak-injective;*
(3) *R is a perfect ring.*

Proof. (2) $\Rightarrow$ (1) It is direct as, by Lemma 2.4, any weak-injective module is $\mathcal{P}_1$-injective.
(1) $\Leftrightarrow$ (3) First, note that, by Kaplansky's theorem [3, Kaplansky's theorem, page 466] and by [3, Example 6, page 476], R is perfect if and only if $\operatorname{FPD}(R) = 0$. Therefore we are done by Proposition 3.5.
(3) $\Rightarrow$ (2) Assume that R is perfect. Then $\operatorname{FPD}(R) = 0$ and $\mathcal{F}_1 = \mathcal{P}_1$ so that, by Proposition 2.5, $\mathcal{P}_1\mathcal{I}(R) = \mathcal{W}$. Now, Proposition 3.5 completes the proof. □

We deduce the following characterization of self-injective rings R on which the class of $\mathcal{P}_1$-injective modules is stable under direct limit.

Corollary 3.8. *Let R be a self-injective ring. Then the following assertions are equivalent.*

(1) *$\mathcal{P}_1\mathcal{I}(R)$ is stable under direct limit;*
(2) *Any R-module is $\mathcal{P}_1$-injective;*
(3) $\mathrm{FPD}(R) = 0$.
Moreover, if R is commutative, then the above assertions are equivalent to the following:
(4) *R is a perfect ring.*

Recall that R is a Quasi-Frobenius ring (QF-ring for short) if R is left Artinian and ${}_RR$ is injective, equivalently if any projective left R-module is injective (see [1, Theorem 31.9]).

Corollary 3.9. *Let R be a QF-ring. Then any left R-module is $\mathcal{P}_1$-injective.*

Proof. Let R be a QF-ring. We prove that $\mathrm{FPD}(R) = 0$. In fact, as $\text{l-silp}(R) := \max\{\mathrm{id}_R(M) : M \in \mathcal{P}(R)\} = 0$ and $\text{l-spli}(R) = \max\{\mathrm{pd}_R(M) : M \in \mathcal{I}(R)\} = 0$, we get, by [6, Theorem 3.3], that

$$\text{G-gldim}(R) = \max\{\text{l-silp}(R), \text{l-spli}(R)\} = 0.$$

Then $\mathrm{FPD}(R) = \mathrm{GFPD}(R) = 0$ [17, Theorem 2.28], where $\mathrm{GFPD}(R) := \sup\{\mathrm{Gpd}_R(M) : M \in \mathrm{Mod}(R) \text{ and } \mathrm{Gpd}_R(M) < +\infty\}$ denotes the finitistic Gorenstein projective dimension of R. Hence $\mathrm{FPD}(R) = 0$, so that, by Proposition 3.5, $\mathcal{P}_1\mathcal{I}(R) = \mathrm{Mod}(R)$, as desired. $\square$

The following result characterizes rings of finite global dimension such that $\mathcal{P}_1\mathcal{I}(R) = \mathrm{Mod}(R)$.

Corollary 3.10. *Let R be a ring such that* $\text{l-gl-dim}(R) < +\infty$. *Then the following conditions are equivalent.*

(1) *Any R-module is $\mathcal{P}_1$-injective;*
(2) *R is a semi-simple;*
(3) *R is a QF-ring.*

A ring R is called a PP-ring if each principal ideal of R is projective. Note that any left semi-hereditary ring is a PP-ring. Next, we prove that any von Neumann regular ring is a left semi-hereditary ring and thus a PP-ring and if R is moreover self $\mathcal{P}_1$-injective, then the converse holds as well.

