Turk. J. Math. Comput. Sci.
13(1)(2021) 25–43
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DOI : 10.47000/tjmcs.787578

# On Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers

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Received: 29-08-2020 • Accepted: 12-02-2021

Abstract. In this paper, we present Binet's formulas, generating functions, and the summation formulas for generalized Hexanacci numbers, and as special cases, we investigate Hexanacci and Hexanacci-Lucas numbers with their properties. Also, we define Gaussian generalized Hexanacci numbers and as special cases, we investigate Gaussian Hexanacci and Gaussian Hexanacci-Lucas numbers with their properties. Moreover, we give some identities for these numbers. Furthermore, we present matrix formulations of generalized Hexanacci numbers and Gaussian generalized Hexanacci numbers.

*2010 AMS Classification:* 11B37, 11B39, 11B83

**Keywords:** Hexanacci numbers, Gaussian generalized Hexanacci numbers, Gaussian Hexanacci numbers, Gaussian Hexanacci-Lucas numbers.

## 1. Introduction and Preliminaries

In this paper, we investigate generalized Hexanacci numbers and give properties of Hexanacci and Hexanacci-Lucas numbers as special cases. We also define Gaussian generalized Hexanacci numbers and give properties of Gaussian Hexanacci and Gaussian Hexanacci-Lucas numbers as special cases. First, in this section, we present some background about generalized Hexanacci numbers.

There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Hexanacci and Hexanacci-Lucas which are special case of generalized Hexanacci numbers. A generalized Hexanacci sequence $\{V_n\}_{n\geq0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n\geq0}$ is defined by the sixth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6}, \tag{1.1}$$

with the initial values $V_0 = c_0$, $V_1 = c_1$, $V_2 = c_2$, $V_3 = c_3$, $V_4 = c_4$, $V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} - V_{-(n-5)} + V_{-(n-6)},$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (1.1) holds for all integer $n$.

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The first few generalized Hexanacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few generalized Hexanacci numbers

| $n$ | $V_n$ | $V_{-n}$ |
|---|---|---|
| 0 | $c_0$ | $c_0$ |
| 1 | $c_1$ | $-c_0 - c_1 - c_2 - c_3 - c_4 + c_5$ |
| 2 | $c_2$ | $2c_4 - c_5$ |
| 3 | $c_3$ | $2c_3 - c_4$ |
| 4 | $c_4$ | $2c_2 - c_3$ |
| 5 | $c_5$ | $2c_1 - c_2$ |
| 6 | $c_0 + c_1 + c_2 + c_3 + c_4 + c_5$ | $2c_0 - c_1$ |
| 7 | $c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5$ | $-3c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 + 2c_5$ |
| 8 | $2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4 + 4c_5$ | $c_0 + c_1 + c_2 + c_3 + 5c_4 - 3c_5$ |
| 9 | $4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4 + 8c_5$ | $4c_3 - 4c_4 + c_5$ |
| 10 | $8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4 + 16c_5$ | $4c_2 - 4c_3 + c_4$ |
| 11 | $16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4 + 32c_5$ | $4c_1 - 4c_2 + c_3$ |

We consider two special cases of $V_n$ : $V_n(0,1,1,2,4,8) = H_n$ is the sequence of Hexanacci numbers (sequence $A001592$ in [20]) and $V_n(5,1,3,7,15,31) = E_n$ is the sequence of Hexanacci-Lucas numbers ($A074584$ in [20]). In other words, Hexanacci sequence $\{H_n\}_{n\geq 0}$ and Hexanacci-Lucas (or companian Hexanacci or Esanacci ) sequence $\{E_n\}_{n\geq 0}$ are defined by the sixth-order recurrence relations

$$H_n = H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6},\ H_0 = 0,\ H_1 = 1,\ H_2 = 1,\ H_3 = 2,\ H_4 = 4,\ H_5 = 8,$$

and

$$E_n = E_{n-1} + E_{n-2} + E_{n-3} + E_{n-4} + E_{n-5} + E_{n-6},\ E_0 = 6,\ E_1 = 1,\ E_2 = 3,\ E_3 = 7,\ E_4 = 15,\ E_5 = 31,$$

respectively. Hexanacci sequence has been studied by many authors, see [14, 18].

Next, we present the first few values of the Hexanacci and Hexanacci-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few Hexanacci and Hexanacci-Lucas Numbers

| $n$ | $-10$ | $-9$ | $-8$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H_n$ | 0 | 0 | 0 | 0 | $-1$ | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 63 | 125 | 248 |
| $E_n$ | $-1$ | $-1$ | $-1$ | $-8$ | 11 | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | 6 | 1 | 3 | 7 | 15 | 31 | 63 | 120 | 239 | 475 | 943 |

Exact formulas for the $n$th Hexanacci number and the $n$th Hexanacci-Lucas number can be given explicitly in terms of the six roots of of the equation

$$x^6 - x^5 - x^4 - x^3 - x^2 - x - 1 = 0,$$

as follows: for all integers $n$, usual Hexanacci and Hexanacci-Lucas numbers can be expressed using Binet's formulas

$$\begin{aligned} H_n \quad = \quad & \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ & + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} + \frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ & + \frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} + \frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}, \end{aligned}$$

