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THE TOTAL GRAPH OF ANNIHILATING ONE-SIDED IDEALS OF A RING

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Abstract. Let R be an associative ring with $1 \neq 0$ which is not a domain. Let $A(R)^* = \{I \subseteq R \mid I \text{ is a left or right ideal of } R \text{ and } \mathrm{l.ann}(I) \cup \mathrm{r.ann}(I) \neq 0\} \setminus \{0\}$. The total graph of annihilating one-sided ideals of R, denoted by $\Omega(R)$, is a graph with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if $\mathrm{l.ann}(I+J) \cup \mathrm{r.ann}(I+J) \neq 0$. In this paper, we study the relations between the graph-theoretic properties of this graph and some algebraic properties of rings. We characterize all rings whose graphs are disconnected. Also, we study diameter, girth, independence number, domination number and planarity of this graph.

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1. Introduction

In recent years, using graph theoretical tools in the study of algebraic structures attracted many researchers, see, for instance, [1,2,11]. I. Beck in [2] introduced the idea of a zero-divisor graph of a commutative ring, where he was mainly interested in colorings. Authors in [1] introduced the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, as the graph with vertices $Z(R)^*$, the set of all nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if $xy = 0$. They investigate the relations between the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma(R)$. For a commutative ring R, authors in [11] introduced and studied a graph, denoted by $\Omega(R)$, with the vertex-set $\mathrm{A}(R)^*$, the set of all nonzero annihilating ideals of R, and two distinct vertices I and J are adjacent if $I + J$ is an annihilating ideal. They study some connections between R and the graph $\Omega(R)$.

Throughout this paper, R is an associative ring with nonzero identity which is not a domain. The nonzero elements of $X \subseteq R$ will be denoted by X^*. For a nonempty subset $X \subseteq R$, let $\mathrm{l.ann}(X) = \{a \in R \mid aX = 0\}$ be the left annihilator of X, and $\mathrm{r.ann}(X) = \{b \in R \mid Xb = 0\}$ be the right annihilator of X. We write $Z_\ell(R)$, $Z_r(R)$, $Z(R)$, $\mathrm{Min}(R)$ and $J(R)$ for the set of all left zero-divisors of R, the set of all right zero-divisors of R, the set $Z_\ell(R) \cup Z_r(R)$, the set of all minimal prime ideals of R and the Jacobson radical of R, respectively. Moreover, we assume that $\mathbb{I}(R)^* = \{I \subsetneqq R \mid I \text{ is a left or right ideal of } R\} \setminus \{0\}$ and $A(R)^* = \{I \subsetneqq R \mid I \text{ is a left or right ideal of } R \text{ and } \mathrm{l.ann}(I) \cup \mathrm{r.ann}(I) \neq 0\} \setminus \{0\}$. A ring R is called semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Also, a ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. A ring R is called abelian if every idempotent is central, that is, $xe = ex$ for any $e^2 = e, x \in R$. A ring R is called local, if R has a unique maximal left ideal. An ideal P of R is called completely prime if R/P is a domain. We denote the number of elements in a set S by $|S|$.

Let $G = (V, E)$ be a simple graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $|G|$, $\mathrm{diam}(G)$, $\mathrm{gr}(G)$, $\gamma(G)$, $\beta(G)$ and $\omega(G)$, we mean the number of vertices, the diameter, the girth, the domination number, the independence number and the clique number of G, respectively. For two distinct vertices u and v in G, the notation $u - v$ means that u and v are adjacent. The set of neighbors of a vertex v in G is denoted by $N(v)$, that is, $N(v) := \{u \in V \setminus \{v\} \mid \{u, v\} \in E\}$. For any undefined notation or terminology in graph theory, we refer the reader to [12].

Let R be a ring and $P_1, P_2, \ldots, P_n$ a finite number of ideals of R, and S a subring of R that is contained in the set theoretic union $P_1 \cup \cdots \cup P_n$. Then by a similar way as used in [7, Theorem 81], one can prove that if at least $n - 2$ of the P's are completely prime, then S is contained in some P_j.

In this paper, we extend the concept of the graph introduced in [11] to a noncommutative ring with nonzero identity as follows:

Definition 1.1. Let R be an associative ring with nonzero identity which is not a domain. The total graph of annihilating one-sided ideals of R, denoted by $\Omega(R)$, is a simple graph with the vertex-set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $\mathrm{l.ann}(I + J) \cup \mathrm{r.ann}(I + J) \neq 0$.

We study the relations between the graph-theoretic properties of $\Omega(R)$ and some algebraic properties of rings. We characterize all rings whose graphs are disconnected. Also, we study diameter, girth, independence number, domination number and planarity of $\Omega(R)$. It is worth to mention that some of our results in this paper appear at first time for the case of noncommutative rings, for example, see Theorem 2.1(2), Propositions 2.2, 2.11(3), 2.16, 2.19 and 3.4.

2. The diameter and girth of $\Omega(R)$

In this section we study the diameter and girth of $\Omega(R)$. Also, we characterize all rings whose graphs are disconnected. One of our main results in this section is the following theorem.

Theorem 2.1. *Let R be a ring. Then*

(1) $\mathrm{diam}(\Omega(R)) \in \{0, 1, 2, 3, \infty\}$.
(2) $\Omega(R)$ *is disconnected if and only if R is a prime ring or R is a reduced ring with exactly two minimal prime ideals.*

In order to prove the above theorem, we need the following propositions.

