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A GRAPH ASSOCIATED TO A COMMUTATIVE SEMIRING

Shahabaddin Ebrahimi ATANI, Mehdi KHORAMDEL, Saboura Dolati Pish HESARI,
and Mahsa Nikmard Rostam ALIPOUR
Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, IRAN

Abstract. Let R be a commutative semiring with nonzero identity and H be an arbitrary multiplicatively closed subset R. The generalized identity-summand graph of R is the (simple) graph $G_H(R)$ with all elements of R as the vertices, and two distinct vertices x and y are adjacent if and only if $x+y \in H$. In this paper, we study some basic properties of $G_H(R)$. Moreover, we characterize the planarity, chromatic number, clique number and independence number of $G_H(R)$.

1. Introduction

Semirings provide useful instruments to solve problems in many areas of information sciences and applied mathematics such as optimization theory, graph theory, automata theory, coding theory and analysis of computer programs, because the structure of semiring provides a useful algebraic technique for investigating and modelling the key factors in these problems.

Over the last few years, the study of algebraic structures by graphs has been done and several interesting results have been obtained (see [1,2,4,5,10,11,13–17]). For instance, the total graph of a commutative ring R is a simple graph whose vertex set is R, and two distinct vertices a and b are adjacent if $a+b$ is a zero divisor of R (the set of all zero-divisor elements of R is denoted by $Z(R)$)(see [3,18]). Recently, in [9], the authors considered the *identity summand graph* of a commutative semiring R denoted by $\Gamma(R)$, as the simple graph with the set of vertices $\{x \in R \setminus \{1\} : x + y = 1 \text{ for some } y \in R \setminus \{1\}\}$, where two distinct vertices x and y are adjacent if and only if $x + y = 1$. Moreover, the identity-summand graph with respect to co-ideal I denoted by $\Gamma_I(R)$ is a graph with vertices as elements

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ebrahimi@guilan.ac.ir; mehdikhoramdel@gmail.com-Corresponding author;
Saboura_dolati@yahoo.com; mhs.nikmard@gmail.com
0000-0003-0568-9452; 0000-0003-0663-0356; 0000-0001-8830-636X; 0000-0003-3264-7936.

$S_I(R) = \{x \in R \setminus I : x + y \in I \text{ for some } y \in R \setminus I\}$, where two distinct vertices x and y are adjacent if and only if $x + y \in I$ [12].

Let H be a nonempty subset of a semiring R with nonzero identity. H is said to be *multiplicatively closed* if $xy \in H$, for all x and y of H. Also, a subset H of R is called saturated if $xy \in H$ if and only if $x, y \in H$. For a multiplicatively closed subset H of R, we define the *generalized identity-summand graph* of R, denoted by $G_H(R)$, as a simple graph, with vertex set R and two distinct vertices x and y being adjacent if and only if $x + y \in H$. Since the subsets $Z(R)$ of R is multiplicatively closed, $G_H(R)$ is a natural generalization of the total graph of R. Hence the total graph is a well-known graph of this type. Moreover, if H is a co-ideal of R, then $\Gamma_H(R)$ is a subgraph of $G_H(R)$.

We summarize the contents of this article as follows. In Section 2, we investigate the basic properties of generalized identity-summand graph, for instance , the degree of the vertices and connectivity. Also, We consider the possible integers for the diameter and the girth of the graph $G_H(R)$. We investigate the case that H is a saturated multiplicatively closed subset of R. We prove a subset H of R is saturated if and only if $R \setminus H$ is a union of some prime ideals. Therefore $R \setminus H = \bigcup_{j \in J} M_j$ for some prime ideals M_j with $j \in J$. Set $I := \bigcap_{j \in J} M_j$. If I is a Q-ideal of R, then set $\widetilde{H} := \{q + I \ : \ h \in q + I \text{ for some } h \in H\}$. We show that the newly constructed subset $\widetilde{H}$ is a saturated multiplicatively closed subset of R/I and study the relationship between the combinatorial properties of the graphs $G_H(R)$ and $G_{\widetilde{H}}(R/I)$. Further, we consider the graph $G_H(R)$, when it is complete, complete r-partite, complete 2-partite and regular graph. It is proved that $G_H(R)$ is complete 2-partite if and only if it is star graph. In Section 3, we consider and study the planar property, clique number, chromatic number and independence number of $G_H(R)$. We will show that $\omega(G_H(R)) = \chi(G_H(R))$ and completely determine the chromatic number, clique number and independence number of $G_H(R)$.

Now, we are going to recall some notations and definitions of graph theory from [6], which are needed in this paper. Let G be a graph. By $E(G)$ and $V(G)$ we will denote the set of all *edges* and *vertices*, respectively. A graph G is called *connected* provided that there exists a path between any two distinct vertices. Otherwise, G is said to be *disconnected*. The *distance* between two distinct vertices a and b is the length of the shortest path connecting them, denoted by $d(a, b)$, (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The *diameter* of a graph G, denoted by $diam(G)$, is equal to $sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of a graph G denoted $gr(G)$, is the length of a shortest cycle in G, provided that G contains a cycle; otherwise $gr(G) = \infty$. For a given vertex $x \in V(G)$, the *neighborhood set* of x is the set $N(x) = \{a \in V(G) : a \text{ is adjacent to } x\}$. A graph G is called *complete*, if every pair of distinct vertices is connected by a unique edge. The notation K_n will denote the complete graph on n vertices. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. A complete r-partite graph with part sizes $m_1, ..., m_r$ is denoted by $K_{m_1,m_2,...,m_r}$. We will sometimes call $K_{1,n}$ a *star graph*. Let G be a graph. A *coloring* of a graph G is an assignment one color to each vertex of G such that distinct colors are assigned to adjacent vertices. If one used n colors for the coloring of G, then it is referred to as an *n-coloring*. If G has n-coloring, then G is called *n-colorable*. The minimum positive integer n for which a graph G is n-colorable is called the *chromatic number* of G, and is denoted by $\chi(G)$. A graph G is said to be *totally disconnected*, if no two vertices of G are adjacent. Every complete subgraph of a graph G is called a *clique* of G, and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. In a graph $G = (V, E)$, a subset S of V is said to be an *independent set* provided that the subgraph induced by S is totally disconnected. The *independence number* is the maximum size of an independent set in G and denoted by $\alpha(G)$. A graph G is called a *null graph* if whose vertex-set is empty and a graph whose edge-set is empty is said to be an *empty graph*. Let G be a graph with edge set E. Also, suppose that there exists a family of edge-disjoint subgraphs $\{G_i\}_{i\in I}$ of G. Then we put $G = \oplus_{i\in I} G_i$. Furthermore, in the case that $G_i \cong H$ for every $i \in I$, we set $G = \oplus_{|I|} H$.