(see Theorem 2.2) or

$$H_n = \frac{\alpha-1}{7\alpha-12}\alpha^{n-1} + \frac{\beta-1}{7\beta-12}\beta^{n-1} + \frac{\gamma-1}{7\gamma-12}\gamma^{n-1} + \frac{\delta-1}{7\delta-12}\delta^{n-1} + \frac{\lambda-1}{7\lambda-12}\lambda^{n-1} + \frac{\mu-1}{7\mu-12}\mu^{n-1}, \tag{1.2}$$

(see [5]) or

$$\begin{aligned} H_n &= \frac{\alpha^n}{-\alpha^5+\alpha^3+2\alpha^2+10\alpha-1}+\frac{\beta^n}{-\beta^5+\beta^3+2\beta^2+10\beta-1}+\frac{\gamma^n}{-\gamma^5+\gamma^3+2\gamma^2+10\gamma-1} \\ &\quad +\frac{\delta^n}{-\delta^5+\delta^3+2\delta^2+10\delta-1}+\frac{\lambda^n}{-\lambda^5+\lambda^3+2\lambda^2+10\lambda-1}+\frac{\mu^n}{-\mu^5+\mu^3+2\mu^2+10\mu-1}, \end{aligned}$$

(see, for example, http://mathworld.wolfram.com/HexanacciNumber.html)
and

$$E_n = \alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n+\mu^n,$$

respectively, where $x_1=\alpha, x_2=\beta, x_3=\gamma, x_4=\delta, x_5=\lambda$ and $x_6=\mu$ are the roots of the equation

$$x^6-x^5-x^4-x^3-x^2-x-1=0. \tag{1.3}$$

Note that we can write the Binet's formulas for $H_n$ and $E_n$ as follows:

$$\begin{aligned} H_n &= \sum_{p=1}^{6}\frac{x_p^{n+4}}{\prod\limits_{\substack{q=1\\p\neq q}}^{6}(x_p-x_q)}, \\ H_n &= \sum_{p=1}^{6}\frac{x_p-1}{7x_p-12}x_p^{n-1}, \\ H_n &= \sum_{p=1}^{6}\frac{x_p^n}{-x_p^5+x_p^3+2x_p^2+10x_p-1}, \end{aligned}$$

and

$$E_n=\sum_{p=1}^{6}x_p^n,$$

respectively.

Moreover, the approximate value of $\alpha, \beta, \gamma, \delta, \lambda$ and $\mu$ are given by

$$\begin{aligned} \alpha &= 1.9836, \\ \beta &= -0.84031, \\ \gamma &= 0.39029+0.81786\,i, \\ \delta &= 0.39029-0.81786\,i, \\ \lambda &= -0.46193+0.71914\,i, \\ \mu &= -0.46193-0.71914\,i. \end{aligned}$$

In fact, there are no solutions of the characteristic equation (1.3) in terms of radicals, see [28].

Note that we have the following identities:

$$\begin{aligned} \alpha+\beta+\gamma+\delta+\lambda+\mu &= 1, \\ \alpha\beta\gamma\delta\lambda\mu &= -1. \end{aligned}$$

## 2. Properties of Generalized Hexanacci Numbers

In this section, we present Binet's formulas, the generating functions, and the summation formulas for generalized Hexanacci numbers.

First, we give the ordinary generating function $\sum\limits_{n=0}^{\infty}V_nx^n$ of the sequence $V_n$.

**Lemma 2.1.** *Suppose that* $f_{V_n}(x)=\sum_{n=0}^{\infty}V_nx^n$ *is the ordinary the generating function of the generalized Hexanacci sequence* $\{V_n\}_{n\geq 0}$.*Then* $f_{V_n}(x)$ *is given by*

$$f_{V_n}(x)=\frac{\begin{array}{c}V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3\\+(V_4-V_3-V_2-V_1-V_0)x^4+(V_5-V_4-V_3-V_2-V_1-V_0)x^5\end{array}}{1-x-x^2-x^3-x^4-x^5-x^6}. \tag{2.1}$$

*Proof.* Using (1.1) and some calculation, we obtain

$$\begin{array}{rl} & f_{V_n}(x)-xf_{V_n}(x)-x^2f_{V_n}(x)-x^3f_{V_n}(x)-x^4f_{V_n}(x)-x^5f_{V_n}(x)-x^6f_{V_n}(x)\\ = & V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3\\ & +(V_4-V_3-V_2-V_1-V_0)x^4+(V_5-V_4-V_3-V_2-V_1-V_0)x^5,\end{array}$$

which gives (2.1). □

The previous Lemma gives the following results as particular examples. The generating function of the Hexanacci sequence $H_n$ is

$$f_{H_n}(x)=\sum_{n=0}^{\infty}H_nx^n=\frac{x}{1-x-x^2-x^3-x^4-x^5-x^6}, \tag{2.2}$$

and the generating function of the Hexanacci-Lucas sequence $E_n$ is

$$f_{E_n}(x)=\sum_{n=0}^{\infty}E_nx^n=\frac{6-5x-4x^2-3x^3-2x^4-x^5}{1-x-x^2-x^3-x^4-x^5-x^6}.$$

We next find Binet's formula for Hexanacci numbers by the use of the generating function for $H_n$.