Proposition 2.2. *Let R be a prime ring. Then $\Omega(R)$ is not connected.*

Proof. Since R is not a domain, there exist $a, b \in R^*$ such that $ab = 0$. We show that $Ra \neq bR$. To see this, let $Ra = bR$. Then $RabR = RaRa = 0$ which is a contradiction. Thus Ra and bR are two distinct vertices of $\Omega(R)$. Suppose that $Ra - I_1 - \cdots - I_n - bR$ is a path between Ra and bR. Since Ra is adjacent to I_1 and R is a prime ring, I_1 is not a right ideal. By a similar method, one can see that I_n is not a right ideal. Now since bR is adjacent to I_n, we have $\mathrm{l.ann}(I_n) \neq 0$ or $\mathrm{r.ann}(bR) \neq 0$ which is a contradiction (because R is a prime ring). Therefore, $\Omega(R)$ is not connected. □

Recall that a ring R is said to be simple if $\{0\}$ and R are the only ideals in R. Now, by Proposition 2.2, we conclude that if R is a simple ring, then $\Omega(R)$ is not connected.

Example 2.3. Let K be a division ring and $\mathbb{M}_n(K)$ be the ring of $n \times n$ matrices over K. Then by [10, Theorem 3.3], $\mathbb{M}_n(K)$ is a simple ring. Thus $\Omega(\mathbb{M}_n(K))$ is disconnected.

We will use the following proposition in the sequel.

Proposition 2.4. *Let R be a nonprime and nonreduced ring. Let I and J be two distinct vertices of $\Omega(R)$. If I and J are right ideals, then $d(I, J) \in \{1, 2, 3\}$.*

Proof. Let I and J be two distinct vertices of $\Omega(R)$ and let I and J be right ideals. If I is adjacent to J, then $d(I, J) = 1$. Thus we may suppose that I is not adjacent to J. By our assumption, there exists $x \in R^*$ such that $x^2 = 0$. Now we consider the following three cases:

Case 1: Suppose that $I \cap Rx = 0$ and $J \cap Rx = 0$. Then we have $Rx \subseteq \text{r.ann}(I)$ and $Rx \subseteq \text{r.ann}(J)$. Thus $I - Rx - J$ is a path of length two in $\Omega(R)$, since $x^2 = 0$. Hence $d(I, J) = 2$.

Case 2: Suppose that $I \cap Rx = 0$ and $J \cap Rx \neq 0$. Then we have $Rx \subseteq \text{r.ann}(I)$ and $I \subseteq \text{l.ann}(Rx)$. Thus I is adjacent to Rx. Let ax, where $a \in R$, be a nonzero element of $J \cap Rx$. Thus either $J = axR$ or J is adjacent to axR. Now since $I \subseteq \text{l.ann}(Rx)$, we have either $Rx = axR$ or Rx is adjacent to axR. Then the set $\{I, Rx, axR, J\}$ forms a path of length at most three between I and J. Hence $d(I, J) \leq 3$.

Case 3: Suppose that $I \cap Rx \neq 0$ and $J \cap Rx \neq 0$. Then there exist elements $a, b \in R$ such that $0 \neq ax \in I \cap Rx$ and $0 \neq bx \in J \cap Rx$. Thus either $axR = I$ or I is adjacent to axR. Also, either $bxR = J$ or J is adjacent to bxR. Now we consider the following three subcases:

Subcase 1: Assume that $\text{r.ann}(I) \neq 0$ and $\text{r.ann}(J) \neq 0$. If $\text{r.ann}(axR) \cap xR = 0$, then $\text{r.ann}(axR) \subseteq \text{r.ann}(xR)$. Thus, one can see that the set $\{I, axR, bxR, J\}$ forms a path of length at most three between I and J. Hence $d(I, J) \leq 3$. Otherwise, we can suppose that $\text{r.ann}(axR) \cap xR \neq 0$. Then there exists $y \in R$ such that $0 \neq xy \in \text{r.ann}(axR) \cap xR$. Thus, either $axR = Rbx$ or axR is adjacent to Rbx. So the set $\{I, axR, Rbx, J\}$ forms a path of length at most three between I and J. Hence $d(I, J) \leq 3$.

Subcase 2: Assume that $\text{r.ann}(I) \neq 0$ and $\text{r.ann}(J) = 0$. Then $I - JI - J$ is a path of length two in $\Omega(R)$. Hence $d(I, J) = 2$.

Subcase 3: Assume that $\text{r.ann}(I) = \text{r.ann}(J) = 0$. Since R is a nonprime ring, there exists a nonzero two-sided ideal K such that $\text{l.ann}(K) \neq 0$. Then it is easy to see that $I \cap K \neq 0$ and $J \cap K \neq 0$. Thus, the set $\{I, K \cap I, K \cap J, J\}$ forms a path of length at most three between I and J. Hence $d(I, J) \leq 3$. □

By a method similar to that we used in the proof of Proposition 2.4, we have the following proposition.

Proposition 2.5. *Let R be a nonprime and nonreduced ring. Let I and J be two distinct vertices of $\Omega(R)$. If I and J are left ideals, then $d(I, J) \in \{1, 2, 3\}$.*

Proposition 2.6. *Let R be a ring which is not reduced. Then* $\text{diam}(\Omega(R)) \in \{0, 1, 2, 3, \infty\}$.

Proof. If R is a prime ring, then by Proposition 2.2, $\Omega(R)$ is disconnected and hence $\text{diam}(\Omega(R)) = \infty$. Thus we may assume that R is not a prime ring. Also, suppose that I and J are two distinct vertices of $\Omega(R)$. Now, if I and J are right ideals (or left ideals), then by Proposition 2.4 (Proposition 2.5), we have $d(I,J) \in \{1,2,3\}$ ($d(I,J) \in \{1,2,3\}$). Hence, we can suppose that I is a right ideal and J is a left ideal. If I is adjacent to J, then $d(I,J)=1$. Thus, we may suppose that I is not adjacent to J. Since R is not reduced, there exists $x \in R^*$ such that $x^2 = 0$. Now we consider the following four cases:

Case 1: Suppose that $I \cap Rx = 0$ and $J \cap xR = 0$. Then we have $Rx \subseteq \mathrm{r.ann}(I)$, $I \subseteq \mathrm{l.ann}(Rx)$, $J \subseteq \mathrm{r.ann}(xR)$ and $xR \subseteq \mathrm{l.ann}(J)$. Thus, the set $\{I, Rx, xR, J\}$ forms a path of length at most three between I and J. Hence $d(I,J) \leq 3$.