An algebraic system $(R, +, .)$ is called a commutative *semiring* provided that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exist $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. Throughout this paper, all semirings considered will be assumed to be commutative semirings with a non-zero identity. Let R be a semiring. A non-empty subset I of R is called *co-ideal* (resp. *ideal*), if it is closed under multiplication (rep. under addition) and satisfies the condition $r + a \in I$ (resp. $ra \in I$) for all $a \in I$ and $r \in R$ (so $0 \in I$ (resp. $1 \in I$) if and only if $I = R$). A co-ideal I of a semiring R is said to be a *strong co-ideal*, if $1 \in I$. A co-ideal (resp. ideal) I of R is called *k-ideal* or *subtractive*, if $ab \in I$ and $b \in I$ imply that $a \in I$ (resp. $a+b \in I$ and $a \in I$ imply that $b \in I$), for each $a, b \in R$. A proper ideal P of R is called *prime* if $xy \in P$, then $x \in P$ or $y \in P$. A proper co-ideal M of R is said to be *prime*, if $x + y \in M$, then $x \in M$ or $y \in M$ [8]. A semiring R is called *I-semiring*, if $r + 1 = 1$ for all $r \in R$. A semiring R is called *idempotent* if $x^2 = x$ for all $x \in R$. Let I be a proper ideal of R. Then I is said to be *maximal* if R is the only ideal having I. The notation $Jac(R)$ will denote the *jacobson radical* of R which is the intersection of all maximal ideals of R. Let I be an ideal of a semiring R. Then I is said to be a *partitioning ideal* (= Q-ideal) provided that there exists a subset Q of R such that

(1) $R = \cup\{q + I \ : \ q \in Q\}$,

(2) *If* $q_1, q_2 \in Q$, *then* $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ *if and only if* $q_1 = q_2$.

If I is a Q-ideal of a semiring R, the we set

$$R/I := \{q + I \ : \ q \in Q\}.$$

Thus R/I is a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

$(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$.

$(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_4 + I$. Semiring R/I is said to be the quotient semiring of R by I. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is a zero element of R/I. Clearly, if R is an idempotent I-semiring, then so is R/I ([7]). Dual notion of Q-ideal (Q-co-ideal) was defined in [8].

2. Basic structure $G_H(R)$

Throughout this paper, R is a I-semiring and H is a multiplicatively closed subset of R.

Lemma 1. *The following statements hold:*

(i) If $0 \in H$, then $N(0) = H \setminus \{0\}$ and if $0 \notin H$, then $N(0) = H$.

(ii) If $1 \in H$, then $N(1) = R \setminus \{1\}$ and if $1 \notin H$, then $N(1) = \emptyset$.

Proof. (i) Since $0 + x = x \in H$ for all $x \in H$ and $0 \in H$, $N(0) = H \setminus \{0\}$. Otherwise, $N(0) = H$. This proves (i). Since $1 + x = 1$ for all $x \in R$, the statement (ii) holds. □

Theorem 1. *$G_H(R)$ is connected if and only if $1 \in H$. Moreover, if $G_H(R)$ is connected, then $diam(G_H(R)) \leq 2$.*

Proof. If $1 \in H$, then $deg(1) = |R| - 1$ by Lemma 1 (ii); so $G_H(R)$ is connected. Conversely, if $G_H(R)$ is connected, then $deg(1) \neq 0$ which implies that $1 \in H$ by Lemma 1 (ii). Finally, let x and y be distinct elements of R. If $x + y \in H$, then $x - y$ is a path in $G_H(R)$. So we may assume that $x + y \notin H$. Now the assertion follows the fact that $x - 1 - y$ is a path in $G_H(R)$. □

Proposition 1. *The following statements hold:*

(1) $G_H(R)$ is complete if and only if $R = H$ or $H = R \setminus \{0\}$.

(2) $G_H(R)$ is regular if and only if it is either complete or totally disconnected.

Proof. (1) Let $G_H(R)$ be complete. Thus 0 is connected to every element of $R \setminus \{0\}$, and so $0 + x \in H$ for every $x \in R \setminus \{0\}$. So $R \setminus \{0\} \subseteq H$. Therefore $R = H$ or $H = R \setminus \{0\}$. The converse is clear. Note that if $x + y = 0$, then $x = x + x + y = x(1 + 1) + y = x + y = 0$, because R is an I-semiring.

(2) Assume that $G_H(R)$ is regular and that is not totally disconnected. By Theorem 1, $1 \in H$; so $deg(1) = |R| - 1$. Then $G_H(R)$ is regular gives $deg(y) = |R| - 1$ for all $y \in R$; hence $G_H(R)$ is complete. The other implication is clear. □

In the following, the notation $max(R)$ denotes the set of all maximal ideals of R.