Proposition 3.11. *Let R be a ring. Then*

(1) *If R is a von Neumann regular ring, then R is left semi-hereditary and thus a PP-ring.*

(2) *If moreover R is self $\mathcal{P}_1$-injective ring, then the following assertions are equivalent:*
   (i) *R is a von Neumann regular ring;*
   (ii) *R is left semi-hereditary;*
   (iii) *R is a PP-ring.*

Proof. (1) Assume that R is a von Neumann regular ring. Let I be a finitely generated ideal of R. As I is finitely generated, $\dfrac{R}{I}$ is finitely presented. Now, since R is von Neumann regular, $\dfrac{R}{I}$ is a flat R-module. Hence $\dfrac{R}{I}$ is projective over R and thus the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \dfrac{R}{I} \longrightarrow 0$ splits, so that I is projective over R. Consequently, R is left semi-hereditary.
(2) Assume that R is self $\mathcal{P}_1$-injective.
(i) $\Rightarrow$ (ii) It holds by (1).
(ii) $\Rightarrow$ (iii) It is straightforward.
(iii) $\Rightarrow$ (i) Assume that R is a PP-ring. Let $I = aR$ be a principal ideal of R. Then, by Proposition 2.2(1), I, being a homomorphic image of R, is $\mathcal{P}_1$-injective and, as R is a PP-ring, I is projective. Consider the following exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \dfrac{R}{I} \longrightarrow 0$. Then $\mathrm{pd}_R\Big(\dfrac{R}{I}\Big) \leq 1$. Since I is $\mathcal{P}_1$-injective, we get $\mathrm{Ext}^1_R\Big(\dfrac{R}{I}, I\Big) = 0$. Therefore, the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \dfrac{R}{I} \longrightarrow 0$ splits and thus I is a direct summand of R. It follows that I is generated by an idempotent element, that is, there exists $e \in R$ such that $e = e^2$ and $I = eR$. Then there exist p, $q \in R$ such that $e = pa$, and $a = qe$. Hence $ae = (qe)e = qe = a$ and $a = ae = apa$. Consequently, R is von Neumann regular. $\square$

Recall that a ring R is left Noetherian if and only if each left R-module is FP-projective [20, Proposition 2.6]. Our next proposition refines this theorem via $\mathcal{P}_1$-injective modules.

Proposition 3.12. *Let R be a ring. Then the following conditions are equivalent.*
(1) *R is left Noetherian;*
(2) *Every $\mathcal{P}_1$-injective left R-module is FP-projective.*

Proof. (1) $\Rightarrow$ (2) It is obvious by [20, Proposition 2.6].
(2) $\Rightarrow$ (1) Let N be an FP-injective left module. Then, for each $\mathcal{P}_1$-injective R-module M, we get, by assumption, $\mathrm{Ext}^1_R(M, N) = 0$. Hence $N \in (\mathcal{P}_1\mathcal{I}(R))^{\perp}$. As, by Proposition 2.3, $(\mathcal{P}_1\mathcal{I}(R))^{\perp} = \mathcal{I}(R)$, it follows that N is injective. Consequently, R is Noetherian, by [21, Theorem 3]. $\square$

Corollary 3.13. *Let R be a domain. Then the following conditions are equivalent.*

(1) *R is Noetherian;*
(2) *Every divisible R-module is FP-projective.*

It is well known that a ring R is left coherent if and only if FP-$\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$ [7, Theorem 1]. Also, FP-$\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is projective$\}$ if and only if R is left coherent and right perfect [7, Theorem 3]. The following theorem and corollary discuss the class of rings R such that $\mathcal{P}_1\mathcal{I}(R) \subseteq \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$ as well as those rings R satisfying $\mathcal{P}_1\mathcal{I}(R) \subseteq \{M \in \mathrm{Mod}(R) : M^+$ is projective$\}$.

Recall that a module M over a ring R is said to be coflat if $\mathrm{Ext}^1_R\left(\frac{R}{A}, M\right) = 0$ for each finitely generated left ideal A of R. The class of coflat R-modules is denoted by co-$\mathcal{F}(R)$.

