Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (3) (2021), 754 – 769
DOI : 10.15672/hujms.750244

Research Article

The structure of k-Lucas cubes

Ömer Eğecioğlu[1], Elif Saygı[2], Zülfükar Saygı*[3]

[1] Department of Computer Science, University of California Santa Barbara, Santa Barbara, California 93106, USA
[2] Department of Mathematics and Science Education, Hacettepe University, Ankara 06800, Turkey
[3] Department of Mathematics, TOBB University of Economics and Technology, Ankara 06560, Turkey

Abstract

Fibonacci cubes and Lucas cubes have been studied as alternatives for the classical hypercube topology for interconnection networks. These families of graphs have interesting graph theoretic and enumerative properties. Among the many generalization of Fibonacci cubes are k-Fibonacci cubes, which have the same number of vertices as Fibonacci cubes, but the edge sets determined by a parameter k. In this work, we consider k-Lucas cubes, which are obtained as subgraphs of k-Fibonacci cubes in the same way that Lucas cubes are obtained from Fibonacci cubes. We obtain a useful decomposition property of k-Lucas cubes which allows for the calculation of basic graph theoretic properties of this class: the number of edges, the average degree of a vertex, the number of hypercubes they contain, the diameter and the radius.

Mathematics Subject Classification (2020). 05C31, 05A15, 11B37, 11B39

Keywords. hypercube, Fibonacci cube, Lucas cube, k-Fibonacci cube, Fibonacci number, Lucas number

1. Introduction

An n-dimensional hypercube Q_n is the graph whose vertices are the all binary strings of length n, adjacent when their string representations differ in exactly one position. Hypercubes are one of the basic models for interconnection networks. In [4] and [13] Fibonacci cubes Γ_n and Lucas cubes Λ_n were defined as alternative topologies for the interconnection networks. Both of these networks are special subgraphs of Q_n with interesting properties.

A binary string $b_1 b_2 \ldots b_n$ such that $b_i \cdot b_{i+1} = 0$ for $1 \le i \le n-1$ is called a Fibonacci string of length n. For $n \ge 1$ the Fibonacci cube Γ_n is the subgraph of Q_n induced by vertices indexed by the Fibonacci strings of length n. By convention $\Gamma_0 = Q_0$. By removing all the vertices that start and end with 1 from the vertex set of Γ_n, Lucas cubes Λ_n are obtained. This additional requirement corresponds to the Fibonacci strings $b_1 b_2 \ldots b_n$ satisfying $b_1 \cdot b_n = 0$ for $n \ge 2$.

Graph theoretic and enumerative properties of Fibonacci cubes and Lucas cubes have been extensively studied in the literature. A survey of the some of the properties of Γ_n is

*Corresponding Author.
Email addresses: omer@cs.ucsb.edu (Ö. Eğecioğlu), esaygi@hacettepe.edu.tr (E. Saygı), zsaygi@etu.edu.tr (Z. Saygı)
Received: 09.06.2020; Accepted: 18.12.2020

presented in [8]. Basic graph theoretic properties of Λ_n appear in [13]. The average degree of a vertex in Γ_n and Λ_n are computed in [10] and the induced d-dimensional hypercubes Q_d in Γ_n and Λ_n are studied in [3, 9, 11, 14–16].

There are also other variants of interest inspired by these families of graphs. In [5] and [6], the generalized Fibonacci cube $Q_n(f)$ and the generalized Lucas cube $Q_d(\overleftarrow{f})$ are defined by removing all the vertices that contain some forbidden string f, and by removing all vertices that have a circular rearrangement containing f as a substring, respectively. With this formulation one has $\Gamma_n = Q_n(11)$ and $\Lambda_n = Q_d(\overleftarrow{11})$. The matchable Lucas cubes and their basic properties are studied in [17]. A new family of graphs akin to the Fibonacci cubes called Pell graphs are introduced in [12]. The k-Fibonacci cubes Γ_n^k which are obtained by eliminating certain edges from Γ_n are considered in [2] (see, Section 2 also).

In this work, we consider the subgraph of Γ_n^k which is obtained by removing all the vertices that start and end with 1. The idea is analogous to the construction of Λ_n from Γ_n and $Q_d(\overleftarrow{f})$ from $Q_d(f)$. The graphs Λ_n^k we obtain from Γ_n^k (called k-Lucas cubes) depend on a parameter k just like k-Fibonacci cubes. We obtain basic graph theoretic properties of k-Lucas cubes including the number of edges, the average degree of a vertex, the number of induced hypercubes, the diameter and the radius.

2. Preliminaries

Fibonacci numbers and Lucas numbers are defined by the same recursion $f_n = f_{n-1} + f_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $f_0 = 0$, $f_1 = 1$; $L_0 = 2$ and $L_1 = 1$. It is known that using the Zeckendorf or canonical representation, any positive integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers. For a given positive integer i with $0 < i \leq f_{n+2} - 1$ writing $i = \sum_{j=1}^{n} b_j \cdot f_{n-j+2}$, where $b_j \in \{0,1\}$ and no two consecutive b_j's are 1 one obtains the Zeckendorf representation of i corresponding to the Fibonacci string $b_1 b_2 \dots b_n$ as $(b_1, b_2, \dots, b_n)$. We assume that 0 has Zeckendorf representation $0^n = (0, 0, \dots, 0)$.

The distance between two vertices u and v in a connected graph G is defined as the length of a shortest path between u and v in G. For Q_n, Γ_n and Λ_n this distance coincides with the Hamming distance d_H, which is the number of different bits in the binary string representation of the vertices. Let $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Then the vertex set and the edge set of Γ_n and Λ_n can be written as

$$
\begin{aligned}
V(\Gamma_n) &= \{b_1 b_2 \dots b_n \mid b_i \in \{0,1\} \text{ with } b_i \cdot b_{i+1} = 0, 1 \leq i < n\} \\
E(\Gamma_n) &= \{\{u,v\} \mid u,v \in V(\Gamma_n) \text{ and } d_H(u,v) = 1\}
\end{aligned}
$$

and

$$
\begin{aligned}
V(\Lambda_n) &= \{b_1 b_2 \dots b_n \mid b_i \in \{0,1\} \text{ with } b_i \cdot b_{i+1} = 0, 1 \leq i < n \text{ and } b_1 \cdot b_n = 0\} \\
E(\Lambda_n) &= \{\{u,v\} \mid u,v \in V(\Lambda_n) \text{ and } d_H(u,v) = 1\}.
\end{aligned}
$$

Note that the number of vertices of Γ_n is f_{n+2} and the number of vertices of Λ_n is L_n.