**Theorem 2.2.** *(The Binet's formula for Hexanacci numbers)*

$$\begin{array}{rcl} H_n & = & \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)}\\ & & +\frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)}\\ & & +\frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}.\end{array} \tag{2.3}$$

*Proof.* Let

$$h(x)=1-x-x^2-x^3-x^4-x^5-x^6.$$

Then, for some $\alpha,\beta,\gamma,\delta,\lambda$ and $\mu$, we write

$$h(x)=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)(1-\mu x),$$

i.e.,

$$1-x-x^2-x^3-x^4-x^5-x^6=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)(1-\mu x). \tag{2.4}$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\delta},\frac{1}{\lambda}$ ve $\frac{1}{\mu}$ are the roots of $h(x)$. This gives $\alpha,\beta,\gamma,\delta,\lambda$ and $\mu$ as the roots of

$$h(\frac{1}{x})=1-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{x^4}-\frac{1}{x^5}-\frac{1}{x^6}=0.$$

This implies $x^6-x^5-x^4-x^3-x^2-x-1=0$. Now, by (2.2) and (2.4), it follows that

$$f_{H_n}(x)=\sum_{n=0}^{\infty}H_nx^n=\frac{x}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)(1-\mu x)}.$$

Then, we write

$$\begin{array}{rcl}\frac{x}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)(1-\mu x)} & = & \frac{B_1}{(1-\alpha x)}+\frac{B_2}{(1-\beta x)}+\frac{B_3}{(1-\gamma x)}\\ & & +\frac{B_4}{(1-\delta x)}+\frac{B_5}{(1-\lambda x)}+\frac{B_6}{(1-\mu x)}.\end{array} \tag{2.5}$$

So,

$$\begin{aligned} x &= B_1(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)(1-\mu x)+B_2(1-\alpha x)(1-\gamma x)(1-\delta x)(1-\lambda x)(1-\mu x) \\ &\quad +B_3(1-\alpha x)(1-\beta x)(1-\delta x)(1-\lambda x)(1-\mu x)+B_4(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\lambda x)(1-\mu x) \\ &\quad +B_5(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\mu x)+B_6(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x). \end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $\frac{1}{\alpha}=B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})(1-\frac{\lambda}{\alpha})(1-\frac{\mu}{\alpha})$. This gives $B_1=\frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}$. Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)},\ B_3=\frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}, \\ B_4 &= \frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)},\ B_5=\frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}, \\ B_6 &= \frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}. \end{aligned}$$

Thus, (2.5) can be written as

$$f_{H_n}(x)=B_1(1-\alpha x)^{-1}+B_2(1-\beta x)^{-1}+B_3(1-\gamma x)^{-1}+B_4(1-\delta x)^{-1}+B_5(1-\lambda x)^{-1}+B_6(1-\mu x)^{-1}.$$

This gives

$$\begin{aligned} f_{H_n}(x) &= B_1\sum_{n=0}^{\infty}\alpha^n x^n+B_2\sum_{n=0}^{\infty}\beta^n x^n+B_3\sum_{n=0}^{\infty}\gamma^n x^n+B_4\sum_{n=0}^{\infty}\delta^n x^n+B_5\sum_{n=0}^{\infty}\lambda^n x^n+B_6\sum_{n=0}^{\infty}\mu^n x^n \\ &= \sum_{n=0}^{\infty}(B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n+B_5\lambda^n+B_6\mu^n)x^n. \end{aligned}$$

Using the values of $B_1$, $B_2$, $B_3$, $B_4$, $B_5$ and $B_6$, we get

$$\begin{aligned} f_{H_n}(x) &= \sum_{n=0}^{\infty}H_n x^n \\ &= \sum_{n=0}^{\infty}\Big(\frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &\quad +\frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &\quad +\frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}\Big)x^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we get (2.3). □

Next, we give an identity related with generalized Hexanacci numbers and Hexanacci numbers.

**Theorem 2.3.** *For $n\geq 0$ and $m\geq 0$, the following identity holds:*

$$\begin{aligned} V_{m+n} &= H_{m-5}V_n+(H_{m-5}+H_{m-6})V_{n+1}+(H_{m-5}+H_{m-6}+H_{m-7})V_{n+2} \\ &\quad +(H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8})V_{n+3} \\ &\quad +(H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8}+H_{m-9})V_{n+4}+H_{m-4}V_{n+5} \\ &= H_{n-5}V_0+\left(\sum_{s=5}^{6}H_{n-s}\right)V_1+\left(\sum_{s=5}^{7}H_{n-s}\right)V_2+\left(\sum_{s=5}^{8}H_{n-s}\right)V_3+\left(\sum_{s=5}^{9}H_{n-s}\right)V_4+H_{n-4}V_5. \end{aligned} \tag{2.6}$$