Theorem 2. *If $1 \in H$, then $gr(G_H(R)) \in \{3, \infty\}$.*

Proof. Assume that $|max(R)| \geq 2$ and let $M_1, M_2 \in max(R)$. Since $x + y = 1$, for some $x \in M_1$ and $y \in M_2$, we have $1 - x - y - 1$ is a cycle in $G_H(R)$; hence $gr(G_H(R)) = 3$. So we may assume that $|max(R)| = 1$. If $H = \{1\}$, then the graph $G_H(R)$ is a star graph which implies that $gr(G_H(R)) = \infty$. Now suppose that $|H| = 2$. If $H = \{0, 1\}$, then the graph $G_H(R)$ is a star graph which implies that $gr(G_H(R)) = \infty$, because $x + y = 0$ implies $x = 0$ and $y = 0$ for each $x, y \in R$. Otherwise, $H = \{1, r\}$, where $r \neq 0$. Then the cycle $1 - r - 0 - 1$ is the shortest cycle in the graph $G_H(R)$. So $gr(G_H(R)) = 3$. If $|H| \geq 3$, then there is an element $r \in H$ such that $r \neq 0, 1$. Now the cycle $1 - r - 0 - 1$ is the shortest cycle in the graph $G_H(R)$ which implies that $gr(G_H(R)) = 3$. □

The remaining of this section, we assume that R is an idempotent I-semiring, H is a saturated subset of R and $H \neq R$. Note that if $0 \in H$, then $H = R$, and so, by Proposition 1, the graph $G_H(R)$ is complete.

Proposition 2. *Assume that $|R| \geq 3$. If $|H| \geq 2$, then every vertex of the graph $G_H(R)$ lies in a cycle of length 3, and so $gr(G_H(R)) = 3$.*

Proof. By assumption, there is an element $x \in H$ with $x \neq 1$. If $y \neq 1, x$ is an arbitrary element in R, then $x(x + y) = x + xy = x \in H$. Therefore $x + y \in H$ and we have the cycle $1 - y - x - 1$, as required. □

Theorem 3. *Let $|H| = 1$. Then the following hold:*

(i) $deg(a) = 1$ for all $a \in Jac(R)$.

(ii) If $|max(R)| \geq 2$, then every vertex in graph $G_H(R) \setminus Jac(R)$ lies in a cycle of length 3.

Proof. (i) Since $|H| = 1$, we have $H = \{1\}$. Let $x \in Jac(R)$. Since $1 + y = 1 \in H$, 1 is adjacent to every vertex y in $G_H(R)$ which implies that $deg(x) \geq 1$. Suppose the result is false. Let $deg(x) \geq 2$. So there is $1 \neq y \in R$ such that x and y are adjacent (note that $1 + x = 1 \in H = \{1\}$), so $x + y = 1$. One can find a maximal ideal M of R such that $y \in M$. Hence $1 = x + y \in M$, which is impossible. So $deg(a) = 1$ for all $a \in Jac(R)$.

(ii) Assume that x is an arbitrary vertex in $G_H(R) \setminus Jac(R)$. Thus $x \notin M$, for some maximal ideal M of R. Thus $xR + M = R$, and so there exist $r \in R, m \in M$ such that $xr + m = 1$. Hence $x + m = x + xr + m = 1 + x = 1$. If $m \in Jac(R)$, then $x + m \in M'$, for some maximal ideal M' of R (we can find the maximal ideal M' such that $x \in M'$), which is a contradiction. Hence, we can consider the cycle $x - m - 1 - x$ in $G_H(R) \setminus Jac(R)$. □

Lemma 2. *The following statements hold:*

(1) If I is an ideal of R and $a+b \in I$, for some $a, b \in R$, then $a, b \in I$.

(2) Every ideal of R is k-ideal.

Proof. (1) Let I be an ideal of R and $a+b \in I$, for some $a, b \in R$. Then

$$a = a(1+b) = a + ab = a(a+b) \in I.$$

Similarly, $b \in I$.

(2) It is clear from (1). □

Proposition 3. *(1) The following statements are equivalent on a subset H of R:*

(i) H is saturated.

(ii) $R \setminus H = \bigcup_{i \in \Lambda} M_i$, for some prime ideals M_i of R.

(2) H is a saturated multiplicatively closed subset of R if and only if H is a co-ideal of R. Moreover, $H = \bigcap_{j \in J} P_j$, where $\{P_j\}_{j \in J}$ is the set of all prime co-ideals of R containing H.

(3) P is a prime co-ideal of R if and only if $R \setminus P$ is a prime ideal of R.

(4) Let H be a subset of R. Then P is a minimal prime co-ideal of R containing H if and only if $R \setminus P$ is an ideal of R which is maximal with disjoint from H.

Proof. (1) $(i) \Rightarrow (ii)$ Let $x \in R \setminus H$. Set $\sum = \{I : I$ is an ideal of R, $I \cap H = \emptyset$ and$x \in I\}$. Since $Rx \in \sum$, $\sum \neq \emptyset$. By Zorn's Lemma, $\sum$ has a maximal element P. It can be easily seen that P is a prime ideal. Therefore every $x \notin H$ has been inserted in a prime ideal disjoint from H. This proves (2).

$(ii) \Rightarrow (i)$ It is clear.

(2) Let H be saturated. Then $R \setminus H = \bigcup_{i \in \Lambda} M_i$, for some prime ideals M_i of R, by (1). Let $a \in H$ and $r \in R$. If $r + a \notin H$, then $r + a \in M_i$, for some $i \in \Lambda$. Therefore by Lemma 2(1), $a \in M_i$, a contradiction. Therefore H is a co-ideal of R. The converse is clear from [12, Proposition 2.1(1)]. Therefore $H = \bigcap_{j \in J} P_j$, where $\{P_j\}_{j \in J}$ is the set of all prime strong co-ideals of R containing S, by [12, Theorem 4.6].

(3) Assume that P is a prime co-ideal of R. Let $x \in R - P$ and $r \in R$. If $rx \in P$, then $r, x \in P$, by [12, Proposition 2.1(1)], a contradiction. Thus $rx \in R - P$. Let $x, y \in R - P$. If $x + y \in P$, then either $x \in P$ or $y \in P$, which is impossible. Therefore $x + y \in R - P$. This implies that $R - P$ is an ideal of R. It is clear that $R - P$ is a prime ideal. Conversely, let T be a prime ideal of R. Let $x \in R - T$ and $r \in R$. If $r + x \in T$, then $r, x \in T$, by Lemma 2. Thus $r + x \in R - T$. Let $x, y \in R - T$. If $xy \in T$, then either $x \in T$ or $y \in T$. Therefore $xy \in R - T$. This implies that $R - T$ is a co-ideal of R. Also, It is clear that $R - T$ is a prime co-ideal. Therefore, if $R - P$ is a prime ideal of R, then P is a prime co-ideal of R.