Γ_n can be decomposed into the subgraphs induced by the vertices that start with 0 and 10 respectively. The vertices that start with 0 constitute a graph isomorphic to Γ_{n-1} and the vertices that start with 10 constitute a graph isomorphic to Γ_{n-2}. This can be written symbolically as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2} \tag{2.1}$$

and is usually referred to as the fundamental decomposition of Γ_n [8]. In (2.1), there is a perfect matching between $10\Gamma_{n-2}$ and its copy $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$. We call the f_n edges of the perfect matching between $10\Gamma_{n-2}$ and $00\Gamma_{n-2}$ *link edges*. We note that in (2.1) the vertices of $0\Gamma_{n-1}$ are labeled with 0α where α runs all the Fibonacci strings of length $n-1$ and the vertices of $10\Gamma_{n-2}$ are labeled with $10\beta_1 0$ and $10\beta_2 01$ where β_1 and β_2 run over all the Fibonacci strings of length $n-3$ and $n-4$, respectively. Similarly, the decomposition of Γ_n can also be written in the form

$$\Gamma_n = \Gamma_{n-1}0 + \Gamma_{n-2}01 \; . \tag{2.2}$$

Using (2.1) and (2.2) we can write

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2} = 0\Gamma_{n-1} + \left(10\Gamma_{n-3}0 + 10\Gamma_{n-4}01\right),$$

for $n \geq 4$ and consequently

$$\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0 \; . \tag{2.3}$$

Note that in the decomposition (2.3) of Λ_n in terms of Fibonacci cubes, there are f_{n-1} link edges between $10\Gamma_{n-3}0$ and its copy $00\Gamma_{n-3}0 \subset 0\Gamma_{n-1}$.

2.1. k-Fibonacci cubes

In this section we recall some of the basic properties of k-Fibonacci cubes Γ_n^k introduced in [2]. We first remark that the vertices of Γ_n^k are labeled using the same strings as in Γ_n. Throughout the paper instead of the string label of a vertex, sometimes we will use the number whose Zeckendorf representation corresponds to that string to label the same vertex.

Given a positive integer k, let $\Gamma_n^k = \Gamma_n$ for $f_n \leq k$. Let $n_0(k)$ be the smallest integer for which $f_{n_0(k)} > k$. For a given k, $n_0(k)$ is the smallest integer n for which $\Gamma_n^k \neq \Gamma_n$. For $n \geq n_0(k)$, Γ_n^k is defined in terms of Γ_{n-1}^k and Γ_{n-2}^k similar to the fundamental decomposition of Γ_n as follows:

$$\Gamma_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-2}^k \tag{2.4}$$

where there are k link edges between the vertices with labels $0, 1, \ldots, k-1$ in $0\Gamma_{n-1}^k$ and the corresponding vertices with labels $f_{n+1}, f_{n+1}+1, \ldots, f_{n+1}+k-1$ in $10\Gamma_{n-2}^k$. Using the well known identity $f_n = \lfloor \frac{\Phi^n}{\sqrt{5}} + \frac{1}{2} \rfloor$ it is shown in [2] that

$$n_0(k) = 1 + \left\lfloor \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2} \right) \right\rfloor$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This sequence starts as

$$3, 4, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, \ldots$$

If k is clear from the context we will use n_0 for $n_0(k)$.

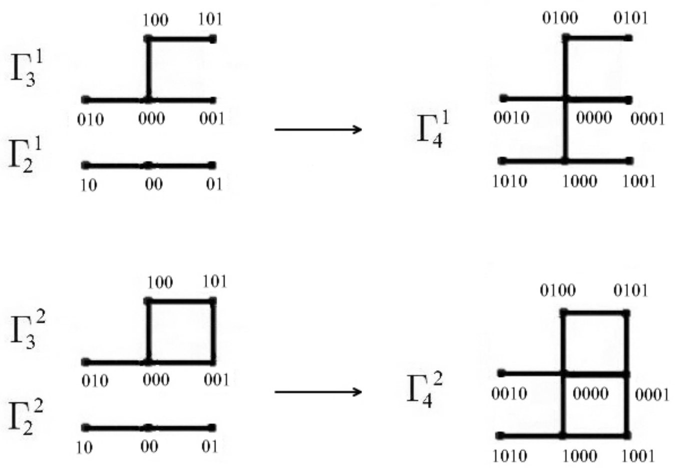

Figure 1. Construction of the k-Fibonacci cubes Γ_4^1 and Γ_4^2.

In Figure 1, we illustrate the constructions of Γ_4^1 and Γ_4^2 from the previous k-Fibonacci cubes. Note that in Figure 1, there is only one link edge between the vertices having labels 0000 and 1000 in Γ_4^1 as $k=1$ and there are two link edges between the vertices having labels 0000 and 1000; 0001 and 1001 in Γ_4^2 as $k=2$.

3. k-Lucas cubes

In this section we introduce k-Lucas cubes, a special subgraph of k-Fibonacci cubes. We will indicate the dependence on k by a superscript and denote these graphs by Λ_n^k. Similar to the definition of Λ_n as the subgraph of Γ_n obtained by eliminating the vertices with $b_1 = b_n = 1$, we define the k-Lucas cube Λ_n^k from the k-Fibonacci cube Γ_n^k by eliminating the vertices with $b_1 = b_n = 1$. In other words, Λ_n^k is obtained from Γ_n^k as the induced subgraph of Γ_n^k in which the binary labels of the vertices satisfy the additional requirement $b_1 \cdot b_n = 0$.

For $k=1$, the graphs Λ_n^1 are all trees. The height h_n of Λ_n^1 satisfies $h_1 = 0$, $h_2 = 1$ and $h_n = \max\{h_{n-1}, 1 + h_{n-2}\}$. Therefore the height of the tree Λ_n^1 with the 0 vertex as the root is given by $h_n = \lfloor n/2 \rfloor$. Note that the height of the tree Γ_n^1 with the 0 vertex as the root is $h_n = \lceil n/2 \rceil$ [2]. Figure 2 shows the first five k-Lucas cubes (trees) for $k=1$.

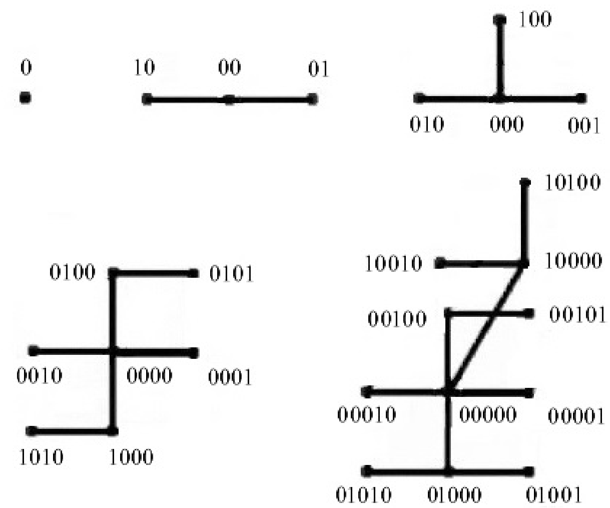

Figure 2. The first five k-Lucas cubes $\Lambda_1^1, \Lambda_2^1, \ldots, \Lambda_5^1$ for $k=1$.