*Proof.* We prove the identity by induction on $m$. If $m=0$ then

$$\begin{aligned} V_n &= H_{-5}V_n+(H_{-5}+H_{-6})V_{n+1}+(H_{-5}+H_{-6}+H_{-7})V_{n+2}+(H_{-5}+H_{-6}+H_{-7}+H_{-8})V_{n+3} \\ &\quad +(H_{-5}+H_{-6}+H_{-7}+H_{-8}+H_{-9})V_{n+4}+H_{-4}V_{n+5} \end{aligned}$$

which is true because $H_{-4} = 0, H_{-5} = 1, H_{-6} = -1, H_{-7} = 0, H_{-8} = 0, H_{-9} = 0.$ Assume that the equaliy holds for all $m \leq k$. For $m = k + 1$, we have

$$\begin{aligned}
V_{(k+1)+n} &= V_{n+k} + V_{n+k-1} + V_{n+k-2} + V_{n+k-3} + V_{n+k-4} + V_{n+k-5} \\
&= H_{k-5}V_n + (H_{k-5} + H_{k-6})V_{n+1} + (H_{k-5} + H_{k-6} + H_{k-7})V_{n+2} + (H_{k-5} + H_{k-6} + H_{k-7} + H_{k-8})V_{n+3} \\
&\quad +(H_{k-5} + H_{k-6} + H_{k-7} + H_{k-8} + H_{k-9})V_{n+4} + H_{k-4}V_{n+5} + H_{k-1-5}V_n \\
&\quad +H_{k-1-5}V_n + (H_{k-1-5} + H_{k-1-6})V_{n+1} + (H_{k-1-5} + H_{k-1-6} + H_{k-1-7})V_{n+2} \\
&\quad +(H_{k-1-5} + H_{k-1-6} + H_{k-1-7} + H_{k-1-8})V_{n+3} \\
&\quad +(H_{k-1-5} + H_{k-1-6} + H_{k-1-7} + H_{k-1-8} + H_{k-1-9})V_{n+4} + H_{k-1-4}V_{n+5} \\
&\quad +H_{k-2-5}V_n + (H_{k-2-5} + H_{k-2-6})V_{n+1} + (H_{k-2-5} + H_{k-2-6} + H_{k-2-7})V_{n+2} \\
&\quad +(H_{k-2-5} + H_{k-2-6} + H_{k-2-7} + H_{k-2-8})V_{n+3} \\
&\quad +(H_{k-2-5} + H_{k-2-6} + H_{k-2-7} + H_{k-2-8} + H_{k-2-9})V_{n+4} + H_{k-2-4}V_{n+5} \\
&\quad +H_{k-3-5}V_n + (H_{k-3-5} + H_{k-3-6})V_{n+1} + (H_{k-3-5} + H_{k-3-6} + H_{k-3-7})V_{n+2} \\
&\quad +(H_{k-3-5} + H_{k-3-6} + H_{k-3-7} + H_{k-3-8})V_{n+3} \\
&\quad +(H_{k-3-5} + H_{k-3-6} + H_{k-3-7} + H_{k-3-8} + H_{k-3-9})V_{n+4} + H_{k-3-4}V_{n+5} \\
&\quad +H_{k-4-5}V_n + (H_{k-4-5} + H_{k-4-6})V_{n+1} + (H_{k-4-5} + H_{k-4-6} + H_{k-4-7})V_{n+2} \\
&\quad +(H_{k-4-5} + H_{k-4-6} + H_{k-4-7} + H_{k-4-8})V_{n+3} \\
&\quad +(H_{k-4-5} + H_{k-4-6} + H_{k-4-7} + H_{k-4-8} + H_{k-4-9})V_{n+4} + H_{k-4-4}V_{n+5} \\
&\quad +H_{k-5-5}V_n + (H_{k-5-5} + H_{k-5-6})V_{n+1} + (H_{k-5-5} + H_{k-5-6} + H_{k-5-7})V_{n+2} \\
&\quad +(H_{k-5-5} + H_{k-5-6} + H_{k-5-7} + H_{k-5-8})V_{n+3} \\
&\quad +(H_{k-5-5} + H_{k-5-6} + H_{k-5-7} + H_{k-5-8} + H_{k-5-9})V_{n+4} + H_{k-5-4}V_{n+5} \\
&= H_{k-4}V_n + (H_{k-4} + H_{k-5})V_{n+1} + (H_{k-4} + H_{k-5} + H_{k-6})V_{n+2} \\
&\quad +(H_{k-4} + H_{k-5} + H_{k-6} + H_{k-7})V_{n+3} \\
&\quad +(H_{k-4} + H_{k-5} + H_{k-6} + H_{k-7} + H_{k-8})V_{n+4} + H_{k-3}V_{n+5} \\
&= H_{(k+1)-5}V_n + (H_{(k+1)-5} + H_{(k+1)-6})V_{n+1} + (H_{(k+1)-5} + H_{(k+1)-6} + H_{(k+1)-7})V_{n+2} \\
&\quad +(H_{(k+1)-5} + H_{(k+1)-6} + H_{(k+1)-7} + H_{(k+1)-8})V_{n+3} \\
&\quad +(H_{(k+1)-5} + H_{(k+1)-6} + H_{(k+1)-7} + H_{(k+1)-8} + H_{(k+1)-9})V_{n+4} + H_{(k+1)-4}V_{n+5}.
\end{aligned}$$