Theorem 3.14. *Let R be a ring. Then the following assertions are equivalent:*

(1) $\mathcal{P}_1\mathcal{I}(R) \subseteq \{M \in \mathrm{Mod}(R) : M^+$ *is flat*$\}$*;*
(2) $\mathcal{P}_1\mathcal{I}(R) \subseteq$ *FP-*$\mathcal{I}(R)$*;*
(3) $\mathcal{P}_1\mathcal{I}(R) \subseteq$ *co-*$\mathcal{F}(R)$*.*
(4) *R is left semi-hereditary.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds, that is, $\mathcal{P}_1\mathcal{I}(R) \subseteq \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$. First, let us prove that a right R-module M is flat if and only if so is M^{++}. In fact, suppose that M is a flat right R-module. Then M^+ is injective and thus a $\mathcal{P}_1$-injective left R-module, and so, using (1), M^{++} is a flat right R-module. Conversely, suppose that M^{++} is a flat right R-module. Then M is flat since M is a pure submodule of M^{++} and it is known that the class of flat modules is closed under pure submodules. This proves our claim and yields that R is left coherent, by [7, Theorem 1]. Therefore, FP-$\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$, by [7, Theorem 1]. It follows that $\mathcal{P}_1\mathcal{I}(R) \subseteq$ FP-$\mathcal{I}(R)$ proving (2).
(4) $\Rightarrow$ (2) Assume that R is left semi-hereditary. Then $\mathrm{pd}_R(M) \leq 1$ for each finitely presented left R-module M, that is, $\mathcal{P}_1$ includes all finitely presented modules. Therefore, every $\mathcal{P}_1$-injective left R-module is FP-injective proving (2).
(2) $\Rightarrow$ (4) Let I be a finitely generated ideal I of R. Then $\frac{R}{I}$ is finitely presented. Hence $\mathrm{Ext}^1_R\left(\frac{R}{I}, N\right) = 0$ for any $\mathcal{P}_1$-injective R-module N. Therefore, by Proposition 2.11, $\mathrm{pd}_R\left(\frac{R}{I}\right) \leq 1$. It follows that I is projective. Consequently, R is left semi-hereditary.
(3) $\Leftrightarrow$ (4) It follows easily from Proposition 3.1.
(4) $\Rightarrow$ (1) Assume that R is semi-hereditary. Then, in particular, (2) holds and R

is left coherent and thus, by [7, Theorem 1], FP-$\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$. Hence (1) holds using (2). This completes the proof. $\square$

Corollary 3.15. *Let R be a ring. Then*

(1) *If $\mathcal{P}_1\mathcal{I}(R) \subseteq \{M \in \mathrm{Mod}(R) : M^+$ is projective$\}$, then R left semi-hereditary.*
(2) *If, moreover, R is right perfect, then the following assertions are equivalent:*
  (i) $\mathcal{P}_1\mathcal{I}(R) \subseteq \{M \in \mathrm{Mod}(R) : M^+$ *is flat* $\}$;
  (ii) $\mathcal{P}_1\mathcal{I}(R) \subseteq$ *FP-*$\mathcal{I}(R)$;
  (iii) $\mathcal{P}_1\mathcal{I}(R) \subseteq$ *co-*$\mathcal{F}(R)$;
  (iv) *R is left semi-hereditary;*
  (v) $\mathcal{P}_1\mathcal{I}(R) \subseteq \{M \in \mathrm{Mod}(R) : M^+$ *is projective*$\}$.

Proof. (1) It is direct using Theorem 3.14 as $\{M \in \mathrm{Mod}(R) : M^+$ is projective$\} \subseteq \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$.
(2) It is straightforward since, when R is right perfect, a right module is flat if and only if it is projective and thus $\{M \in \mathrm{Mod}(R) : M^+$ is projective$\} = \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$. $\square$

It is worth reminding that a ring R is left Artinian if and only if $\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is projective$\}$ (see [7, Theorem 4]). Also, by [21, Theorem 3], a ring R is Noetherian if and only if $\mathcal{I}(R) =$ FP-$\mathcal{I}(R)$. Our focus next is to provide the analog of these theorems for the $\mathcal{P}_1$-injective modules, namely, we seek the class of rings R such that $\mathcal{P}_1\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is projective$\}$ as well as those rings R such that $\mathcal{P}_1\mathcal{I}(R) =$ FP-$\mathcal{I}(R)$.