Recall that for a given k, $n_0(k)$ is the smallest integer n for which $\Gamma_n^k \neq \Gamma_n$. By definition of Λ_n^k and Γ_n^k, $n_0(k)$ is again the smallest integer n for which $\Lambda_n^k \neq \Lambda_n$, except when $k=1$. From Figure 2 one can see that $\Lambda_n^1 \neq \Lambda_n$ for $n \geq 4 = n_0(1)+1$.

By removing the vertex having label 1001 from Γ_4 and Γ_4^2 shown in Figure 1, we obtain the Lucas cube Λ_4 and the 2-Lucas cube Λ_4^2 given in Figure 3.

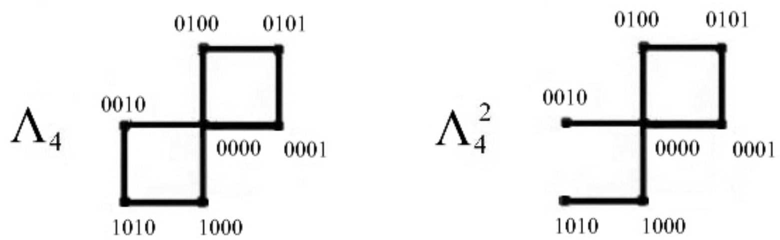

Figure 3. The Lucas cube Λ_4 and the 2-Lucas cube Λ_4^2.

The first eight k-Lucas cubes $\Lambda_1^k, \Lambda_2^k, \ldots, \Lambda_8^k$ for the arbitrarily picked values $k = 1, 3, 6$ and 12 are presented in the Appendix.

We start with a useful result that we need for the analysis of k-Lucas cubes.

Lemma 3.1. *Given a positive integer k, the number of integers N with $0 < N < k$ whose Zeckendorf representation $b_1 b_2 \ldots b_r$ satisfies $b_r = 1$ is*

$$\left\lfloor \frac{k+1}{\phi^2} \right\rfloor \tag{3.1}$$

where ϕ is the golden ratio.

Proof. The integers $N > 0$ with $b_r = 1$ are those with "odd" Zeckendorf expansions. This sequence $1, 4, 6, 9, 12, 14, 17, 19, 22, \ldots$ forms the first column of the Wythoff array [7], and its mth term is given explicitly by

$$\lfloor \phi^2 m \rfloor - 1 \ .$$

Therefore for the lemma we need to count the the number of m satisfying the inequalities

$$0 < \lfloor \phi^2 m \rfloor - 1 < k \ .$$

Using the properties of the floor function, this is equal to the number of positive integers m satisfying

$$\frac{2}{\phi^2} < m < \frac{k+1}{\phi^2} \ . \tag{3.2}$$

Since $\frac{2}{\phi^2} = 0.7639\ldots$, and the upper bound in (3.2) is not an integer, these m are exactly

$$1, 2, \ldots, \left\lfloor \frac{k+1}{\phi^2} \right\rfloor \ .$$

□

For the rest of the paper for a given positive integer k we will always assume that

$$\ell = \ell(k) = k - \left\lfloor \frac{k+1}{\phi^2} \right\rfloor . \tag{3.3}$$

Next we consider a decomposition for Λ_n^k that will be useful in our calculations.

Theorem 3.2. *Let ℓ be as in (3.3). For $n \geq n_0$ the k-Lucas cube Λ_n^k has the decomposition*

$$\Lambda_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-3}^{\ell} 0$$

in which there are ℓ link edges between $10\Gamma_{n-3}^{\ell}0$ and its copy $00\Gamma_{n-3}^{\ell}0 \subset 0\Gamma_{n-1}^k$.

Proof. From the fundamental decomposition (2.4) of k-Fibonacci cubes, we know that $\Gamma_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-2}^k$ with k link edges between the vertices with labels $0, 1, \ldots, k-1$ in $0\Gamma_{n-1}^k$ and the corresponding vertices with labels $f_{n+1}, f_{n+1}+1, \ldots, f_{n+1}+k-1$ in $10\Gamma_{n-2}^k$. Now we consider the effect of eliminating all vertices in Γ_n^k which start and end with 1. This elimination has no effect on $0\Gamma_{n-1}^k$, so all of these vertices are also in Λ_n^k. For $10\Gamma_{n-2}^k$, we need to consider which vertices survive in this subgraph itself, how does the elimination change this graph, and in addition the effect of this elimination on the original k link edges. Any link edge of the original Γ_n^k which has an end vertex in $10\Gamma_{n-2}^k$ which has been eliminated, is no longer a link edge in Λ_n^k. From Lemma 3.1, we know that the number of the first k vertices in $10\Gamma_{n-2}^k$ that end with 1 is given by (3.1) since these vertices corresponds to the first k vertices in Γ_{n-2}^k that end with 1. Therefore only ℓ of the original link edges survive in Λ_n^k.

f_{n-1} of the vertices in $10\Gamma_{n-2}^k$ end with 0 and f_{n-2} of them end with 1. For Λ_n^k the f_{n-2} ending with 1 are removed. Now $10\Gamma_{n-2}^k \subseteq 10\Gamma_{n-2} = 10\Gamma_{n-3}0 + 10\Gamma_{n-4}01$. Therefore, after removing the vertices ending with 1 from $10\Gamma_{n-2}^k$, this has the effect of reducing the number of the previous link edges that appear in the construction of $10\Gamma^k_{n-2}$ itself to ℓ. In other words, the resulting graph is $10\Gamma^{\ell}_{n-3}0 \subseteq 10\Gamma_{n-3}0$. This completes the proof. $\square$

Example 3.3. Consider Λ^2_6 obtained from Γ^2_6. We have the decomposition of Γ^2_6 as

$$\Gamma^2_6 = 0\Gamma^2_5 + 10\Gamma^2_4 \ .$$

The link edges in Γ^2_6 are between the vertices labeled $000000, 000001$ in $0\Gamma^2_5$, and 100000, 100001 respectively in $10\Gamma^2_4$. Of these two link edges, the second one is eliminated because the vertex 100001 is not in Λ^2_6. We note that the vertices labeled $100001, 100101, 101001$ are eliminated from $10\Gamma^2_4$ in the construction of Λ^2_6. In this case $\ell = 1$ and the subgraph of $10\Gamma^2_4$ obtained after the elimination of these vertices is isomorphic to Γ^1_3, which gives $\Lambda^2_6 = 0\Gamma^2_5 + 10\Gamma^1_3 0$.