(4) It is straightforward. □

Throughout the paper, by $min(H)$ and $max(H)$, we show the set of minimal prime co-ideals of R containing H and the set of ideals of R which are maximal with disjoint from H, respectively.

Proposition 4. *If $G_H(R)$ is complete r-partite, then $r = |H| + 1$.*

Proof. Assume that $G_H(R)$ is complete r-partite with parts V_i $(1 \le i \le r)$. Since H is a clique in $G_H(R)$, every element of H is in a part V_i, where $|V_i| = 1$. Let V_1 and V_2 be two parts of $G_H(R)$ and $a, b \in R \setminus H$ such that $a \in V_1$ and $b \in V_2$. As 0 is not adjacent to a, $0 \in V_1$. Therefore 0 and b are adjacent, which is a contradiction. Therefore every element of $R \setminus H$ is in one part and $R \setminus H$ is an ideal. Thus $r = |H| + 1$. □

Theorem 4. *The following statements are equivalent:*

(1) $gr(G_H(R)) = \infty$.

(2) $G_H(R)$ *is a star graph.*

(3) $H = \{1\}$ *and* $max(H) = \{R - \{1\}\}$.

(4) $G_H(R)$ *is a complete bipartite.*

Proof. (1) $\Rightarrow$ (3) Assume that $|H| \ge 2$ and $a, b \in H$. Then $a - b - 0 - a$ is a cycle in $G_H(R)$, a contradiction. Hence $H = \{1\}$. Let $|max(H)| \ge 2$ and $M_1, M_2 \in max(H)$. As $H = \{1\}$, every ideal which is maximal with respect to disjoint from H, is a maximal ideal of R. Therefore $M_1 + M_2 = R$ and $a + b = 1$ for some $a \in M_1, b \in M_2$. Therefore $a - b - 1 - a$ is a cycle in $G_H(R)$, which is a contradiction. Therefore $H = \{1\}$ and $max(H) = \{R - \{1\}\}$.

The implications (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) By Proposition 4, $r = 2$. It is clear that $H = \{1\}$ and $R - \{1\}$ is a maximal ideal of R. Therefore $gr(G_H(R)) = \infty$. □

In the rest of this section, we will assume that $R \setminus H = \bigcup_{i \in \Lambda} M_i$ for some prime ideals M_i of R and $I := \bigcap_{i \in \Lambda} M_i$. Let I be a Q-ideal and $\widetilde{H} := \{q + I \ : \ h \in q + I \text{ for some } h \in H\}$.

Lemma 3. *Let I be a Q-ideal of R. Then $\widetilde{H}$ is a saturated multiplicatively closed subset of R/I.*

Proof. Let $q_1 + I$ and $q_2 + I$ be two elements of $\widetilde{H}$, where $h_1 \in q_1 + I$ and $h_2 \in q_2 + I$, for some $h_1, h_2 \in H$. If $(q_1 + I) \odot (q_2 + I) = q_3 + I$, where $q_1q_2 + I \subseteq q_3 + I$ and $q_3 \in Q$, then we have $h_1h_2 \in q_1q_2 + I \subseteq q_3 + I$. Thus $q_3 + I \in \widetilde{H}$. We show $\widetilde{H}$ is saturated. Let $(q_1 + I) \odot (q_2 + I) = q_3 + I \in \widetilde{H}$, where $q_1q_2 + I \subseteq q_3 + I$ and $q_3 \in Q$. Since $q_3 + I \in \widetilde{H}$, there exists $h \in H$ such that $h \in q_3 + I$. Thus $h = q_3 + i$ for some $i \in I$. As $h \in H$ and $i \in I$, $q_3 \in H$. Let $q_1q_2 = q_3 + j$ for some $j \in I$. Then $q_1q_2 \in H$, because H is a co-ideal, by Lemma 2. Therefore $q_1, q_2 \in H$ and so $q_1 + I, q_2 + I \in \widetilde{H}$. □

Lemma 4. *Let I be a Q-ideal of R. Then the following statements hold:*

(1) Let p_1 and p_2 be two elements of R with $p_1 \in q_1 + I$ and $p_2 \in q_2 + I$, where $q_1 + I \ne q_2 + I$. Then the following statements are equivalent:

(i) p_1 is adjacent to p_2 in $G_H(R)$.

(ii) $q_1 + I$ *is adjacent to* $q_2 + I$ *in* $G_{\widetilde{H}}(R/I)$.

(iii) each element of $q_1 + I$ *is adjacent to* $q_2 + I$.

(iv) there exists an element of $q_1 + I$ *which is adjacent to an element of* $q_2 + I$.

(2) If $q + I \in \widetilde{H}$, *then* $q \in Q \cap H$ *and* $q + I$ *is a clique in* $G_H(R)$.

(3) If $q + I \notin \widetilde{H}$, *then* $q + I$ *is an independent set in* $G_H(R)$.

Proof. (1) $(i) \Rightarrow (ii)$ By (i), $p_1 + p_2 \in H$. Let $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$. Therefore $p_1 + p_2 \in q_1 + q_2 + I \subseteq q_3 + I$ gives $q_3 + I \in \widetilde{H}$.

$(ii) \Rightarrow (iii)$ Let $q_1 + i_1 \in q_1 + I$ and $q_2 + i_2 \in q_2 + I$, where $i_1, i_2 \in I$. Assume that $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$. By (ii), $q_3 + I \in \widetilde{H}$. Thus there exists $h \in H$ such that $h \in q_3 + I$. Hence $h = q_3 + j$ for some $j \in I$. Therefore $q_3 \in H$. Let $q_1 + q_2 = q_3 + i$ for some $i \in I$. Then $q_1 + i_1 + q_2 + i_2 \in H$, because H is a co-ideal.