Case 2: Suppose that $I \cap Rx \neq 0$ and $J \cap xR = 0$. Then we have $xR \subseteq \mathrm{l.ann}(J)$ and $J \subseteq \mathrm{r.ann}(xR)$. Let ax, where $a \in R$, be a nonzero element of $I \cap Rx$. Then it is easy to see that the set $\{I, axR, xR, J\}$ forms a path of length at most three between I and J. Hence $d(I,J) \leq 3$.

Case 3: Suppose that $I \cap Rx = 0$ and $J \cap xR \neq 0$. Then we have $Rx \subseteq \mathrm{r.ann}(I)$ and $I \subseteq \mathrm{l.ann}(Rx)$. Let xb, where $b \in R$, be a nonzero element of $J \cap xR$. Then the set $\{I, Rx, Rxb, J\}$ forms a path of length at most three between I and J. Hence $d(I,J) \leq 3$.

Case 4: Suppose that $I \cap Rx \neq 0$ and $J \cap Rx \neq 0$. Now we consider the following two subcases:

Subcase 1: Assume that $\mathrm{r.ann}(I) \neq 0$ or $\mathrm{l.ann}(J) \neq 0$. If $I \cap J \neq 0$, then we can choose a nonzero element $t \in I \cap J$ and so $I - Rt - J$ or $I - tR - J$ is a path of length two in $\Omega(R)$. Hence $d(I,J) = 2$. Otherwise, we may assume that $I \cap J = 0$. Thus $IJ = 0$. Since RI is a two-sided ideal of R, by Proposition 2.5, there exists a path of length at most three between RI and J. Thus since $N(RI) \subseteq N(I) \cup \{I\}$, we have $d(I,J) \leq 3$.

Subcase 2: Assume that $\mathrm{r.ann}(I) = 0$ and $\mathrm{l.ann}(J) = 0$. Then since I and J are vertices of $\Omega(R)$, we have $\mathrm{l.ann}(I) \neq 0$ and $\mathrm{r.ann}(J) \neq 0$. Now since R is not a prime ring, there exists a two-sided ideal K such that K is a vertex of $\Omega(R)$. Then it is easy to see that $I \cap K \neq 0$ and $J \cap K \neq 0$. Thus, the set $\{I, I \cap K, J \cap K, J\}$ forms a path of length at most three between I and J. Hence $d(I,J) \leq 3$. This completes the proof. □

Let R be a reversible ring and X a nonempty subset of R. Then $\mathrm{l.ann}(X) = \mathrm{r.ann}(X)$ and we use $\mathrm{ann}(X)$ to denote the annihilator of X. In the following lemma, we study the case that R is a reversible ring. Note that by [8, Lemma 1.4], reversible rings are semicommutative.

Lemma 2.7. *Let R be a reversible ring. If I is a vertex of $\Omega(R)$, then $N(I) \cup \{I\} = N(RIR) \cup \{RIR\}$.*

Proof. Let I be a vertex of $\Omega(R)$. Since R is a reversible ring, $\mathrm{ann}(I) = \mathrm{ann}(RIR)$. Now we conclude that $N(I) \cup \{I\} = N(RIR) \cup \{RIR\}$. □

Recall that a ring R is said to be reduced if R has no nonzero nilpotent elements. It is easy to see that reduced rings are reversible. In the next proposition, we determine the diameter of $\Omega(R)$ when R is a reduced ring. Before that, the following lemma is necessary.

Lemma 2.8. *Let R be a reduced ring. Then* $\mathrm{diam}(\Omega(R)) \notin \{0, 1\}$.

Proof. Let R be a reduced ring and $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$. We consider the following two cases:

Case 1: Suppose that $\mathrm{diam}(\Omega(R)) = 0$. Then, we may assume that I is the unique vertex of $\Omega(R)$. Thus I is a minimal left ideal of R. Now by [10, Lemma 10.22], we have $I = Re$, where e is a nontrivial idempotent element of R. Then one can see that $R(1-e) \neq Re$ and $R(1-e)$ is a vertex of $\Omega(R)$, which is a contradiction.

Case 2: Suppose that $\mathrm{diam}(\Omega(R)) = 1$. Let I be a vertex of $\Omega(R)$. Then $\mathrm{ann}(I)$ is a vertex of $\Omega(R)$. On the other hand, since R is a reduced ring, we have $I \neq \mathrm{ann}(I)$. Now since $\mathrm{diam}(\Omega(R)) = 1$, $x(I + \mathrm{ann}(I)) = 0$ for some $x \in R^*$. Thus we have $x^2 = 0$ which is a contradiction. □

Proposition 2.9. *Let R be a reduced ring. Then* $\mathrm{diam}(\Omega(R)) \in \{2, \infty\}$.