In Γ^k_n we have k link edges between the vertices with labels $0, 1, \ldots, k-1$ and the corresponding vertices with labels $f_{n+1}, f_{n+1}+1, \ldots, f_{n+1}+k-1$ for $n \geq n_0$. Similar to the decomposition (2.2) if we consider the vertices ending with 0 and 01 in Γ^k_n we see that ℓ of the vertices with labels $0, 1, \ldots, k-1$ ($f_{n+1}, f_{n+1}+1, \ldots, f_{n+1}+k-1$) end with 0 and $k-\ell$ of them end with 01. Then by modifying the proof of Theorem 3.2, we obtain the following decomposition of Γ^k_n which we state here for the record.

Proposition 3.4. *k-Fibonacci cube Γ^k_n has the decomposition*

$$\Gamma^k_n = \Gamma^{\ell}_{n-1}0 + \Gamma^{k-\ell}_{n-2}01$$

where ℓ is as in (3.3), Γ^0_{n-2} is the graph with f_n vertices and no edges and there is a matching between $\Gamma^{k-\ell}_{n-2}01$ and $\Gamma^{k-\ell}_{n-2}00 \subset \Gamma^{\ell}_{n-1}0$.

4. Basic properties of k-Lucas cubes Λ^k_n

By definition of Λ^k_n we know that $|V(\Lambda^k_n)| = |V(\Lambda_n)| = L_n$. Next we consider basic graph theoretical parameters associated with k-Lucas cubes.

4.1. The number of edges

Let $m(G) = |E(G)|$ denote the number of edges of G. It is shown in [13] that $m(\Lambda_n) = nf_{n-1}$ for $n \geq 1$. Since $m(\Lambda^k_n) = m(\Lambda_n)$ for $n < n_0$, we have $m(\Lambda^k_n) = nf_{n-1}$ for $1 \leq n < n_0$.

From Theorem 3.2 we observe that $m(\Lambda^k_n)$ satisfies

$$m(\Lambda^k_n) = m(\Gamma^k_{n-1}) + m(\Gamma^{\ell}_{n-3}) + \min\{\ell, f_{n-1}\} \ , \tag{4.1}$$

and for $n \geq n_0$, (4.1) reduces to

$$m(\Lambda^k_n) = m(\Gamma^k_{n-1}) + m(\Gamma^{\ell}_{n-3}) + \ell. \tag{4.2}$$

Here we need the number of edges of Γ^k_n which is obtained in [4] for $n < n_0$ and in [2] for $n \geq n_0$ as follows.

Lemma 4.1 ([2,4])**.** *The number of edges of Γ^k_n is given by*

$$m(\Gamma^k_n) = \begin{cases} \frac{1}{5}(2(n+1)f_n + nf_{n+1}) & \text{for } n < n_0 \\ \frac{1}{5}\Big(n_0 f_{n_0-1}L_{t+1} + (n_0-1)f_{n_0}L_{t+2}\Big) + (f_{t+3}-1)k & \text{for } n \geq n_0 \end{cases}$$

where $t = n - n_0$.

We need the following useful result that relates the parameters $n_0(\ell)$ and $n_0(k)$.

Lemma 4.2. *Suppose n_0 and ℓ are as defined in* (2.5) *and* (3.3)*. Then $n_0(k) - n_0(\ell) \leq 1$. Moreover, $n_0(\ell) = n_0(k)$ if and only if $k = f_{2p+1} - 1$ for some positive integer p.*

Proof. We know that $n_0(k)$ is the smallest integer n for which $\Gamma_n^k \neq \Gamma_n$, i.e., the smallest integer n for which $f_{n_0(k)} > k$. Using (2.4) with $n = n_0 + 1$ we can write

$$\begin{aligned} \Gamma_{n_0+1}^k &= 0\Gamma_{n_0}^k + 10\Gamma_{n_0-1}^k & (4.3) \\ &= 0\Gamma_{n_0}^k + 10\Gamma_{n_0-1} & (4.4) \end{aligned}$$

Then substituting $n = n_0 + 1$ in Theorem 3.2 we obtain that

$$\Lambda_{n_0+1}^k = 0\Gamma_{n_0}^k + 10\Gamma_{n_0-2}^\ell 0\ . \tag{4.5}$$

By using the decomposition (4.4) in the proof of the Theorem 3.2 we have

$$\Lambda_{n_0+1}^k = 0\Gamma_{n_0}^k + 10\Gamma_{n_0-2} 0\ . \tag{4.6}$$

The right hand sides of (4.5) and (4.6) give $10\Gamma_{n_0-2}^\ell 0 = 10\Gamma_{n_0-2}0$ which means that $n_0(\ell) > n_0 - 2$. Hence we have $n_0(k) - n_0(\ell) \leq 1$.

Similarly we can write

$$\Gamma_{n_0+2}^k = 0\Gamma_{n_0+1}^k + 10\Gamma_{n_0}^k \quad \text{and} \quad \Lambda_{n_0+2}^k = 0\Gamma_{n_0+1}^k + 10\Gamma_{n_0-1}^\ell 0\ .$$

Now if we have $10\Gamma_{n_0-1}^\ell 0 = 10\Gamma_{n_0-1}0$ then it follows that $n_0(\ell) = n_0(k)$. We know that $10\Gamma_{n_0-1}^\ell 0 \subset 10\Gamma_{n_0}^k \neq 10\Gamma_{n_0}$ and at least one of the link edge in $10\Gamma_{n_0}^k$ eliminated during the construction. But if $k = f_{2p+1} - 1$ for some positive integer p then using the classical summation formula for Fibonacci numbers we know that k has odd Zeckendorf expansion and $n_0(k) = 2p + 1$. Therefore, there is only one eliminated link edge in $10\Gamma_{n_0}^k$, namely the link edge between the vertices with labels $f_{n_0} - 1$ and $f_{n_0+2} - 1$. This gives that $10\Gamma_{n_0}^k \neq 10\Gamma_{n_0}$ but $10\Gamma_{n_0-1}^\ell = 10\Gamma_{n_0-1}$, that is $n_0(\ell) = n_0(k)$. If $k \neq f_{2p+1} - 1$ for any positive integer p then we have $10\Gamma_{n_0-1}^\ell \neq 10\Gamma_{n_0-1}$ since at least one of the link edge in $10\Gamma_{n_0-1}^\ell$ between the vertices having even Zeckendorf expansions $f_{n_0} - 1$ and $f_{n_0+2} - 1$, or $f_{n_0} - 2$ and $f_{n_0+2} - 2$ has been eliminated, that is, $n_0(\ell) = n_0 - 1$. □

Now we are ready to present the number of edges $m(\Lambda_n^k)$ of Λ_n^k.