Theorem 3.16. *Let R be a ring.*

(1) *Consider the following assertions.*
  (i) $\mathcal{P}_1\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ *is projective*$\}$.
  (ii) $\mathcal{P}_1\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ *is flat*$\}$.
  (iii) $\mathcal{P}_1\mathcal{I}(R) =$ *FP-*$\mathcal{I}(R)$.
  Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).
(2) *Assume that R is commutative. Then the following assertions are equivalent.*
  (i) $\mathcal{P}_1\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ *is projective*$\}$;
  (ii) *R is a semisimple ring.*

Proof. (1) (ii) $\Leftrightarrow$ (iii) It is straightforward using Theorem 3.14 since R is left coherent in case either (ii) or (iii) is satisfied and thus, by [7, Theorem 1], FP-$\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is flat$\}$.
(i) $\Rightarrow$ (iii) Assume that $\mathcal{P}_1\mathcal{I}(R) = \{M \in \mathrm{Mod}(R) : M^+$ is projective$\}$. Let M be

an FP-injective left R-module and let $0 \longrightarrow M \longrightarrow E \longrightarrow K \longrightarrow 0$ be a short exact sequence with E an injective left R-module. Then it is pure and thus the associated exact sequence of character modules

$$0 \longrightarrow K^+ \longrightarrow E^+ \longrightarrow M^+ \longrightarrow 0$$

splits. As E is injective, it is $\mathcal{P}_1$-injective. Then E^+ is projective and therefore M^+ is projective. Hence, by assumptions, we get M is $\mathcal{P}_1$-injective. Conversely, let M be a $\mathcal{P}_1$-injective left R-module. Then M^+ is projective and thus M^{++} is injective. It follows, by [24, Proposition 2.6], since M is pure in M^{++}, that M is FP-injective. Therefore $\mathcal{P}_1\mathcal{I}(R) = \text{FP-}\mathcal{I}(R)$.

(2) Assume that R is commutative.

(ii) $\Rightarrow$ (i) Suppose that R is semisimple. As R is hereditary, by Corollary 3.2, $\mathcal{P}_1\mathcal{I}(R) = \mathcal{I}(R)$ and as R is Artinian, we have $\{M \in \text{Mod}(R) : M^+ \text{ is projective}\} = \mathcal{I}(R)$. Then $\mathcal{P}_1\mathcal{I}(R) = \{M \in \text{Mod}(R) : M^+ \text{ is projective}\}$ establishing (i).

(i) $\Rightarrow$ (ii) Assume that (i) holds. Then, by (1), $\mathcal{P}_1\mathcal{I}(R) = \text{FP-}\mathcal{I}(R) = \{M \in \text{Mod}(R) : M^+ \text{ is projective}\}$. Hence, by [7, Theorem 3], R is a semi-hereditary and perfect ring. Therefore R is hereditary and thus $\mathcal{P}_1\mathcal{I}(R) = \mathcal{I}(R)$, so that $\mathcal{I}(R) = \{M \in \text{Mod}(R) : M^+ \text{ is projective}\}$. It follows, by [7, Theorem 4], that R is Artinian. Now, using [22, Theorem 3.2.6], we get $0 = \dim(R) = \text{FPD}(R) = \text{gl-dim}(R)$ as $\text{gl-dim}(R) \leq 1$. This means that R is semisimple establishing (ii) and completing the proof. □

It is well known that if R is a Prüfer domain, then, by [25, Theorem 4.9], $\mathcal{P}_1\mathcal{I}(R) = \mathcal{P}_1^{\perp} = \text{FP-}\mathcal{I}(R)$. Next, we prove that this last condition totally characterizes Prüfer domains.

Corollary 3.17. *Let R be an integral domain. Then the following statements are equivalent.*

(1) *R is Prüfer;*
(2) $\mathcal{P}_1\mathcal{I}(R) = FP\text{-}\mathcal{I}(R)$;
(3) $\mathcal{P}_1\mathcal{I}(R) = \{M \in \text{Mod}(R) : M^+ \textit{ is flat}\}$.