$(iii) \Rightarrow (iv)$ This implication is clear.

$(iv) \Rightarrow (i)$ Assume that $q_1 + i \in q_1 + I$ and $q_2 + i' \in q_2 + I$ are adjacent in $G_H(R)$, where $i, i' \in I$. Let $q_1 + i_1 \in q_1 + I$ and $q_2 + i_2 \in q_2 + I$, where $i_1, i_2 \in I$. As $q_1 + i + q_2 + i' \in H$ and $i, i' \in I$, we have $q_1 + q_2 \in H$. Therefore $p + q \in H$.

(2) Let $q + I \in \widetilde{H}$. Then $h = q + i$ for some $h \in H$ and $i \in I$. Therefore $q \in H$. Also, it is clear that $q + I$ is a clique in $G_H(R)$.

(3) If $q + I \notin \widetilde{H}$, then $q \notin H$. Let $q + i$ and $q + i'$ be arbitrary elements of $q + I$. Then $q + i + q + i' \notin H$, because $q, i, i' \in M$ for some $M \in max(H)$. Therefore $q + I$ is an independent set in $G_H(R)$. $\square$

In the following, we investigate the relationship between the diameter and the girth of the graphs $G_H(R)$ and $G_{\widetilde{H}}(R/I)$.

Theorem 5. *The following statements hold:*

(1) $gr(G_H(R)) \leq gr(G_{\widetilde{H}}(R/I))$.

(2) $diam(G_{\widetilde{H}}(R/I)) \leq diam(G_H(R))$.

Proof. (1) If $G_{\widetilde{H}}(R/I)$ has no cycle, then there is nothing to prove. Hence assume that $q_1 + I - q_2 + I - ... - q_n + I - q_1 + I$ is a cycle in $G_{\widetilde{H}}(R/I)$. Then we have the cycle $q_1 - q_2 - ... - q_n - q_1$ in $G_H(R)$, by Lemma 4, which implies that $gr(G_H(R)) \leq gr(G_{\widetilde{H}}(R/I))$.

(2) If $n := diam(G_{\widetilde{H}}(R/I))$, then there are two vertices $q_1 + I$ and $q_2 + I$ of $G_{\widetilde{H}}(R/I)$ with $d(q_1 + I, q_2 + I) = n$. Assume that $q_1 + I - p_1 - ... - p_{n-2} + I - q_2 + I$ is a corresponding path of length n between $q_1 + I$ and $q_2 + I$ in $G_{\widetilde{H}}(R/I)$. In view of Lemma 4, $q_1 - p_1 - ... - p_{n-2} - q_2$ is a path of length n in $G_H(R)$. Therefore $diam(G_{\widetilde{H}}(R/I)) \leq diam(G_H(R))$. $\square$

The following example shows that we may have strict inequality in parts (1), (2) of Theorem 5.

Example 1. *Let* $X = \{a, b, c\}$ *and* $R = (P(X), \cup, \cap)$ *a semiring, where* $P(X)$ *is the set of all subsets of* X. *If* $H = \{\{a\}, \{a, b\}, \{a, c\}, X\}$, *then* H *is a saturated* *multiplicatively closed subset of R and a minimal prime co-ideal of R. Therefore $I = R \setminus H$ is a maximal ideal of R. It can be verified that I is a Q-ideal of R and $Q = \{\emptyset, \{a\}\}$. By drawing $G_H(R)$ and $G_{\widetilde{H}}(R/I)$, one can see that $1 = diam(G_{\widetilde{H}}(R/I)) < diam(G_H(R)) = 2$ and $3 = gr(G_H(R)) < gr(G_{\widetilde{H}}(R/I)) = \infty$.*

In the following theorem, we provide a characterization of $G_H(R)$ in terms of $G_{\widetilde{H}}(R/I)$.

Theorem 6. *Let I be a Q-ideal of R. Then*

$$G_H(R) = (\oplus_{|I|^2} G_{\widetilde{H}}(R/I)) \oplus (\oplus_{|I|} K_{|Q\cap H|}).$$

Proof. If there exist $p, q \in Q$ such that $p + I$ and $q + I$ are adjacent in $G_{\widetilde{H}}(R/I)$, then in view of Lemma 4, each element of $p + I$ is adjacent to each element of $q + I$ in $G_H(R)$. Thus, each edge of $G_{\widetilde{H}}(R/I)$ corresponds to exactly $|I|^2$ edges in $G_H(R)$. Also, for each $p \in Q \cap H$, the coset $p + I$ forms a clique in $G_H(R)$. Hence $G_H(R) = (\oplus_{|I|^2} G_{\widetilde{H}}(R/I)) \oplus (\oplus_{|I|} K_{|Q\cap H|})$. □

3. Planarity, clique number, chromatic number and independence number of $G_H(R)$

In this section, we use the notations already established, so R is an idempotent I-semiring and H is a saturated proper subset of R. We will investigate clique number, independence number and planar property of the graph $G_H(R)$. A graph G is called planar, if it can be drawn in the plane (i.e. its edges intersect only at their ends). A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. An interesting characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [6].

Proposition 5. *The following hold:*

(i) If $0 \in H$, then $G_H(R)$ is planar if and only if $|R| \leq 4$.
(ii) If $|max(H)| \geq 4$ or $|H| \geq 4$, then $G_H(R)$ is not planar.
(iii) If $|H| = 3$, then $G_H(R)$ is planar if and only if $|R| \leq 5$.
(iv) Let $H = \{1\}$. Then $G_H(R)$ is planar if and only if $G_H(R) \setminus \mathrm{Jac}(R)$ is planar.

Proof. (i) Since $0 \in H$, $H = R$. It follows that $G_H(R)$ is complete. Now the assertion follows from Kuratowski's theorem.

(ii) If $|max(H)| \geq 4$, then $|min(H)| \geq 4$. Hence $\Gamma_H(R)$ is not planar, by [12, Theorem 4.10]. Therefore $G_H(R)$ is not planar. The other implication is clear.