Proof. Let R be a reduced ring. Then by Lemma 2.8, $|\Omega(R)| \geq 2$. Assume that I and J are two distinct vertices of $\Omega(R)$. By Lemma 2.8, we can suppose that I is not adjacent to J. Hence $\mathrm{ann}(I) \cap \mathrm{ann}(J) = 0$. Now by Lemma 2.7, we can consider the vertices RIR and RJR instead of the vertices I and J. Since I is not adjacent to J, we have $RIR \neq RJR$. Now if $RIR \cap RJR \neq 0$, then $RIR - RIR \cap RJR - RJR$ is a path of length two in $\Omega(R)$. So $d(I, J) = 2$. Thus, we may assume that $RIR \cap RJR = 0$. Then $IJ = 0$. Now we have the following two cases:

Case 1: Suppose that $\mathrm{Min}(R) = \{P_1, P_2\}$. Then, we may assume that $P_1 = \mathrm{ann}(RIR)$ and $P_2 = \mathrm{ann}(RJR)$. We show that $\Omega(R)$ is not connected. To see this, let $RIR - A_1 - \cdots - A_n - RJR$ be a path between RIR and RJR. Then since A_1 is adjacent to RIR and R is a reduced ring with exactly two minimal prime ideals, we have $\mathrm{ann}(RIR) = \mathrm{ann}(A_1)$. By a similar method, one can see that $\mathrm{ann}(RIR) = \mathrm{ann}(A_1) = \cdots = \mathrm{ann}(A_n) = \mathrm{ann}(RJR)$. Hence $P_1 = P_2$ which is a contradiction. Thus $d(I, J) = \infty$.

Case 2: Suppose that $|\mathrm{Min}(R)| \geq 3$. Then there exists $P \in \mathrm{Min}(R)$ such that $P \nsubseteq \mathrm{ann}(RIR) \cup \mathrm{ann}(RJR)$. Let $x \in P \setminus (\mathrm{ann}(RIR) \cup \mathrm{ann}(RJR))$. Now by [9, Lemma 1.5], we have $\mathrm{ann}(x) \neq 0$. If $RIR \cap \mathrm{ann}(x) \neq 0$, then $RIR - RxRI - RJR$ is a path of length two in $\Omega(R)$ and hence $d(I, J) = 2$ (note that $RxRI \subseteq RIR$ and $IJ = 0$). Otherwise, we may assume that $RIR \cap \mathrm{ann}(x) = 0$. Thus, we have $\mathrm{ann}(x) \subseteq \mathrm{ann}(RIR)$. Then $RIR - RxRJ - RJR$ is a path of length two in $\Omega(R)$ and hence $d(I, J) = 2$. This completes the proof. □

From the proof of Proposition 2.9, we have the following corollary.

Corollary 2.10. *Let R be a reduced ring. Then $\Omega(R)$ is disconnected if and only if R is a reduced ring with exactly two minimal prime ideals.*

Now, from Propositions 2.2, 2.6 and 2.9 and Corollary 2.10, the proof of Theorem 2.1 is complete.

In the next proposition, we characterize all rings R such that the edge set of $\Omega(R)$ is empty. Note that $\mathbb{M}_n(R)$ is the ring of $n \times n$ matrices over a ring R.

Proposition 2.11. *Let R be a ring. Then the edge set of $\Omega(R)$ is empty if and only if one of the following statements holds:*

(1) $|A(R)^*| = 1$.

(2) $R \cong K_1 \times K_2$ *as rings, where K_1 and K_2 are division rings.*

(3) $R \cong \mathbb{M}_2(K)$ *as rings, where K is a division ring.*

Proof. Assume that the edge set of $\Omega(R)$ is empty and $|A(R)^*| \neq 1$. Now we show that R is an Artinian ring. Since the edge set of $\Omega(R)$ is empty, every vertex of $\Omega(R)$ is minimal as a left or right ideal. Thus we may assume that $Rx \in A(R)^*$ is a minimal left ideal. Since $Rx \cong R/\mathrm{l.ann}(x)$ as modules, $R/\mathrm{l.ann}(x)$ is an Artinian left R-module. Also, by our assumption, $\mathrm{l.ann}(x)$ is an Artinian left R-module. Thus by [5, Proposition 3.5], R is an Artinian left R-module. Similarly, one can see that R is an Artinian right R-module. Thus R is an Artinian ring. On the other hand, since the edge set of $\Omega(R)$ is empty and $|A(R)^*| \neq 1$, $\Omega(R)$ is disconnected and hence by Theorem 2.1, R is a prime ring or R is a reduced ring with exactly two minimal prime ideals. We consider the following two cases:

Case 1: Suppose that R is a reduced ring with exactly two minimal prime ideals, say P_1 and P_2. Since R is an Artinian ring, P_1 and P_2 are maximal ideals. Now since $R = P_1 + P_2$, by [10, Exercise 1.7] one can see that $R \cong K_1 \times K_2$ as rings, where K_1 and K_2 are division rings.

Case 2: Suppose that R is a prime ring. Since R is a prime and Artinian ring, by [10, Theorems 10.24 and 3.5] we have $R \cong \mathbb{M}_n(K)$ as rings, where K is a division ring. Now since the edge set of $\Omega(R)$ is empty, we conclude that $n = 2$. Therefore, $R \cong \mathbb{M}_2(K)$, where K is a division ring.

The converse is clear. □

In the next proposition, we determine the diameter of $\Omega(R)$ when R is a semicommutative ring. Before that, we need the following two lemmas.

Lemma 2.12. ([5, Exercise 3T]) *Let R be a ring and I be a nilpotent ideal of R.*

(1) *If K is a nonzero right ideal of R, then* $\mathrm{l.ann}(I) \cap K \neq 0$.
(2) *If K is a nonzero left ideal of R, then* $\mathrm{r.ann}(I) \cap K \neq 0$.

Proof. (1) Assume that I is a nilpotent ideal of R and K a nonzero right ideal of R. Suppose to the contrary that $\mathrm{l.ann}(I) \cap K = 0$. Then since $\mathrm{l.ann}(I)$ is an ideal of R, we have $\mathrm{l.ann}(I) \subseteq \mathrm{r.ann}(K)$. Thus $KI^{n-1} = 0$, where $n \in \mathbb{N}$ is minimum such that $I^n = 0$. Hence $KI = 0$ for $n = 2$, or $KI^{n-2}I = 0$ for $n \geq 3$. Now since K is a nonzero right ideal of R, we have $\mathrm{l.ann}(I) \cap K \neq 0$ which is a contradiction.