Proof. Observe that, by Theorem 3.14 and Theorem 3.16, we have (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (2) holds by [25, Theorem 4.9], as desired. □

Corollary 3.18. *Let R be an integral domain. Then the following assertions are equivalent:*

(1) $\mathcal{P}_1\mathcal{I}(R) = \{M \in \text{Mod}(R) : M^+ \textit{ is projective}\}$;
(2) *R is a field.*

Proof. It is straightforward applying Theorem 3.16. □

4. $\mathcal{P}_1$-injective modules and Hom

This section studies the behavior of the $\mathcal{P}_1$-injective modules with respect to the functor Hom. It is worthwhile recalling in this context that an R-module M is flat if and only if $\mathrm{Hom}_R(M, N)$ is injective for each injective R-module N if and only if the character module M^+ is injective. In this section, we seek properties of the modules M such that $\mathrm{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective module N. Recall, in this aspect, that Fuchs and Lee proved in [12] that, given an integral domain R and an R-module M, $\mathrm{Hom}_R(M, N)$ is weak-injective for any weak-injective module N if and only if M is flat [12, Theorem 4.3].

In [12, Theorem 4.1], Fuchs and Lee proved, in the context of an integral domain R, that a module M weak-injective if and only if $\mathrm{Hom}_R(F, M)$ is weak-injective for any flat module F. We give next an analog of this theorem for $\mathcal{P}_1$-injectivity.

Proposition 4.1. *Let R be a ring. A left R-module M is $\mathcal{P}_1$-injective if and only if $\mathrm{Hom}_R(P, M)$ is a right $\mathcal{P}_1$-injective module for each projective R-module P.*

Proof. It is easy as the class of $\mathcal{P}_1$-injective modules is stable under direct product and direct summand. □

Proposition 4.2. *Let R be a ring and M an R-module. Then the following assertions are equivalent:*

1. $\mathrm{Hom}_R(M, N)$ *is a right $\mathcal{P}_1$-injective module for each left injective R-module N;*
2. $\mathrm{Tor}_1^R(E, M) = 0$ *for each right module $E \in \mathcal{P}_1$;*
3. M^+ *is a right $\mathcal{P}_1$-injective module.*

Proof. (2) ⇔ (3) It is direct using the isomorphism $\mathrm{Ext}_R^1(E, M^+) \cong \mathrm{Tor}_1^R(E, M)^+$ for any right module $E \in \mathcal{P}_1$.
(1) ⇒ (2) Let $E \in \mathcal{P}_1$ be a right module and let $0 \longrightarrow Q \longrightarrow P \longrightarrow E \longrightarrow 0$ be an exact sequence of right modules such that P and Q are projective R-modules. Then the following sequence is exact

$$0 \longrightarrow \mathrm{Hom}_R(E, \mathrm{Hom}_R(M, N)) \longrightarrow \mathrm{Hom}_R(P, \mathrm{Hom}_R(M, N)) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_R(Q, \mathrm{Hom}_R(M, N)) \longrightarrow 0$$

for each injective R-module N. Thus we get the following exact sequence for each injective R-module N

$$0 \longrightarrow \mathrm{Hom}_R(E \otimes_R M, N) \longrightarrow \mathrm{Hom}_R(P \otimes_R M, N) \longrightarrow \mathrm{Hom}_R(Q \otimes_R M, N) \longrightarrow 0.$$

Hence the sequence

$$0 \longrightarrow Q \otimes_R M \longrightarrow P \otimes_R M \longrightarrow E \otimes_R M \longrightarrow 0$$

is exact which means that $\mathrm{Tor}_1^R(E, M) = 0$, as desired.
(2) $\Rightarrow$ (1) It is similar by inverting the order of the proof of (1) $\Rightarrow$ (2). $\square$

Corollary 4.3. *Let R be an integral domain and M an R-module. Then the following assertions are equivalent:*

(1) $\mathrm{Hom}_R(M, N)$ *is $\mathcal{P}_1$-injective for each injective R-module N;*
(2) $\mathrm{Hom}_R(M, N)$ *is weak-injective for each injective R-module N;*
(3) *M is torsion-free.*

Proof. Combine Proposition 4.2 and [18, Lemma 2.3]. $\square$

Corollary 4.4. *Let R be a left hereditary ring. Then the following assertions are equivalent.*

(1) $\mathrm{Hom}_R(M, N)$ *is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective R-module N;*
(2) $\mathrm{Hom}_R(M, N)$ *is injective for each injective R-module N;*
(3) *M is flat.*