Proposition 4.3. *For $n \geq n_0 = n_0(k)$ the number of edges $m(\Lambda_n^k)$ of Λ_n^k is given by*

- $m(\Lambda_n^k) = (n_0 - 1)f_{n_0-1} + \ell$ *if* $n = n_0$
- $m(\Lambda_n^k) = \frac{1}{5}\Big(n_0 f_{n_0-1} L_t + (n_0 - 1) f_{n_0} L_{t+1} + (n - 3)L_{n-2} + 2f_{n-3}\Big) + (f_{t+2} - 1)k + \ell$ *if* $n_0 + 1 \leq n \leq n_0(l) + 2$
- $m(\Lambda_n^k) = \frac{1}{5}\Big(n_0 f_{n_0-1} L_t + (n_0 - 1) f_{n_0} L_{t+1}\Big) + (f_{t+2} - 1)k + \frac{1}{5}\Big(n_0(l) f_{n_0(l)-1} L_{t_\ell-2} + (n_0(l) - 1) f_{n_0(l)} L_{t_\ell-1}\Big) + f_{t_\ell}\ell$ *if* $n \geq n_0(l) + 3$

where $t = n - n_0$ and $t_\ell = n - n_0(\ell)$.

Proof. For a fixed k, using (4.2) we know that

$$m(\Lambda_n^k) = m(\Gamma_{n-1}^k) + m(\Gamma_{n-3}^\ell) + \ell$$

where $n \geq n_0(k)$. Therefore we need to find the cardinalities of edge sets of Γ_{n-1}^k and Γ_{n-3}^ℓ depending on the values of n, $n_0(k)$ and $n_0(\ell)$. Then using Lemma 4.1, Lemma 4.2 and the classical identity $L_n = f_{n+1} + f_{n-1}$ we obtain the desired result. □

4.2. The average degree of a vertex

In [10] the limit average degree of the Fibonacci and Lucas cubes are computed as

$$\lim_{n\to\infty} \frac{2m(\Gamma_n)}{nf_{n+2}} = \lim_{n\to\infty} \frac{2m(\Lambda_n)}{nL_n} = 1 - \frac{1}{\sqrt{5}}$$

which means that the average degree of a vertex in Γ_n and Λ_n is asymptotically given by

$$\left(1 - \frac{1}{\sqrt{5}}\right) n \ . \tag{4.7}$$

The analogous problem for the k-Fibonacci cubes Γ_n^k for a fixed k was considered in [2] where it was proved that the limit average degree of a vertex in Γ_n^k is independent of n. Denoting this limit average degree by $\overline{d_k}$, it is shown in [2] that

$$\overline{d_k} \approx 1.047 + 0.553 \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2}\right) \tag{4.8}$$

where ϕ is the golden ratio. For the limit average degree of k-Lucas cubes we obtain the following result.

Proposition 4.4. *For a fixed k the average degree of a vertex in Λ_n^k is asymptotically given by*

$$1.047 + 0.4 \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2}\right) + 0.153 \log_\phi \left(\sqrt{5}\ell + \sqrt{5} - \frac{1}{2}\right)$$

where ϕ is the golden ratio and ℓ is as in (3.3).

Proof. By the properties of the Fibonacci and Lucas numbers we have

$$\lim_{n\to\infty} \frac{f_{n+1}}{L_n} = \frac{\phi}{\sqrt{5}} \ , \qquad \lim_{n\to\infty} \frac{f_{n-1}}{L_n} = \frac{\phi^{-1}}{\sqrt{5}} \ . \tag{4.9}$$

For a fixed k, using (4.2), (4.8) and (4.9), the average degree of a vertex in Λ_n^k is computed as

$$\begin{aligned}
\lim_{n\to\infty} \frac{2m(\Lambda_n^k)}{L_n} &= \lim_{n\to\infty} \frac{2\left(m(\Gamma_{n-1}^k) + m(\Gamma_{n-3}^\ell) + \ell\right)}{L_n} \\
&= \lim_{n\to\infty} \frac{2m(\Gamma_{n-1}^k)}{f_{n+1}} \cdot \frac{f_{n+1}}{L_n} + \lim_{n\to\infty} \frac{2m(\Gamma_{n-3}^\ell)}{f_{n-1}} \cdot \frac{f_{n-1}}{L_n} \\
&= \overline{d_k} \cdot \frac{\phi}{\sqrt{5}} + \overline{d_\ell} \cdot \frac{\phi^{-1}}{\sqrt{5}}.
\end{aligned}$$

Using the expressions for $\overline{d_k}$ and $\overline{d_\ell}$ from (4.8) and simplifying with Mathematica gives the desired result. □

Remark 4.5. We note that ℓ is a function of k and using the explicit expression in (3.3), for large k we obtain the asymptotic value for the average degree in Λ_n^k as

$$\left(1 - \frac{1}{\sqrt{5}}\right) \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2}\right).$$

This is the main term that appears in (4.8). The factor $1 - \frac{1}{\sqrt{5}}$ is also the coefficient of the limiting values for the Fibonacci and Lucas cubes as given in (4.7).

Remark 4.6. Similar to the degree polynomial of k-Fibonacci cubes given in [2, Section 4.3], we can consider the degree polynomial $D_{\Lambda_n^k}(x)$ of k-Lucas cubes. For small values of n this polynomial can be obtained directly from the definition and figures by inspection. For example, for $k = 1$, we obtain the following from Figure 2.

$$D_{\Lambda_1^1}(x) = 1, \quad D_{\Lambda_2^1}(x) = x^2 + 2x, \quad D_{\Lambda_3^1}(x) = x^3 + 3x, \quad D_{\Lambda_4^1}(x) = x^4 + 2x^2 + 4x \ .$$

For larger values of n and k it becomes more complicated to calculate the degree polynomial for Λ_n^k since we need the degree information of the vertices in the link edges. For the special cases of $k \in \{1,2\}$ we can obtain $D_{\Lambda_n^k}(x)$ using Theorem 3.2 and the results in [2, Corollary 6]. Using Theorem 3.2 for $n \geq 5$ we know that

$$\Lambda_n^1 = 0\Gamma_{n-1}^1 + 10\Gamma_{n-3}^1 0 \quad \text{and} \quad \Lambda_n^2 = 0\Gamma_{n-1}^2 + 10\Gamma_{n-3}^1 0$$

where there are only 1 link edge in these decompositions. That is, for any $k \in \{1,2\}$ the degrees of all the vertices of $0\Gamma_{n-1}^k$ and $10\Gamma_{n-3}^1 0$ remains the same in Λ_n^k except the vertices having labels 0^n and 10^{n-1} whose degrees increase by one due to the link edge. Hence, using [2, Corollary 6] we have the explicit formulas

$$\begin{aligned} D_{\Lambda_n^1}(x) &= x^n + 2x^{n-2} + x^{n-3} + L_{n-1}x + \sum_{i=2}^{n-4} L_{n-i-2}x^i \ , \text{ for } n \geq 5 \\ D_{\Lambda_n^2}(x) &= x^n + 3x^{n-2} + 2x^{n-3} + 2f_{n-2}x + 2\sum_{i=2}^{n-4} f_{n-i-1}x^i \ , \text{ for } n \geq 6. \end{aligned}$$