By induction on $m$, this proves (2.6). □

The previous Theorem gives the following results as particular examples: for $n \geq 0$ and $m \geq 0$, we have (taking $V_n = H_n$)

$$\begin{aligned}
H_{m+n} &= H_{m-5}H_n + (H_{m-5} + H_{m-6})H_{n+1} + (H_{m-5} + H_{m-6} + H_{m-7})H_{n+2} \\
&\quad +(H_{m-5} + H_{m-6} + H_{m-7} + H_{m-8})H_{n+3} + \\
&\quad (H_{m-5} + H_{m-6} + H_{m-7} + H_{m-8} + H_{m-9})H_{n+4} + H_{m-4}H_{n+5},
\end{aligned}$$

and similarly

$$\begin{aligned}
E_{m+n} &= H_{m-5}E_n + (H_{m-5} + H_{m-6})E_{n+1} + (H_{m-5} + H_{m-6} + H_{m-7})E_{n+2} \\
&\quad +(H_{m-5} + H_{m-6} + H_{m-7} + H_{m-8})E_{n+3} \\
&\quad +(H_{m-5} + H_{m-6} + H_{m-7} + H_{m-8} + H_{m-9})E_{n+4} + H_{m-4}E_{n+5}.
\end{aligned}$$

Next, we present the Binet's formula for the generalized Hexanacci sequence.

**Lemma 2.4.** *The Binet's formula for the generalized Hexanacci sequence $\{V_n\}$ is given as*

$$\begin{aligned}
V_n &= H_{n-5}V_0 + (H_{n-5} + H_{n-6})V_1 + (H_{n-5} + H_{n-6} + H_{n-7})V_2 + (H_{n-5} + H_{n-6} + H_{n-7} + H_{n-8})V_3 \\
&\quad +(H_{n-5} + H_{n-6} + H_{n-7} + H_{n-8} + H_{n-9})V_4 + H_{n-4}V_5.
\end{aligned}$$

*Proof.* Take $n = 0$, and then replace $n$ with $m$ in Theorem 2.3. □

For another proof of the Lemma 2.4, see [18]. This Lemma is also a special case of a work on the $n$th $k$-generalized Fibonacci number (which is also called $k$-step Fibonacci number) in ( [2], Theorem 2.2).

**Corollary 2.5.** *The Binet's formula for the generalized Hexanacci sequence* $\{V_n\}$

$$\begin{aligned} V_n &= A_1\alpha^{n-10} + A_2\beta^{n-10} + A_3\gamma^{n-10} + A_4\delta^{n-10} + A_5\lambda^{n-10} + A_6\mu^{n-10} \\ &= \sum_{p=1}^{6} A_p x_p^{n-10}, \end{aligned}$$

*where*

$$\begin{aligned} A_1 &= \frac{\alpha-1}{7\alpha-12}(V_5\alpha^5 + (V_0+V_1+V_2+V_3+V_4)\alpha^4 + (V_1+V_2+V_3+V_4)\alpha^3 \\ &\quad +(V_2+V_3+V_4)\alpha^2 + (V_3+V_4)\alpha + V_4), \\ A_2 &= \frac{\beta-1}{7\beta-12}(V_5\beta^5 + (V_0+V_1+V_2+V_3+V_4)\beta^4 + (V_1+V_2+V_3+V_4)\beta^3 \\ &\quad +(V_2+V_3+V_4)\beta^2 + (V_3+V_4)\beta + V_4), \\ A_3 &= \frac{\gamma-1}{7\gamma-12}(V_5\gamma^5 + (V_0+V_1+V_2+V_3+V_4)\gamma^4 + (V_1+V_2+V_3+V_4)\gamma^3 \\ &\quad +(V_2+V_3+V_4)\gamma^2 + (V_3+V_4)\gamma + V_4), \\ A_4 &= \frac{\delta-1}{7\delta-12}(V_5\delta^5 + (V_0+V_1+V_2+V_3+V_4)\delta^4 + (V_1+V_2+V_3+V_4)\delta^3 \\ &\quad +(V_2+V_3+V_4)\delta^2 + (V_3+V_4)\delta + V_4), \\ A_5 &= \frac{\lambda-1}{7\lambda-12}(V_5\lambda^5 + (V_0+V_1+V_2+V_3+V_4)\lambda^4 + (V_1+V_2+V_3+V_4)\lambda^3 \\ &\quad +(V_2+V_3+V_4)\lambda^2 + (V_3+V_4)\lambda + V_4), \\ A_6 &= \frac{\mu-1}{7\mu-12}(V_5\mu^5 + (V_0+V_1+V_2+V_3+V_4)\mu^4 + (V_1+V_2+V_3+V_4)\mu^3 \\ &\quad +(V_2+V_3+V_4)\mu^2 + (V_3+V_4)\mu + V_4). \end{aligned}$$

*Proof.* The proof follows from Lemma 2.4 and (1.2). □

In fact, Corollary 2.5 is a special case of a result in ( [2], Remark 2.3).
The following Theorem presents some summation formulas of the generalized Hexanacci numbers.

**Theorem 2.6.** *For* $n \geq 0$, *we have the following formulas:*

**(a):** $\sum_{k=0}^{n} V_k = \frac{1}{5}(V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} + V_n - V_5 + V_3 + 2V_2 + 3V_1 + 4V_0)$,
**(b):** $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{5}(3V_{2n+2} + 2V_{2n} - V_{2n-1} + V_{2n-2} - 2V_{2n-3} + 2V_5 - 5V_4 + 3V_3 - 4V_2 + 4V_1 - 3V_0)$,
**(c):** $\sum_{k=0}^{n} V_{2k} = \frac{1}{5}(-2V_{2n+2} + 5V_{2n+1} + 2V_{2n} + 4V_{2n-1} + V_{2n-2} + 3V_{2n-3} - 3V_5 + 5V_4 - 2V_3 + 6V_2 - V_1 + 7V_0)$.