(iii) Assume that $G_H(R)$ is planar and let $V_1 = H = \{x_1, x_2, x_3\}$. Suppose to the contrary that $|R| \geq 6$. Set $V_2 = \{y_1, y_2, y_3\} \subseteq R \setminus H$. It can be easily seen that one can find a copy of $K_{3,3}$ in $G_H(R)$, which is a contradiction. Conversely, assume that $|R| \leq 5$. If $|R| \leq 4$, we are done. If $|R| = 5$, then by Proposition 1, $G_H(R)$ is not K_5; hence $G_H(R)$ is planar.

(iv) Since by Theorem 3 (i), $\deg(a) = 1$ for all $a \in Jac(R)$, the result is clear. □

If $max(H) = \{M_1, M_2, ..., M_n\}$, then we denote $M_i \setminus (\cup_{j=1,j\neq i}^{n} M_j)$ by M'_i and $(M_i \cap M_j) \setminus (\cup_{s=1,s\neq i,j}^{n} M_s)$ by $M_{i,j}$ for each $1 \leq i \neq j \leq n$.

Theorem 7. *Let $H = \{1\}$. Then the graph $G_H(R)$ is planar if and only if one of the following statements holds:*

(1) $max(R) = \{M_1, M_2, M_3\}$, $|M'_i| = 1$ for each $1 \leq i \leq 3$ and $V(G_H(R)) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$, where V_i's are satisfying the following:

(i) $V_1 = M'_1 \cup M'_2 \cup M'_3 \cup \{1\}$ is a clique in $G_H(R)$.

(ii) $V_2 = M_{1,2}$ and every element of V_2 is adjacent to 1 and $a \in M'_3$.

(iii) $V_3 = M_{1,3}$ and every element of V_3 is adjacent to 1 and $b \in M'_2$.

(iv) $V_4 = M_{2,3}$ and every element of V_4 is adjacent to 1 and $c \in M'_1$.

(v) $V_5 = M_1 \cap M_2 \cap M_3$ and every element of V_5 is adjacent to 1.

(2) $max(R) = \{M_1, M_2\}$, $V(G_H(R)) = V_1 \cup V_2 \cup V_3 \cup V_4$ where V_i's are satisfying the following:

(i) $V_1 = \{1\}$, and 1 is adjacent to every vertex of $G_H(R)$.

(ii) $V_2 = M'_1$, $V_3 = M'_2$ and either $|V_i| \geq 3$ and $|V_j| = 1$ $(i \neq j)$ or $|V_i| \leq 2$ for each $i = 1, 2$. Moreover, the subgraph generated by V_2, V_3 is complete 2-partite with parts V_2 and V_3 and every element of $V_2 \cup V_3$ is adjacent to 1.

(iii) $V_4 = M_1 \cap M_2$ and every element of V_4 is adjacent to 1.

(3) $R - \{1\}$ is a maximal ideal of R and $G_H(R)$ is a star graph.

Proof. Assume that the graph $G_H(R)$ is planar. Then $|max(R)| \leq 3$, by Proposition 5. Let $max(R) = \{M_1, M_2, M_3\}$. If $|M'_i| \geq 2$, for some $i \in \{1, 2, 3\}$, then there exist $x, y \in M'_i$. Let $z \in M'_j$ and $t \in M'_k$, where $1 \leq k, j \leq 3$ and $k \neq j$ are distinct from i. Set $S_1 := \{x, y, 1\}$ and $S_2 := \{z, t, zt\}$. As $x + z = x + t = 1$, we have $x+tz = 1$ (note that $zt \neq z$ and $zt \neq t$). Similarly, $y+z = y+t = y+tz = 1$. Hence, one can find a copy of $K_{3,3}$ in $G_H(R)$, which is impossible. Hence $|M'_i| = 1$ for each $i \in \{1, 2, 3\}$. It can be easily verified that (1) holds. If $max(R) = \{M_1, M_2\}$, then we will prove that (2) holds. If $|M'_1| \geq 3$, then there exist $x, y, z \in M'_1$. If $t, s \in M'_2$, then by setting $S_1 := \{x, y, z\}$ and $S_2 := \{t, s, 1\}$, the graph $G_H(R)$ has a subgraph isomorphic to $K_{3,3}$, a contradiction. Hence $|M'_2| = 1$. Similarly, if $|M'_2| \geq 3$, then $|M'_1| = 2$. Hence $|M'_i| \geq 3$ and $|M'_j| = 1$ $(i \neq j)$ or $|M'_i| \leq 2$ for each $i = 1, 2$. It is easy to see that (2) holds. If $|max(R)| = 1$, then by Theorem 4 , $G_H(R)$ is a star graph.

Conversely, if one of the conditions (1) or (2) or (3) holds, then it is easy to show $G_H(R)$ is a planar graph. □

Theorem 8. *Let $H = \{1, a\}$. Then the graph $G_H(R)$ is planar if and only if one of the following statements holds:*

(1) $max(R) = \{M_1, M_2\}$, $V(G_H(R)) = V_1 \cup V_2 \cup V_3 \cup V_4$ where V_i's are satisfying the following:

(i) $V_1 = \{1, a\}$, and every element of V_1 is adjacent to every vertex of $G_H(R)$.

(ii) $V_2 = M'_1$, $V_3 = M'_2$ and either $|V_i| = 1$ for each $i = 1, 2$ or $|V_i| = 2$ and $|V_j| = 1$ for each $i \neq j \in \{1, 2\}$. Moreover, the subgraph generated by V_2, V_3 is

complete 2-partite with parts V_2 and V_3 and every element of $V_2 \cup V_3$ is adjacent to 1 and a.

(iii) $V_4 = M_1 \cap M_2$ and every element of V_4 is adjacent to 1 and $\{a\}$.

(2) $max(H) = \{R - \{1, a\}\}$ and $G_H(R) \cong K_{1,1,|R-\{1\}|}$.