4.3. Number of induced hypercubes

Let $Q_d(G)$ denote the number of d-dimensional hypercubes induced in G and $C(G,x)$ be the cube polynomial [1] of G defined as

$$C(G,x) = \sum_{d \geq 0} Q_d(G)x^d \ .$$

This polynomial is considered for Fibonacci and Lucas cubes in [9]. For k-Fibonacci cubes, it is shown in [2] that $Q_d(\Gamma_n^k)$ satisfies the recursion

$$Q_d(\Gamma_n^k) = Q_d(\Gamma_{n-1}^k) + Q_d(\Gamma_{n-2}^k) + P_{d-1}(k-1)$$

where

$$P_{d-1}(k-1) = \sum_{i=0}^{k-1} \binom{Z(i)}{d-1},$$

and $Z(i)$ denotes the number of 1's in the Zeckendorf representation of i.

Note that for $n \leq n_0$ we know that $\Lambda_n^k = \Lambda_n$ and the cube polynomial of Λ_n which has degree $\lfloor \frac{n}{2} \rfloor$ is obtained in [9]. Therefore for all values of k we have

$$C(\Lambda_1^k, x) = 1 \ , \qquad C(\Lambda_2^k, x) = 2x + 3 \ , \qquad C(\Lambda_3^k, x) = 3x + 4 \ .$$

In Table 1 we list cube polynomials of Λ_n^k for $1 \leq k \leq 4$ and $n \in \{4,5,6\}$.

Table 1. Cube polynomials $C(\Lambda_n^k, x)$ of Λ_n^k for $1 \leq k \leq 4$ and $n \in \{4,5,6\}$.

$n\backslash k$	1	2	3	4
4	$6x+7$	x^2+7x+7	$2x^2+8x+7$	$2x^2+8x+7$
5	$10x+11$	$2x^2+12x+11$	$4x^2+14x+11$	$5x^2+15x+11$
6	$17x+18$	$4x^2+21x+18$	$8x^2+25x+18$	$10x^2+27x+18$

To find the number of d-dimensional hypercubes induced in Λ_n^k, we use an argument similar to the one in [2] for the number of d-dimensional hypercubes in Γ_n^k. From Theorem 3.2 we know that $\Lambda_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-3}^\ell 0$. Therefore, there are three types of d-dimensional hypercubes that contribute to $Q_d(\Lambda_n^k)$: those coming from $0\Gamma_{n-1}^k$, those coming from $10\Gamma_{n-3}^\ell 0$, and those that involve the ℓ link edges used in the construction of Λ_n^k. It is enough to consider the d-dimensional hypercubes of the last type. These can be counted by the number of $(d-1)$-dimensional hypercubes contained in the subgraph of $10\Gamma_{n-3}^\ell 0$ induced by the ℓ vertices with labels in $\{0,1,\ldots,k-1\}$ having even Zeckendorf expansions, that is, whose representations that end with 0. For any of these vertices i again we need to select $d-1$ ones among the $Z(i)$ ones in i. Then by varying these $d-1$ ones we obtain 2^{d-1} vertices with labels in $\{0,1,\ldots,k-1\}$ having even Zeckendorf expansions themselves. Each one of these gives a $(d-1)$-dimensional hypercube in $10\Gamma^{\ell}_{n-3}0$. All of these $(d-1)$-dimensional hypercubes also have a copy in $0\Gamma^{k}_{n-1}$ and there is a matching between the two hypercubes due to the ℓ link edges. This produces a d-dimensional hypercube in Λ^k_n that involves the link edges. We have the following result:

Proposition 4.7. *Let $Q_d(\Lambda^k_n)$ and $Q_d(\Gamma^k_n)$ denote the number of d-dimensional hypercubes in Λ^k_n and Γ^k_n respectively. Then for $n \geq n_0$ and $d \leq \lfloor \frac{n_0}{2} \rfloor$ we have*

$$Q_d(\Lambda^k_n) = Q_d(\Gamma^k_{n-1}) + Q_d(\Gamma^{\ell}_{n-3}) + P_{d-1}(\ell - 1)$$

and $Q_d(\Lambda^k_n) = 0$ for $d > \lfloor \frac{n_0}{2} \rfloor$.

Proof. The bulk of the proof of the proposition has been given above, showing

$$Q_d(\Lambda^k_n) = Q_d(\Gamma^k_{n-1}) + Q_d(\Gamma^{\ell}_{n-3}) + \sum_{i\in S} \binom{Z(i)}{d-1},$$

where S is the ℓ integers in $\{0,1,\ldots,k-1\}$ having even Zeckendorf expansions. To show that

$$\sum_{i\in S} \binom{Z(i)}{d-1} = \sum_{i=0}^{\ell-1} \binom{Z(i)}{d-1} = P_{d-1}(\ell-1) \tag{4.10}$$

we argue as follows. The Zeckendorf expansions of the numbers $\{0,1,\ldots k-1\}$ can be partitioned into the disjoint union of two sets of expansions of the form $(A,0)$ and $(B,0,1)$ where A is the Zeckendorf expansion of the numbers $\{0,1,\ldots,\ell-1\}$ and B is the Zeckendorf expansion of the numbers $\{0,1,\ldots,\lfloor \frac{k+1}{\phi^2} \rfloor - 1\}$. Since the number of ones of the even Zeckendorf numbers in $\{0,1,\ldots,k-1\}$ does not change when we drop the last 0, the sums in (4.10) are identical.