Proof. (1) $\Leftrightarrow$ (2) It holds as $\mathcal{P}_1\mathcal{I}(R) = \mathcal{I}(R)$.
(1) $\Leftrightarrow$ (3) By Proposition 4.2, $\mathrm{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective module N if and only if $\mathrm{Tor}_1^R(E, M) = 0$ for each $E \in \mathcal{P}_1$ if and only if $\mathrm{Tor}_1^R(E, M) = 0$ for each R-module E if and only if M is flat. $\square$

Next, we present the main theorem of this section. A nice duality arises between the behavior of $\mathcal{P}_1$-injective modules with respect to the Hom and Ext functors, on the one hand, and the behavior of modules of projective dimension at most one with respect to the tensor product and Tor functors, on the other.

Theorem 4.5. *Let R be a commutative ring and M an R-module. Then the following assertions are equivalent:*

(1) *$M \in \mathcal{P}_1$ and $\mathrm{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective R-module N;*
(2) *$\mathrm{Ext}_R^1(M, N) = 0$ and $\mathrm{Hom}_R(M, N)$ is $\mathcal{P}_1$-injective for each $\mathcal{P}_1$-injective R-module N;*
(3) *$\mathrm{Tor}_1^R(E, M) = 0$ and $E \otimes_R M \in \mathcal{P}_1$ for each $E \in \mathcal{P}_1$.*

The proof requires the following result of Fuchs and Lee [11, Lemma 2.3].

Lemma 4.6. *Let R be a commutative ring and A, B and C be R-modules. If* $\mathrm{Ext}_R^1(A,B)=0$ *and* $\mathrm{Tor}_1^R(C,A)=0$, *then*

$$\mathrm{Ext}_R^1(C\otimes_R A,B)\cong \mathrm{Ext}_R^1(C,\mathrm{Hom}_R(A,B)).$$

Proof of Theorem 4.5. (1) $\Leftrightarrow$ (2) It is easy as $(\mathcal{P}_1,\mathcal{P}_1\mathcal{I}(R))$ is a cotorsion theory. (2) $\Leftrightarrow$ (3) Observe that if (2) holds, then $\mathrm{Hom}_R(M,I)$ is $\mathcal{P}_1$-injective for each injective R-module I, so that, by Proposition 4.2, $\mathrm{Tor}_1^R(E,M)=0$ for each $E\in\mathcal{P}_1$. Also, if (3) holds, then, taking $E=R$, we get $R\otimes_R M\cong M\in\mathcal{P}_1$ and thus $\mathrm{Ext}_R^1(M,N)=0$ for each $\mathcal{P}_1$-injective R-module N. Therefore either (2) or (3) implies that $\mathrm{Tor}_1^R(E,M)=0$ and $\mathrm{Ext}_R^1(M,N)=0$ for each $E\in\mathcal{P}_1$ and each $\mathcal{P}_1$-injective R-module N. It follows, by Lemma 4.6, that

$$\mathrm{Ext}_R^1(E\otimes_R M,N)\cong \mathrm{Ext}_1^R(E,\mathrm{Hom}_R(M,N))$$

for each $E\in\mathcal{P}_1$ and each $\mathcal{P}_1$-injective module N. This establishes the desired equivalence and completes the proof of the theorem. $\square$

Corollary 4.7. *Let R be a Matlis domain and M an R-module. Then the following assertions are equivalent:*

(1) *$M\in\mathcal{P}_1$ and $\mathrm{Hom}_R(M,N)$ is divisible for each divisible R-module N;*

(2) *M is strongly flat.*

Proof. (1) $\Rightarrow$ (2) By Theorem 4.5, $\mathrm{Tor}_1^R(E,M)=0$ for each $E\in\mathcal{P}_1$. Then, by [18, Lemma 2.3], M is torsion-free. Hence, using [11, Lemma 6.1] and Theorem 4.5, we get that M is strongly flat.

(2) $\Rightarrow$ (1) Combine [11, Lemma 6.1] and Theorem 4.5 noting that any strongly flat module is torsion-free. $\square$

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