(2) By a similar method as one we used in item (1), one can prove it. □

Lemma 2.13. *Let R be a semicommutative ring and I be a nonzero nilpotent ideal of R. Then I is adjacent to every other vertex of $\Omega(R)$.*

Proof. Assume that I is a nonzero nilpotent ideal of R and $J \neq I$ is a vertex of $\Omega(R)$. Without loss of generality, we can suppose that $\mathrm{l.ann}(J) \neq 0$. Since R is a semicommutative ring, $\mathrm{l.ann}(J)$ is an ideal. Now by Lemma 2.12, $\mathrm{l.ann}(J) \cap \mathrm{l.ann}(I) \neq 0$. Thus there exists $x \in R^*$ such that $x(I + J) = 0$. Therefore, we conclude that I is adjacent to every other vertex. □

Recall that a ring R is called semiprime if R contains no nonzero nilpotent ideals [10].

Proposition 2.14. *Let R be a semicommutative ring. Then*

(1) $\mathrm{diam}(\Omega(R)) \in \{0, 1, 2, \infty\}$.
(2) $\Omega(R)$ *is disconnected if and only if R is a reduced ring with exactly two minimal prime ideals.*

Proof. (1) Let R be a semicommutative ring. If R is not a semiprime ring, then by Lemma 2.13, we have $\mathrm{diam}(\Omega(R)) \in \{0, 1, 2\}$. Otherwise, we may assume that R is a semiprime ring. We show that R is a reduced ring. To see this, let $x \in R^*$ such that $x^2 = 0$. Since R is a semicommutative ring, we have $RxRx = 0$ which is a contradiction. Thus R is a reduced ring. Now by Proposition 2.9, we have $\mathrm{diam}(\Omega(R)) \in \{2, \infty\}$.

(2) It follows from Theorem 2.1. □

In the next theorem, we determine the girth of $\Omega(R)$.

Theorem 2.15. *Let R be a ring. Then* $\mathrm{gr}(\Omega(R)) \in \{3, \infty\}$.

Proof. Suppose that $\Omega(R)$ contains three distinct vertices I, J and K such that $K \subsetneqq I$ and $J \subsetneqq I$. Then it is easy to see that $\mathrm{gr}(\Omega(R)) = 3$. Thus, we may assume that every vertex of $\Omega(R)$ contains at most two nonzero one-sided ideals. We show that $\mathrm{gr}(\Omega(R)) = \infty$. To see this, let $I_1 - I_2 - \cdots - I_n - I_1$ be a cycle in $\Omega(R)$. Since I_1 is adjacent to I_2 and every vertex of $\Omega(R)$ contains at most two nonzero one-sided ideals, we can suppose that $I_1 \subseteq I_2$. Now since I_2 is adjacent to I_3, we have $I_2 \subseteq I_3$ or $I_3 \subseteq I_2$, which is a contradiction. $\square$

Recall that if α is an endomorphism of a ring R, then the additive map $\delta : R \to R$ is called an α-derivation if $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for each $a, b \in R$. Let α be an endomorphism of a ring R and let δ be an α-derivation of R. Then the R-module $R[x]$ with associative and distributive multiplication induced by the rule $xr = \alpha(r)x + \delta(r)$ is known as skew polynomial ring, and is denoted by $R[x; \alpha, \delta]$. If $\delta = 0$, then we obtain a skew polynomial ring of endomorphism type $R[x; \alpha]$. If α is the identity, then our α-derivation becomes an ordinary derivation and we obtain a skew polynomial ring of derivation type $R[x; \delta]$. If α is the identity and $\delta = 0$, then we obtain the standard polynomial ring $R[x]$. An endomorphism α of a ring R is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be α-rigid if there exists a rigid endomorphism α of R. Also, a ring R is called α-compatible if $ab = 0 \Leftrightarrow a\alpha(b) = 0$ for $a, b \in R$. In the next proposition, we study the relation between the connectivity of $\Omega(R)$ and $\Omega(R[x; \alpha, \delta])$ when R is an α-rigid ring.

Proposition 2.16. *Let R be an α-rigid ring and let δ be an α-derivation of R. Then $\Omega(R)$ is connected if and only if $\Omega(R[x; \alpha, \delta])$ is connected.*

Proof. Since R is an α-rigid ring, by [6, Page 4] R is a reduced ring. Now by [9, Theorem 3.3], $R[x; \alpha, \delta]$ is a reduced ring and every minimal prime ideal of $R[x; \alpha, \delta]$ is of the form $P[x; \alpha, \delta]$ where P is a minimal prime ideal of R. Thus by Corollary 2.10, we conclude that $\Omega(R)$ is connected if and only if $\Omega(R[x; \alpha, \delta])$ is connected. $\square$

Example 2.17. Let $R = D_1 \times D_2$, where D_1 and D_2 are division rings. Then it is easy to see that $\Omega(R)$ is disconnected. Moreover, by Proposition 2.16, $\Omega(R[x])$ is disconnected. In addition, since $R[x][y] \cong R[x, y]$ as rings, $\Omega(R[x, y])$ is disconnected.

We need the following lemma in the sequel.