*Proof.*

**(a):** Using the recurrence relation

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6},$$

i.e.

$$V_n - V_{n-1} = V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6} = V_{n-6} + V_{n-5} + V_{n-4} + V_{n-3} + V_{n-2},$$

we obtain

$$\begin{aligned}
V_6 - V_5 &= V_0 + V_1 + V_2 + V_3 + V_4, \\
V_7 - V_6 &= V_1 + V_2 + V_3 + V_4 + V_5, \\
V_8 - V_7 &= V_2 + V_3 + V_4 + V_5 + V_6, \\
V_9 - V_8 &= V_3 + V_4 + V_5 + V_6 + V_7, \\
V_{10} - V_9 &= V_4 + V_5 + V_6 + V_7 + V_8, \\
&\vdots \\
V_n - V_{n-1} &= V_{n-6} + V_{n-5} + V_{n-4} + V_{n-3} + V_{n-2}, \\
V_{n+1} - V_n &= V_{n-5} + V_{n-4} + V_{n-3} + V_{n-2} + V_{n-1}, \\
V_{n+2} - V_{n+1} &= V_{n-4} + V_{n-3} + V_{n-2} + V_{n-1} + V_n, \\
V_{n+3} - V_{n+2} &= V_{n-3} + V_{n-2} + V_{n-1} + V_n + V_{n+1}, \\
V_{n+4} - V_{n+3} &= V_{n-2} + V_{n-1} + V_n + V_{n+1} + V_{n+2}, \\
V_{n+5} - V_{n+4} &= V_{n-1} + V_n + V_{n+1} + V_{n+2} + V_{n+3}, \\
V_{n+6} - V_{n+5} &= V_n + V_{n+1} + V_{n+2} + V_{n+3} + V_{n+4}.
\end{aligned}$$

If we add the both sides of the above equations, we get

$$\begin{aligned}
V_{n+6} - V_5 &= \sum_{k=0}^{n} V_k + \left(V_{n+1} - V_0 + \sum_{k=0}^{n} V_k\right) + \left(V_{n+2} + V_{n+1} - V_1 - V_0 + \sum_{k=0}^{n} V_k\right) \\
&\quad + \left(V_{n+3} + V_{n+2} + V_{n+1} - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k\right) \\
&\quad + \left(V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} - V_3 - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k\right),
\end{aligned}$$

or

$$5\sum_{k=0}^{n} V_k = V_{n+6} - V_{n+4} - 2V_{n+3} - 3V_{n+2} - 4V_{n+1} - V_5 + V_3 + 2V_2 + 3V_1 + 4V_0,$$

which maybe reduced easily to (a) by using (1.1) and dividing both sides by 4. Note that

$$\begin{aligned}
V_{n+6} - V_{n+4} - 2V_{n+3} - 3V_{n+2} - 4V_{n+1} &= (V_{n+5} + V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n) \\
&\quad - V_{n+4} - 2V_{n+3} - 3V_{n+2} - 4V_{n+1} \\
&= V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} + V_n.
\end{aligned}$$

**(b),(c):** We have the following equations;

$$\begin{aligned}
V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} - V_{-2}, \\
V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 - V_0, \\
V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 - V_2, \\
V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 - V_4, \\
V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 - V_6, \\
V_{13} &= V_{14} - V_{12} - V_{11} - V_{10} - V_9 - V_8, \\
V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} - V_{10}, \\
&\vdots \\
V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} - V_{2n-4} - V_{2n-5} - V_{2n-6}, \\
V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} - V_{2n-2} - V_{2n-3} - V_{2n-4}.
\end{aligned}$$

Now, adding these equations we have

$$\begin{aligned}-V_1+\sum_{k=0}^{n}V_{2k+1} &= \left(-V_0-V_2+V_{2n+2}+\sum_{k=0}^{n}V_{2k}\right)+\left(V_0-\sum_{k=0}^{n}V_{2k}\right)\\ &\quad+\left(V_{2n+1}-\sum_{k=0}^{n}V_{2k+1}\right)+\left(V_{2n}-\sum_{k=0}^{n}V_{2k}\right)+\left(V_{2n+1}+V_{2n-1}-V_{-1}-\sum_{k=0}^{n}V_{2k+1}\right)\\ &\quad+\left(V_{2n}+V_{2n-2}-V_{-2}-\sum_{k=0}^{n}V_{2k}\right),\end{aligned}$$

or

$$3\sum_{k=0}^{n}V_{2k+1}=-V_2+V_1-V_{-2}-V_{-1}+V_{2n+2}+2V_{2n+1}+2V_{2n}+V_{2n-1}+V_{2n-2}-2\sum_{k=0}^{n}V_{2k},$$

or using $V_{-1}=V_5-V_4-V_3-V_2-V_1-V_0$ and $V_{-2}=V_4-V_3-V_2-V_1-V_0-V_{-1}$

$$3\sum_{k=0}^{n}V_{2k+1}=-V_4+V_3+2V_1+V_0+V_{2n+2}+2V_{2n+1}+2V_{2n}+V_{2n-1}+V_{2n-2}-2\sum_{k=0}^{n}V_{2k}.$$