Proof. If $G_H(R)$ is planar, then $|max(R)| \leq 3$, by Proposition 5. If $max(H) = \{M_1, M_2, M_3\}$, then there exist $d \in M_1',\ b \in M_2',\ c \in M_3'$ such that $\{1, a, b, c, d\}$ is a clique in $G_H(R)$, which is impossible. Hence $|max(H)| \leq 2$. If $|M_i'| \geq 2$ for each $i = 1, 2$, then there exist $x, y \in M_1'$ and $t, z \in M_2'$. By setting $S_1 := \{x, y, a\}$ and $S_2 := \{1, t, z\}$, $G_H(R)$ has a subgraph isomorphic to $K_{3,3}$, which is a contradiction. Hence either $|M_i'| = 1$ for each $i = 1, 2$ or $|M_i'| = 2$ and $|M_j'| = 1$ for each $i \neq j \in \{1, 2\}$. Therefore (1) holds. If $|max(H)| = 1$, then it is easy to verified that $G_H(R)$ is complete 3-partite and $G_H(R) \cong K_{1,1,|R-\{1\}|}$. □

Theorem 9. *In the graph $G_H(R)$ we have the following equality:*

$$\omega(G_H(R)) = \chi(G_H(R)) = |H| + |max(H)|.$$

Proof. It is clear that $\omega(G) \leq \chi(G)$, for each graph G. We consider two cases:

Case 1: $\omega(G_H(R)) = \infty$. Then $\chi(G_H(R)) = \infty$. Assume that H and $max(H)$ are finite and $max(H) = \{M_1, ..., M_n\}$. Let $\mathcal{C}$ be a maximal clique in $G_H(R)$. Set for each $1 \leq i \leq n$, $I_i = \{a \in \mathcal{C} \setminus H :\ a \in M_i\}$. If $|I_i| \geq 2$, for some $1 \leq i \leq n$, then there exist $a, b \in \mathcal{C} \setminus H$. Therefore $a, b \in M_i$ and so $a + b \notin H$ contradicts $a, b \in \mathcal{C}$. Therefore $|I_i| \leq 1$ for each $1 \leq i \leq n$. As $\mathcal{C} \setminus H = \bigcup_{i=1}^{n} I_i$ and I_i is a finite set for each $1 \leq i \leq n$, $\mathcal{C} \setminus H$ is a finite set. Therefore $\mathcal{C}$ is a finite set, a contradiction. Therefore either H is infinite or $max(H)$ is infinite. This gives $\omega(G_H(R)) = \chi(G_H(R)) = |H| + |max(H)| = \infty$.

Case 2: $\omega(G_H(R)) < \infty$. As $\omega(\Gamma_H(R)) < \infty$ and H is a clique in $G_H(R)$, H is a finite set. Moreover, $\omega(\Gamma_H(R)) < \infty$, because $\Gamma_H(R)$ is a subgraph of $G_H(R)$. Therefore $min(H)$ is finite, and so $max(H)$ is finite. Assume that $max(H) = \{M_1, ..., M_n\}$. Let $a_i \in M_i \setminus (\bigcup_{i \neq j, j=1}^{n} M_j)$. If $a_i + a_j \notin H$, for some $1 \leq i, j \leq n$, then $a_i + a_j \in M_k$, for some $M_k \in max(H)$, and so by Lemma 2 we have $a_i, a_j \in M_k$, a contradiction. Therefore $a_i + a_j \in H$. Hence $|H| + |max(H)| \leq \omega(G_H(R))$. Let $|H| = m$ and $H = \{a_1, ..., a_m$. Define $f :\ V(G_H(R)) \to \{1, ..., n, n+1, ..., m\}$ by

$$f(a) = \begin{cases} n + i, & \text{if } a = a_i \in \{a_1, ..., a_m\} \\ i, & \text{if } a = a_i \in M_i - (\bigcup_{i \neq j, j=1}^{n} M_j) \\ j, & \text{if } a \in M_j \cap M_{j+s_1} \cap ... \cap M_{j+s_t}, \text{where } s_1, ..., s_t \in \mathbb{N}. \end{cases}$$

Let $a, b \in R$ be adjacent in $G_H(R)$. Then it is clear that $f(a) \neq f(b)$ provided that $(a, b \in H)$ or $(a \notin H, b \in H)$ or $(a \in H, b \notin H)$. Let $a \notin H$ and $b \notin H$. Then $a \in M_i$ and $b \in M_j$ for some $M_i, M_j \in max(H)$. If $i = j$, then $a + b \in M_i$ and $a + b \notin H$, a contradiction. Let $I = \{i :\ a \in M_i,\ 1 \leq i \leq n\}$ and $J = \{j :\ b \in M_j,\ 1 \leq j \leq n\}$. As $a + b \in H$, we have $I \cap J = \emptyset$. Therefore $f(a)$ and $f(b)$ are the least element of I and J, respectively. Thus $f(a) \neq f(b)$. This implies that $\chi(G_S(R)) \leq |H| + |max(H)|$ and so we have $\omega(G_H(R)) = \chi(G_H(R)) = |H| + |max(H)|$. □

Let $T \subseteq P(\{1,2,...,n\})$, where $P(\{1,2,...,n\}$ denotes the power set of $\{1,2,...,n\}$. We say that T satisfies the property (P), provided that:
(1) For each $I \in T$, $|I| \geq 2$.
(2) For each $I, J \in T$, $I \cap J \neq \emptyset$.
Set $\sum = \{T \subseteq P(\{1,2,...,n\} : \ T$ satisfies the property $(P)\}$.