Lemma 2.18. *Let R be a reversible and α-compatible ring. Let $I \in A(R[x;\alpha])^*$.*

(1) *If* $\mathrm{l.ann}(I) \neq 0$, *then there exists $a \in R^*$ such that $aI = 0$ and $Ia = 0$.*
(2) *If* $\mathrm{r.ann}(I) \neq 0$, *then there exists $b \in R^*$ such that $bI = 0$ and $Ib = 0$.*

Proof. (1) Let $I \in A(R[x;\alpha])^*$ and $\mathrm{l.ann}(I) \neq 0$. We choose $g = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha]^*$ of least degree n such that $gI = 0$. Let $f = b_0 + b_1x + \cdots + b_mx^m \in I^*$, where $b_m \neq 0$. Since $gf = 0$, we have $a_nx^nb_mx^m = 0$ and hence $a_n\alpha^n(b_m) = 0$. Thus $a_nb_m = 0$. Now since $gI = 0$, we have $b_mgI = 0$. Hence, since R is a reversible ring and g is of least degree n such that $gI = 0$, we have $b_mg = 0$. Thus, $b_ma_i = a_ib_m = 0$ for $i = 0, 1, \ldots, n$. Note that if $s \in R$ and $sa_n = 0$, then $sa_i = 0$ for $i = 0, 1, \ldots, n$. Now assume that $j \in \{0, 1, \ldots, m\}$ is maximum such that $a_nb_j \neq 0$. Then we have $(a_n\alpha^n(b_j) + a_{n-1}\alpha^{n-1}(b_{j+1}) + \cdots)x^{n+j} \neq 0$ which is a contradiction, since $gf = 0$. Thus $a_nf = 0$ and hence $a_nI = 0$. Now since R is a reversible and α-compatible ring, we have $Ia_n = 0$.

(2) Use a method similar to that we used in item (1). □

Proposition 2.19. *Let R be a reversible and α-compatible ring. Then* $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$ *if and only if* $\mathrm{diam}(\Omega(R[x;\alpha])) = 1$.

Proof. Let $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$. It is easy to see that $|A(R[x;\alpha])^*| \geq 2$. Thus, we may assume that I and J are two distinct vertices of $\Omega(R[x;\alpha])$. Let Δ be the set of all coefficients of elements of I and Λ be the set of all coefficients of elements of J. Then by Lemma 2.18, $R\Delta$ and $R\Lambda$ are vertices of $\Omega(R)$. Since $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$, $c(R\Delta + R\Lambda) = 0$ for some $c \in R^*$. Thus, I is adjacent to J in $\Omega(R[x;\alpha])$ and hence $\mathrm{diam}(\Omega(R[x;\alpha])) = 1$.

Conversely, let $\mathrm{diam}(\Omega(R[x;\alpha])) = 1$. If $|A(R)^*| = 1$, then $\mathrm{diam}(\Omega(R)) = 0$. Thus, we may assume that I and J are two distinct vertices of $\Omega(R)$. Let $S = \Sigma_{i=0}^{\infty} R\alpha^i(I)R$ and $T = \Sigma_{i=0}^{\infty} R\alpha^i(J)R$. Then $S[x;\alpha]$ and $T[x;\alpha]$ are vertices of $\Omega(R[x;\alpha])$. Since $\mathrm{diam}(\Omega(R[x;\alpha])) = 1$, $c(S[x;\alpha] + T[x;\alpha]) = 0$ for some $c \in R^*$. Thus $c(I + J) = 0$ for some $c \in R^*$ and hence $\mathrm{diam}(\Omega(R)) = 1$. □

3. Some combinatorial properties of $\Omega(R)$

In this section we study some combinatorial properties of $\Omega(R)$ such as independence number, domination number and planarity. We start this section with the following proposition.

Proposition 3.1. *Let R be a reduced ring such that* $|\mathrm{Min}(R)| < \infty$. *Then* $\beta(\Omega(R)) = |\mathrm{Min}(R)|$.

Proof. Assume that $\mathrm{Min}(R) = \{P_1, P_2, \ldots, P_n\}$. Then every P_i is a vertex of $\Omega(R)$, for $i = 1, 2, \ldots, n$. We show that $\mathrm{Min}(R)$ is an independent set in $\Omega(R)$. To see this, without loss of generality, assume that P_1 is adjacent to P_2. Then $x(P_1 + P_2) = 0$ for some $x \in R^*$. Thus, by [10, Lemma 12.6], we have $x \in \bigcap_{i=1}^{n} P_i$ which is a contradiction. Hence $\mathrm{Min}(R)$ is an independent set in $\Omega(R)$. Now we show that $\beta(\Omega(R)) = |\mathrm{Min}(R)|$. To see this, let $S = \{I_1, I_2, \ldots, I_{n+1}\}$ be an independent set in $\Omega(R)$ with $n+1$ vertices. Since R is a reduced ring, by [10, Lemma 12.6], we have $Z(R) = \bigcup_{P \in \mathrm{Min}(R)} P$. Hence, there exist $P_k \in \mathrm{Min}(R)$ and distinct $I_i, I_j \in S$ such that $I_i + I_j \subseteq P_k$. Thus, I_i is adjacent to I_j which is a contradiction. Therefore, we conclude that $\beta(\Omega(R)) = |\mathrm{Min}(R)|$. □

We use the following lemma in the sequel.

Lemma 3.2. *Let R be a semicommutative ring.*

(1) *If R is left Noetherian, then $Z_\ell(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and left annihilator of a nonzero element of $Z_r(R)$.*

(2) *If R is right Noetherian, then $Z_r(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and right annihilator of a nonzero element of $Z_\ell(R)$.*

(3) *If R is Noetherian, then $Z(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and left or right annihilator of a nonzero element of $Z(R)$.*

Proof. By a similar way as used in the proof of [7, Theorem 80], we can prove it. □

Let R be a ring. By $\mathbb{P}(R)$, we denote the set of prime ideals of R which are maximal with respect to the property of being contained in $Z(R)$.