Note that

$$\begin{aligned}-V_{-2}-V_{-1} &= -V_4+V_3+V_2+V_1+V_0,\\ -V_2+V_1-V_{-2}-V_{-1} &= -V_2+V_1+(-V_4+V_3+V_2+V_1+V_0)=-V_4+V_3+2V_1+V_0.\end{aligned}$$

Similarly, we have the following equations;

$$\begin{aligned}V_2 &= V_3-V_1-V_0-V_{-1}-V_{-2}-V_{-3},\\ V_4 &= V_5-V_3-V_2-V_1-V_0-V_{-1},\\ V_6 &= V_7-V_5-V_4-V_3-V_2-V_1,\\ V_8 &= V_9-V_7-V_6-V_5-V_4-V_3,\\ V_{10} &= V_{11}-V_9-V_8-V_7-V_6-V_5,\\ V_{12} &= V_{13}-V_{11}-V_{10}-V_9-V_8-V_7,\\ V_{14} &= V_{15}-V_{13}-V_{12}-V_{11}-V_{10}-V_9,\\ &\vdots\\ V_{2n-2} &= V_{2n-1}-V_{2n-3}-V_{2n-4}-V_{2n-5}-V_{2n-6}-V_{2n-7},\\ V_{2n} &= V_{2n+1}-V_{2n-1}-V_{2n-2}-V_{2n-3}-V_{2n-4}-V_{2n-5}.\end{aligned}$$

Now, adding these equations, we have

$$\begin{aligned}-V_0+\sum_{k=0}^{n}V_{2k} &= \left(-V_1+\sum_{k=0}^{n}V_{2k+1}\right)+\left(V_{2n+1}-\sum_{k=0}^{n}V_{2k+1}\right)+\left(V_{2n}-\sum_{k=0}^{n}V_{2k}\right)\\ &\quad+\left(V_{2n+1}+V_{2n-1}-V_{-1}-\sum_{k=0}^{n}V_{2k+1}\right)+\left(V_{2n-2}+V_{2n}-V_{-2}-\sum_{k=0}^{n}V_{2k}\right)\\ &\quad+\left(V_{2n+1}+V_{2n-1}+V_{2n-3}-V_{-1}-V_{-3}-\sum_{k=0}^{n}V_{2k+1}\right),\end{aligned}$$

or

$$3\sum_{k=0}^{n}V_{2k}=(-V_{-3}-V_{-2}-2V_{-1}+V_0-V_1)+3V_{2n+1}+2V_{2n}+2V_{2n-1}+V_{2n-2}+V_{2n-3}-2\sum_{k=0}^{n}V_{2k+1},$$

or using $V_{-3} = V_3 - V_2 - V_1 - V_0 - V_{-1} - V_{-2}$,

$$3\sum_{k=0}^{n} V_{2k} = (-V_5 + V_4 + 2V_2 + V_1 + 3V_0) + 3V_{2n+1} + 2V_{2n} + 2V_{2n-1} + V_{2n-2} + V_{2n-3} - 2\sum_{k=0}^{n} V_{2k+1}.$$

Note that

$$\begin{aligned} -V_{-3} - V_{-2} - 2V_{-1} + V_0 - V_1 &= -(V_3 - V_2 - V_1 - V_0 - V_{-1} - V_{-2}) - V_{-2} - 2V_{-1} + V_0 - V_1 \\ &= -V_3 + V_2 + 2V_0 - V_{-1} = -V_3 + V_2 + 2V_0 - (V_5 - V_4 - V_3 - V_2 - V_1 - V_0) \\ &= -V_5 + V_4 + 2V_2 + V_1 + 3V_0. \end{aligned}$$

Solving the following system

$$\begin{aligned} 3\sum_{k=0}^{n} V_{2k+1} &= -V_4 + V_3 + 2V_1 + V_0 + V_{2n+2} + 2V_{2n+1} + 2V_{2n} + V_{2n-1} + V_{2n-2} - 2\sum_{k=0}^{n} V_{2k}, \\ 3\sum_{k=0}^{n} V_{2k} &= (-V_5 + V_4 + 2V_2 + V_1 + 3V_0) + 3V_{2n+1} + 2V_{2n} + 2V_{2n-1} + V_{2n-2} + V_{2n-3} - 2\sum_{k=0}^{n} V_{2k+1}, \end{aligned}$$

we find that

$$\begin{aligned} \sum_{k=0}^{n} V_{2k+1} &= \frac{1}{5}(3V_{2n+2} + 2V_{2n} - V_{2n-1} + V_{2n-2} - 2V_{2n-3} + 2V_5 - 5V_4 + 3V_3 - 4V_2 + 4V_1 - 3V_0), \\ \sum_{k=0}^{n} V_{2k} &= \frac{1}{5}(-2V_{2n+2} + 5V_{2n+1} + 2V_{2n} + 4V_{2n-1} + V_{2n-2} + 3V_{2n-3} - 3V_5 + 5V_4 - 2V_3 + 6V_2 - V_1 + 7V_0). \end{aligned}$$

Letting $V_k = H_k(resp.\ V_k = E_k)$ in Theorem 2.3, we have the following two Corollaries which present some summation formulas for Hexanacci and Hexanacci-Lucas numbers.