Theorem 10. *Let* $max(H) = \{M_1, M_2, ..., M_n\}$. *Then*

$$\alpha(G_H(R)) = max\{\{|M_i|\}_{i=1}^n \cup \{| \cup_{I\in T} (\cap_{i\in I} M_i)|\}_{T\in\sum}\}.$$

Proof. It can be easily seen that M_i and $\cup_{I\in T}(\cap_{j\in I} M_j)$ are independent sets in $G_H(R)$, for each $1 \leq i \leq n$ and $T \in \sum$. Therefore, $\alpha(G_H(R)) \geq max\{\{|M_i|\}_{i=1}^n \cup \{| \cup_{I\in T} (\cap_{i\in I} M_i)|\}_{T\in\sum}\}$. Assume that Y is a maximal independent set of $G_H(R)$. For each $a \in Y$, set

$$I_a = \{i : \ a \in M_i, \ 1 \leq i \leq n\}.$$

Let $a \in Y$ and $I_a = \{i\}$, for some $1 \leq i \leq n$. If $b \in Y$, then $b+a \notin H$. Hence $b+a \in M_k$ for some $1 \leq k \leq n$. Hence $a, b \in M_k$, by Lemma 2. This implies that $b \in M_i$. Therefore, $Y \subseteq M_i$. As Y is a maximal independent set, we have $Y = M_i$ (M_i is independent set). Now, let $|I_a| \geq 2$, for each $a \in Y$. If there exist $a, b \in Y$ such that $I_a \cap I_b = \emptyset$, then $a + b \in H$, a contradiction. Thus, $I_a \cap I_b \neq \emptyset$. Set $T = \{I_a\}_{a\in Y}$. Then $T \in \sum$ and $Y \subseteq \cup_{I\in T}(\cap_{i\in I} M_i)$. Since Y is maximal, $Y = \cup_{I\in T}(\cap_{i\in I} M_i)$. This proves that $\alpha(G_H(R)) = max\{\{|M_i|\}_{i=1}^n \cup \{| \cup_{I\in T} (\cap_{i\in I} M_i)|\}_{T\in\sum}\}$. □

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References

[1] Afkhami, M., Barati, Z., Khashyarmanesh, K., A graph associated to a lattice, *Ricerche Mat.*, 63 (2014), 67–78. https://doi.org/10.1007/s11587-013-0164-6
[2] Anderson, D. F., Livingston, P. S., The zero-divisor graph of a commutative rings, *J. Algebra*, 217 (1999), 434-447. https://doi.org/10.1006/jabr.1998.7840
[3] Anderson, D. F., Badawi, A., The total graph of a commutative ring, *J. Algebra*, 320(7) (2008), 2706–2719. https://doi.org/10.1016/j.jalgebra.2008.06.028
[4] Barati, Z., Khashyarmanesh, K., Mohammadi, F., Nafar, Kh., On the associated graphs to a commutative ring, *J. Algebra Appl.,* 11(2) (2012), 1250037 (17 pages). https://doi.org/10.1142/S0219498811005610
[5] Beck, I., Coloring of commutative rings, *J. Algebra,* 116 (1988), 208-226. https://doi.org/10.1016/0021-8693(88)90202-5

[6] Bondy, J. A., Murty, U. S. R., Graph Theory, Graduate Texts in Mathematics, 244, Springer, New York, 2008.

[7] Atani, S. E., The ideal theory in quotients of commutative semirings, *Glas. Math.*, 42 (2007), 301–308. https://doi.org/10.3336/gm.42.2.05

[8] Atani, S. E., Hesari, S.D.P., Khoramdel, M., Strong co-ideal theory in quotients of semirings, *J. of Advanced Research in Pure Math.*, 5(3) (2013), 19–32. https://doi.org/10.5373/jarpm.1482.061212

[9] Atani, S. E., Hesari, S.D.P., Khoramdel, M., The identity-summand graph of commutative semirings, *J. Korean Math. Soc.*, 51 (2014), 189–202. https://doi.org/10.4134/JKMS.2014.51.1.189

[10] Atani, S. E., Hesari, S.D.P., Khoramdel, M., Total graph of a commutative semiring with respect to identity-summand elements, *J. Korean Math. Soc.*, 51(3) (2014), 593– 607. https://doi.org/10.4134/JKMS.2014.51.3.593

[11] Atani, S. E., Hesari, S.D.P., Khoramdel, M., Total identity-summand graph of a commutative semiring with respect to a co-ideal, *J. Korean Math. Soc.*, 52(1) (2015), 159-176. https://doi.org/10.4134/JKMS.2015.52.1.159

[12] Atani, S. E., Hesari, S.D.P., Khoramdel, M., A co-ideal based identity-summand graph of a commutative semiring, *Comment. Math. Univ. Carolin.*, 56(3) (2015), 269–285. https://doi.org/10.14712/1213-7243.2015.124

[13] Atani, S. E., Hesari, S.D.P., Khoramdel, M., A graph associated to proper non-small ideals of a commutative ring, *Comment. Math. Univ. Carolin.*, 58(1) (2017), 1-12. https://doi.org/10.14712/1213-7243.2015.189

[14] Atani, S. E., Hesari, S.D.P., Khoramdel, M., Sedghi Shanbeh Bazari, M., Total graph of a 0-distributive lattice, *Categories and General Algebraic Structures with Applications*, 9(1) (2018), 15-27. https://doi.org/10.29252/cgasa.9.1.15

[15] Atani, S. E., Hesari, S.D.P., Khoramdel, M., Sedghi Shanbeh Bazari, M., A semi-prime filter based identity-summand graph of a lattice, *LE Matematich*, Vol. LXXIII (2018), 297–318. https://doi.org/10.4418/2018.73.2.5

[16] Atani, S. E., Hesari, S.D.P., Khoramdel, M., Sarvandi, Z. E., Intersection graphs of co-ideals of semirings, *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics*, 68(1) (2019), 840–851. https://doi.org/10.31801/cfsuasmas.481603

[17] Atani, S. E., Hesari, S.D.P., Khoramdel, M., On a graph of ideals of a commutative ring, *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics*, 68(2) (2019), 2283-2297. https://doi.org/10.31801/cfsuasmas.534944

[18] Atani S. E., Esmaeili Khalil Saraei, F., The total graph of a commutative semiring, *An. Stiint. Univ. Ovidius Constanta Ser. Mat.*, 21(2) (2013), 21-33. https://doi.org/10.2478/auom-2013-0021

[19] Golan, J. S., Semirings and Their Applications, Kluwer Academic Publisher Dordrecht, 1999. https://doi.org/10.1007/978-94-015-9333-5