Proposition 3.3. *Let R be a semicommutative and Noetherian ring. Then $\beta(\Omega(R)) = |\mathbb{P}(R)| < \infty$.*

Proof. Since R is a semicommutative and Noetherian ring, by Lemma 3.2, we have $Z(R) = \bigcup_{P \in \Delta} P$ where Δ is a finite set and $\Delta = \mathbb{P}(R)$. Now by a method similar to that we used in the proof of Proposition 3.1, we conclude that $\beta(\Omega(R))$ is finite and $\beta(\Omega(R)) = |\Delta|$. □

Let R be a semicommutative ring and $I \in A(R)^*$. Then one can see that $\mathrm{l.ann}(I) = \mathrm{l.ann}(RIR)$ and $\mathrm{r.ann}(I) = \mathrm{r.ann}(RIR)$. Thus, we have the following proposition.

Proposition 3.4. *Let R be a semicommutative ring. Then the following statements are equivalent:*

(1) $\Omega(R)$ *is a complete graph.*
(2) *The subgraph induced by two-sided ideals is complete.*
(3) *The subgraph induced by left ideals is complete.*
(4) *The subgraph induced by right ideals is complete.*

In the following two propositions, we study the case that $\Omega(R)$ is a complete graph.

Proposition 3.5. *Let R be a semicommutative ring. If $\Omega(R)$ is a complete graph, then $Z(R)$ is a completely prime ideal.*

Proof. Let $x, y \in Z(R)$. Since R is a semicommutative ring, we have $RxR \subseteq Z(R)$ and $RyR \subseteq Z(R)$. On the other hand, since $\Omega(R)$ is a complete graph, there exists $t \in R^*$ such that $t(RxR + RyR) = 0$ or $(RxR + RyR)t = 0$. Thus, we have $x + y \in Z(R)$ and hence $Z(R)$ is an ideal of R. Now we show that $Z(R)$ is a completely prime ideal. To see this, let $a, b \in R$ such that $ab \in Z(R)$. Then $abx = 0$ or $xab = 0$, for some $x \in R^*$. Without loss of generality, we can suppose that $abx = 0$. If $bx \neq 0$, then we have $a \in Z(R)$. Thus we conclude that $Z(R)$ is a completely prime ideal. □

Proposition 3.6. *Let R be an abelian and left Artinian ring. Then $\Omega(R)$ is a complete graph if and only $Z(R)$ is a vertex of $\Omega(R)$.*

Proof. Let $Z(R)$ be a vertex of $\Omega(R)$. Then it is easy to see that $\Omega(R)$ is a complete graph.

Conversely, assume that $\Omega(R)$ is a complete graph. Now we consider the following two cases:

Case 1: Suppose that R is a local ring. Since R is a left Artinian ring, by [10, Theorem 4.12] we have $Z(R)^n = 0$ for some positive integer n. Thus $\Omega(R)$ is a complete graph.

Case 2: Suppose that R is not a local ring. Since R is a left Artinian ring, by [10, Lemma 19.19] R contains a nontrivial idempotent element, say e. Thus $R = Re \oplus R(1-e)$. Now since R is an abelian ring, Re and $R(1-e)$ are ideals. Thus by [10, Exercise 1.7], we have $R \cong R_1 \times R_2$ as rings, where R_i is a ring for $i = 1, 2$. Then, $R_1 \times 0$ is not adjacent to $0 \times R_2$ which is a contradiction. □

In the next proposition, we determine the domination number of $\Omega(R)$ when R is a semicommutative ring.

Proposition 3.7. *Let R be a semicommutative ring. Then $\gamma(\Omega(R)) \leq 2$.*

Proof. Since R is a semicommutative ring, there exists a nonzero ideal I of R such that $\mathrm{l.ann}(I) \cup \mathrm{r.ann}(I) \neq 0$. Without loss of generality, assume that $\mathrm{l.ann}(I) \neq 0$. If $I = \mathrm{l.ann}(I)$, then by Lemma 2.12, we have $\gamma(\Omega(R)) = 1$. Otherwise, we can suppose that $I \neq \mathrm{l.ann}(I)$. We show that the set $\{I, \mathrm{l.ann}(I)\}$ is a dominating set. To see this, let J be a vertex of $\Omega(R)$ distinct from I and $\mathrm{l.ann}(I)$. Now we consider the following two cases:

Case 1: Suppose that $\mathrm{l.ann}(J) \neq 0$. If J is not adjacent to I, then we have $\mathrm{l.ann}(I) \cap \mathrm{l.ann}(J) = 0$. Thus $\mathrm{l.ann}(J) \subseteq \mathrm{l.ann}(\mathrm{l.ann}(I))$ and hence J is adjacent to $\mathrm{l.ann}(I)$.

Case 2: Suppose that $\mathrm{r.ann}(J) \neq 0$. If J is not adjacent to $\mathrm{l.ann}(I)$, then we have $\mathrm{r.ann}(J) \cap \mathrm{r.ann}(\mathrm{l.ann}(I)) = 0$. Thus $\mathrm{r.ann}(J) \cap I = 0$ and hence $\mathrm{r.ann}(J) \subseteq \mathrm{r.ann}(I)$. Then J is adjacent to I. □

In the next proposition, we study the case that $\omega(\Omega(R)) < \infty$.

Proposition 3.8. *Let R be a ring such that $\omega(\Omega(R)) < \infty$. Then R is an Artinian ring.*

Proof. Since R is not a domain, there exist $x, y \in Z(R)^*$ such that $xy = 0$. Thus Rx is a vertex of $\Omega(R)$. Now since $\omega(\Omega(R)) < \infty$, Rx contains a minimal left ideal as Rx_1. Then $Rx_1 \cong R/\mathrm{l.ann}(x_1)$ as left R-modules. Thus $R/\mathrm{l.ann}(x_1)$ is an Artinian left R-module. Also, by our assumption, $\mathrm{l.ann}(x_1)$ is an Artinian left R-module. Thus by [5, Proposition 3.5], R is an Artinian left R-module. Similarly, one can see that R is an Artinian right R-module. Thus R is an Artinian ring. □

We will use the following lemma in the sequel.