By the definition of n_0 we know that $f_{n_0-1} \leq k < f_{n_0}$ and $\Gamma^k_n \neq \Gamma_n$ for $n \geq n_0$. Then using Theorem 3.2 we can say that Λ^k_n has a subgraph isomorphic to $\Gamma^k_{n_0-1} = \Gamma_{n_0-1}$ but it doesn't contain any subgraph isomorphic $\Gamma_{n_0} \neq \Gamma^k_{n_0}$. We know that the degree of the cube polynomial of Γ_n is $\left\lfloor \frac{n+1}{2} \right\rfloor$ [9]. Therefore we can say for $n \geq n_0$ that $Q_d(\Lambda^k_n) = 0$ for $d > \left\lfloor \frac{n_0-1+1}{2} \right\rfloor = \lfloor \frac{n_0}{2} \rfloor$ and it is nonzero otherwise since $\Gamma_{n_0-1} \subset \Lambda^k_n$. □

4.4. Diameter and radius

The k-Fibonacci cubes Γ^k_n has the nested structure

$$\Gamma^1_n \subseteq \cdots \subseteq \Gamma^k_n \subseteq \cdots \subseteq \Gamma_n.$$

as shown in [2]. Since we define Λ^k_n by removing certain vertices in Γ^k_n, one can easily observe that k-Lucas cubes have a similar nested structure,

$$\Lambda^1_n \subseteq \cdots \subseteq \Lambda^k_n \subseteq \cdots \subseteq \Lambda_n. \tag{4.11}$$

We know that Λ^1_n is a tree with root 0^n (the vertex with integer label 0). It follows that for $u,v \in V(\Lambda^1_n)$

$$d(u,v) \leq d(u,0^n) + d(v,0^n) = w_H(u) + w_H(v), \tag{4.12}$$

where w_H denotes the Hamming weight. We always have

$$w_H(u) + w_H(v) \leq \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd} \end{cases}$$

for the vertices of Λ_n and it is shown in [13] that

$$diam(\Lambda_n) = \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd.} \end{cases}$$

Λ_n^k is a subgraph of Λ_n with the same vertex set and fewer edges for $n \geq n_0$. This directly gives the inequality $diam(\Lambda_n^k) \geq diam(\Lambda_n)$. On the other hand, using (4.11) and (4.12), for any $u, v \in V(\Lambda_n^k)$ we have

$$d(u,v) \leq w_H(u) + w_H(v) \leq \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd,} \end{cases}$$

which gives $diam(\Lambda_n^k) \leq diam(\Lambda_n)$. Therefore, for all $n \geq 1$

$$diam(\Lambda_n^k) = diam(\Lambda_n) = \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd.} \end{cases}$$

By a similar argument we see that the radius of Λ_n^k is equal to the radius of Λ_n. Since the latter radius was obtained in [13] as $\lfloor \frac{n}{2} \rfloor$, this is also the radius of Λ_n^k.

Acknowledgment. We would like to thank the reviewer for his/her numerous useful comments and suggestions, as well as for pointing out an error in one of our lemmas. The reviewer's careful reading of the manuscript greatly improved the content and the readability of this paper. The work of the second author is supported by TÜBİTAK under grant no. 117R032.

References

[1] B. Brešar, S. Klavžar and R. Škrekovski, *The cube polynomial and its derivatives: the case of median graphs*, Electron. J. Combin. **10**, #R3, 2003.

[2] Ö. Eğecioğlu, E. Saygı and Z. Saygı, *k-Fibonacci cubes: A family of subgraphs of Fibonacci cubes*, Int. J. Found. Comput. Sci. **31** (5), 639–661, 2020.

[3] S. Gravier, M. Mollard, S. Špacapan and S.S. Zemljič, *On disjoint hypercubes in Fibonacci cubes*, Discrete Appl. Math. **190-191**, 50–55, 2015.

[4] W.-J. Hsu, *Fibonacci cubes–a new interconnection technology*, IEEE Trans. Parallel Distrib. Syst. **4** (1), 3–12, 1993.

[5] A. Ilić, S. Klavžar and Y. Rho, *Generalized Fibonacci cubes*, Discrete Math. **312**, 2–11, 2012.

[6] A. Ilić, S. Klavžar and Y. Rho, *Generalized Lucas cubes*, Appl. Anal. Discrete Math. **6** (1), 82–94, 2012.

[7] C. Kimberling, *The Zeckendorf array equals the Wythoff array*, The Fibonacci Quarterly **33**, 3–8, 1995.

[8] S. Klavžar, *Structure of Fibonacci cubes: a survey*, J. Comb. Optim. **25**, 505–522, 2013.

[9] S. Klavžar and M. Mollard, *Cube polynomial of Fibonacci and Lucas cube*, Acta Appl. Math. **117**, 93–105, 2012.

[10] S. Klavžar and M. Mollard, *Asymptotic properties of Fibonacci cubes and Lucas cubes*, Ann. Comb. **18**, 447–457, 2014.

[11] M. Mollard, *Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes*, Discrete Appl. Math. **219**, 219–221, 2017.

[12] E. Munarini, *Pell graphs*, Discrete Math. **342** (8), 2415–2428, 2019.

[13] E. Munarini, C.P. Cippo and N. Zagaglia Salvi, *On the Lucas cubes*, Fibonacci Quart. **39**, 12–21, 2001.

[14] E. Saygı and Ö. Eğecioğlu, *Counting disjoint hypercubes in Fibonacci cubes*, Discrete Appl. Math. **215**, 231–237, 2016.

[15] E. Saygı and Ö. Eğecioğlu, *q-cube enumerator polynomial of Fibonacci cubes*, Discrete Appl. Math. **226**, 127–137, 2017.

[16] E. Saygı and Ö. Eğecioğlu, *q-counting hypercubes in Lucas cubes*, Turk. J. Math. **42**, 190–203, 2018.

[17] X. Wang, X. Zhao and H. Yao, *Structure and enumeration results of matchable Lucas cubes*, Discrete Appl. Math. **277**, 263–279, 2020.

Appendix A. Figures of some k-Lucas cubes

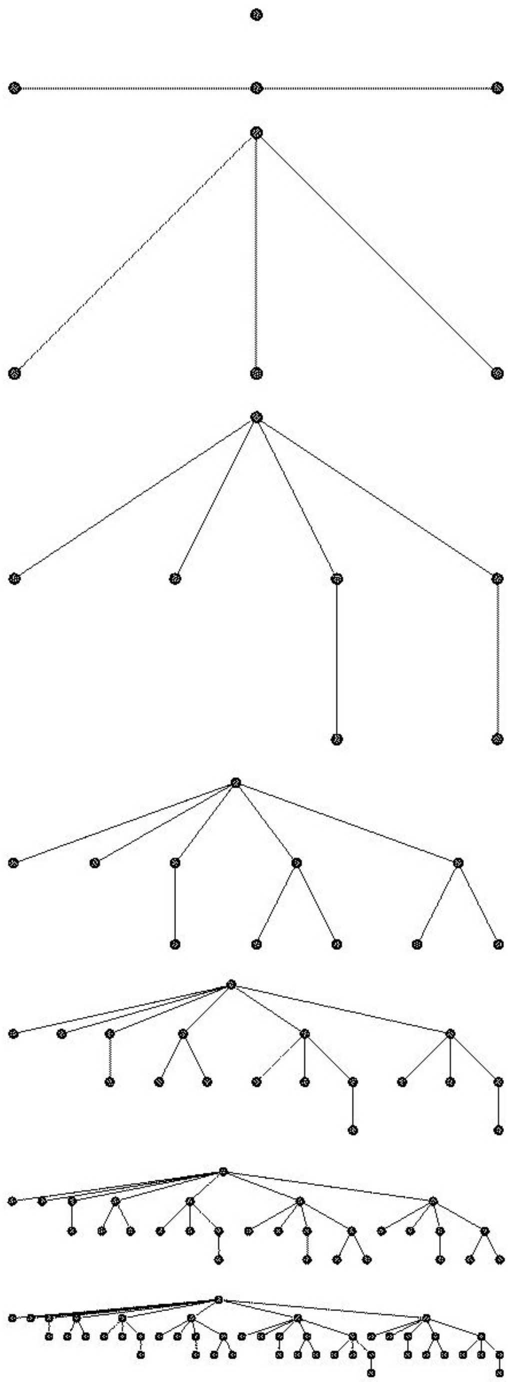

Figure 4. The first eight k-Lucas cubes for $k = 1$.