Lemma 3.9. *Let R be an abelian and left Artinian ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings, where every R_i is a left Artinian local ring for $i = 1, 2, \ldots, n$.*

Proof. If R contains no nontrivial idempotents, then by [10, Lemma 19.19], R is a left Artinian local ring. Thus, we may assume that R contains a nontrivial idempotent element, say e. Then since R is an abelian ring, Re and $R(1-e)$ are ideals. Then by [10, Exercise 1.7], we have $R \cong R_1 \times R_2$ as rings, where R_i is a left Artinian ring for $i = 1, 2$. Now, since R_1 and R_2 are left Artinian and abelian rings, by a similar method, one can see that $R \cong R'_1 \times R'_2 \times \cdots \times R'_n$ as rings, where every R'_i is a left Artinian local ring for $i = 1, 2, \ldots, n$. □

Recall that a graph is said to be planar if it can drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. Note that by [12, Theorem 6.2.2], if a graph contains a subdivision of K_5 or $K_{3,3}$, then it is not planar. We use this fact frequently. In the next proposition, we study the planarity of $\Omega(R)$.

Proposition 3.10. *Let R be an abelian ring. Then $\Omega(R)$ is planar if and only if R is a local ring with $1 \leq |\mathbb{I}(R)^*| \leq 4$, or R is isomorphic to one of the following rings:*

$$A_1 \times A_2 \times A_3 \text{ or } A \times S,$$

where A and each A_i, for $i = 1, 2, 3$, are division rings, and S is an abelian ring with $|\mathbb{I}(S)^*| \leq 1$.

Proof. Suppose that $\Omega(R)$ is planar. Then since $\omega(\Omega(R)) \leq 4$, by Proposition 3.8 and Lemma 3.9 we have $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings, where every R_i is a left Artinian local ring for $i = 1, 2, \ldots, n$. Now we consider the following four cases:

Case 1: Assume that $n \geq 4$. Then the set $\{R_1 \times R_2 \times R_3 \times 0 \times \cdots \times 0, R_1 \times R_2 \times 0 \times 0 \times \cdots \times 0, R_1 \times 0 \times R_3 \times 0 \times \cdots \times 0, 0 \times R_2 \times R_3 \times 0 \times \cdots \times 0, R_1 \times 0 \times 0 \times 0 \times \cdots \times 0\}$ forms K_5 and hence $\Omega(R)$ is not planar.

Case 2: Assume that $n = 3$. We show that $R \cong A_1 \times A_2 \times A_3$, where A_i is a division ring for $i = 1, 2, 3$. To see this, suppose that $I_1 \in A(R_1)^*$. Then the set $\{I_1 \times R_2 \times R_3, I_1 \times 0 \times R_3, I_1 \times R_2 \times 0, I_1 \times 0 \times 0, 0 \times R_2 \times R_3\}$ forms K_5 which is a contradiction. Thus $R \cong A_1 \times A_2 \times A_3$, where A_i is a division ring for $i = 1, 2, 3$.

Case 3: Assume that $n = 2$. Suppose that R_1 and R_2 are not division rings. Then we can choose $I_i \in A(R_i)^*$ for $i = 1, 2$. Thus the set $\{R_1 \times I_2, I_1 \times I_2, R_1 \times 0, 0 \times I_2, I_1 \times 0\}$ forms K_5 which is a contradiction. Hence we may assume that R_1 is a division ring. Now we show that R_2 is a ring with $|\mathbb{I}(R_2)^*| \leq 1$. To see this, suppose that $I, J \in A(R_2)^*$ are distinct. Then the set $\{R_1 \times I, R_1 \times J, R_1 \times 0, 0 \times I, 0 \times J\}$ forms K_5 which is a contradiction. Therefore, we conclude that $R \cong A \times S$, where A is a division ring and S is an abelian ring with $|\mathbb{I}(S)^*| \leq 1$.

Case 4: Assume that $n = 1$. Then it is easy to see that R is a local ring with $1 \leq |\mathbb{I}(R)^*| \leq 4$.

The converse is clear. □

Let G be a simple graph with n vertices and q edges. Recall that a chord is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say C is a primitive cycle if it has no chords. Also, a graph G has the primitive cycle property, say PCP, if any two primitive cycles intersect in at most one edge. The number $frank(G)$ is called the free rank of G and it is the number of primitive cycles of G. Also, the number $rank(G) = qn + r$ is called the cycle rank of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [4, Proposition 2.2], we have $rank(G) \leq frank(G)$. According to [4], a graph G is called a ring graph, if it satisfies in one of the following equivalent conditions:

(1) $rank(G) = frank(G)$,

(2) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

A graph is called outerplanar graph, if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$ (see [3, Theorem 1]). Now, every outerplanar graph is a ring graph and every ring graph is a planar graph. From Proposition 3.10, we have the following corollary.

Corollary 3.11. *Let R be an abelian nonlocal ring. Then the following statements are equivalent:*

(1) $\Omega(R)$ *is planar.*

(2) $\Omega(R)$ *is outerplanar.*

(3) $\Omega(R)$ *is a ring graph.*

Recall that a unicyclic graph is a connected graph with a unique cycle. We conclude by giving a characterization of all abelian rings whose graphs are unicyclic.

Proposition 3.12. *Let R be an abelian ring. Then $\Omega(R)$ is unicyclic if and only if R is a local ring with $|\mathbb{I}(R)^*| = 3$, or $R \cong A \times S$ as rings, where A is a division ring and S is an abelian ring with $|\mathbb{I}(S)^*| = 1$.*

Proof. The proof is similar to that of Proposition 3.10, and hence is excluded. □

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