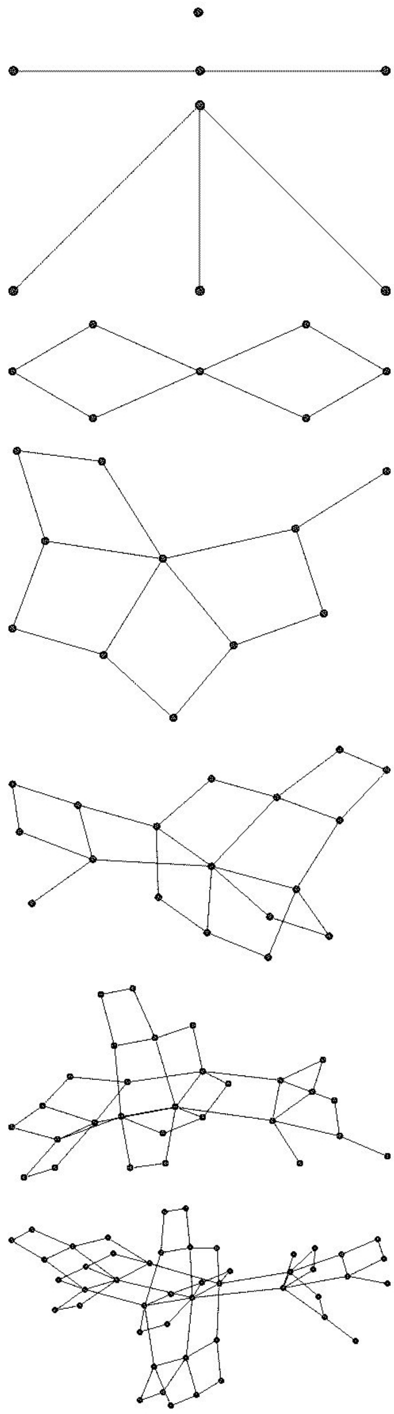

Figure 5. The first eight k-Lucas cubes for $k = 3$.

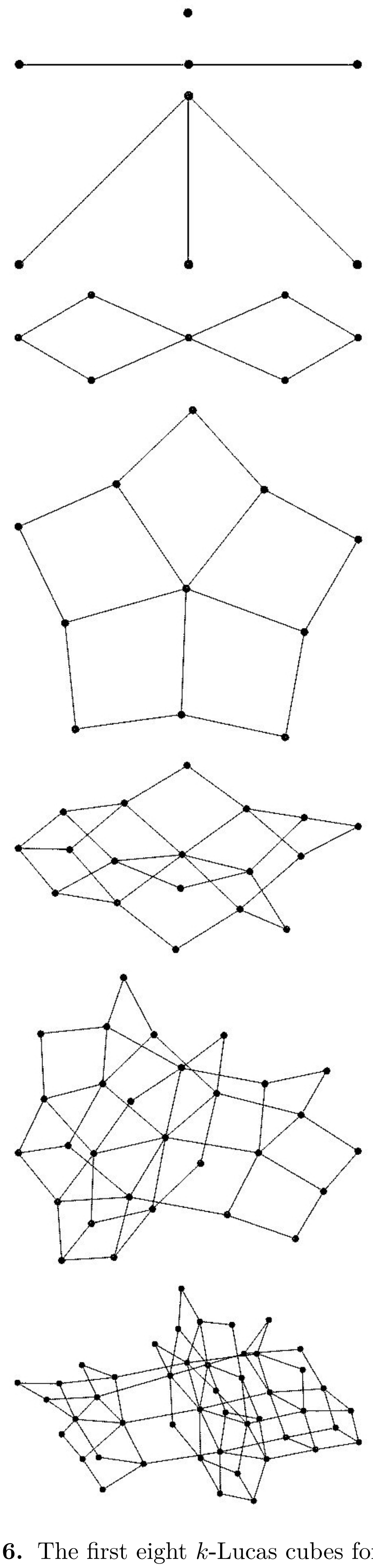

Figure 6. The first eight k-Lucas cubes for $k = 6$.

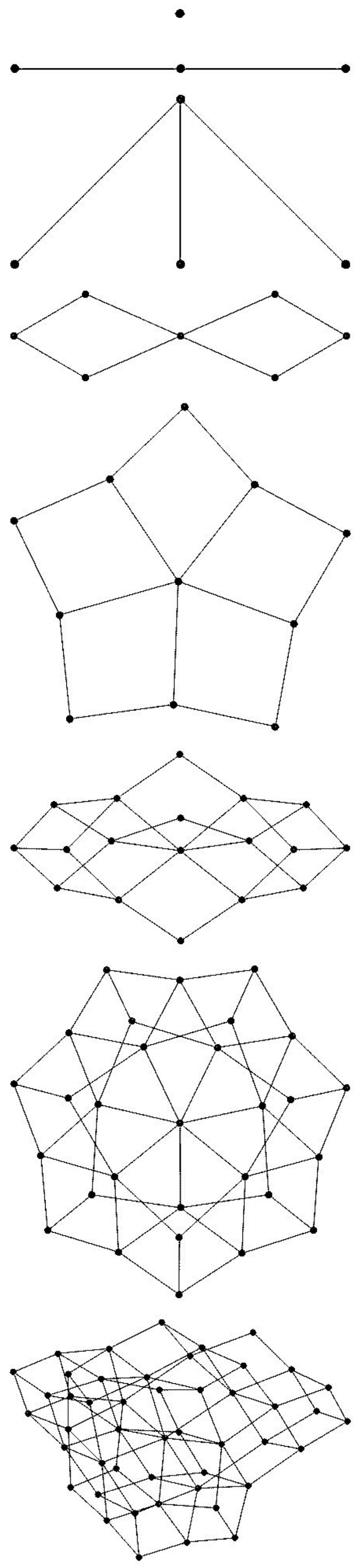

Figure 7. The first eight k-Lucas cubes for $k = 12$.