**Corollary 2.7.** *For $n \geq 0$, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} H_k = \frac{1}{5}(H_{n+5} - H_{n+3} - 2H_{n+2} - 3H_{n+1} + H_n - 1)$.
- **(b):** $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{5}(3H_{2n+2} + 2H_{2n} - H_{2n-1} + H_{2n-2} - 2H_{2n-3} + 2)$.
- **(c):** $\sum_{k=0}^{n} H_{2k} = \frac{1}{5}(-2H_{2n+2} + 5H_{2n+1} + 2H_{2n} + 4H_{2n-1} + H_{2n-2} + 3H_{2n-3} - 3)$.

**Corollary 2.8.** *For $n \geq 0$, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} E_k = \frac{1}{5}(E_{n+5} - E_{n+3} - 2E_{n+2} - 3E_{n+1} + E_n + 9)$.
- **(b):** $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{5}(3E_{2n+2} + 2E_{2n} - E_{2n-1} + E_{2n-2} - 2E_{2n-3} - 18)$.
- **(c):** $\sum_{k=0}^{n} E_{2k} = \frac{1}{5}(-2E_{2n+2} + 5E_{2n+1} + 2E_{2n} + 4E_{2n-1} + E_{2n-2} + 3E_{2n-3} + 27)$.

## 3. Gaussian Generalized Hexanacci Numbers

In this section, we introduce Gaussian generalized Hexanacci numbers and present Binet's formulas, the generating functions, and the summation formulas for Gaussian generalized Hexanacci numbers.

First, we recall Gaussian integers. A Gaussian integer $z$ is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. These numbers are denoted by $\mathbb{Z}[i]$. For more information about this kind of integers, see [6].

If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.

In 1963, Horadam [11] introduced the concept of complex Fibonacci number called as the Gaussian Fibonacci number. Pethe [17] defined the complex Tribonacci numbers at Gaussian integers, see [7]. There are other several studies dedicated to these sequences of Gaussian numbers such as the works in [1,3,4,8–10,12,13,15,16,21–23,25–27], among others.

Gaussian generalized Hexanacci numbers $\{GV_n\}_{n\geq 0} = \{GV_n(GV_0, GV_1, GV_2, GV_3, GV_4, GV_5)\}_{n\geq 0}$ are defined by

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4} + GV_{n-5} + GV_{n-6}, \tag{3.1}$$

with the initial conditions

$$\begin{aligned} GV_0 &= c_0 + (-c_0 - c_1 - c_2 - c_3 - c_4 + c_5)i,\ GV_1 = c_1 + c_0 i,\ GV_2 = c_2 + c_1 i, \\ GV_3 &= c_3 + c_2 i,\ GV_4 = c_4 + c_3 i,\ GV_5 = c_5 + c_4 i, \end{aligned}$$

not all being zero. The sequences $\{GV_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} - GV_{-(n-3)} - GV_{-(n-4)} - GV_{-(n-5)} + GV_{-(n-6)},$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (3.1) hold for all integer $n$. Note that for $n \geq 0$

$$GV_n = V_n + iV_{n-1},$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}.$$

Some Gaussian generalized Hexanacci numbers are with positive and negative integers are given in the following two tables.

Table 3. A few Gaussian generalized Hexanacci numbers with positive subscript

| $n$ | $GV_n$ |
|---|---|
| 0 | $c_0 + (-c_0 - c_1 - c_2 - c_3 - c_4 + c_5)i$ |
| 1 | $c_1 + c_0 i$ |
| 2 | $c_2 + c_1 i$ |
| 3 | $c_3 + c_2 i$ |
| 4 | $c_4 + c_3 i$ |
| 5 | $c_5 + c_4 i$ |
| 6 | $(c_0 + c_1 + c_2 + c_3 + c_4 + c_5) + c_5 i$ |
| 7 | $(c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5) + (c_0 + c_1 + c_2 + c_3 + c_4 + c_5)i$ |
| 8 | $(2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4 + 4c_5) + (c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5)i$ |
| 9 | $(4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4 + 8c_5) + (2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4 + 4c_5)i$ |
| 10 | $(8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4 + 16c_5) + (4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4 + 8c_5)i$ |
| 11 | $(16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4 + 32c_5) + (8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4 + 16c_5)i$ |

Table 4. A few Gaussian generalized Hexanacci numbers with negative subscript

| $n$ | $GV_{-n}$ |
|---|---|
| 0 | $c_0 + (-c_0 - c_1 - c_2 - c_3 - c_4 + c_5)i$ |
| 1 | $(-c_0 - c_1 - c_2 - c_3 - c_4 + c_5) + (2c_4 - c_5)i$ |
| 2 | $(2c_4 - c_5) + (2c_3 - c_4)i$ |
| 3 | $(2c_3 - c_4) + (2c_2 - c_3)i$ |
| 4 | $(2c_2 - c_3) + (2c_1 - c_2)i$ |
| 5 | $(2c_1 - c_2) + (2c_0 - c_1)i$ |
| 6 | $(2c_0 - c_1) + (-3c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 + 2c_5)i$ |
| 7 | $(-3c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 + 2c_5) + (c_0 + c_1 + c_2 + c_3 + 5c_4 - 3c_5)i$ |
| 8 | $(c_0 + c_1 + c_2 + c_3 + 5c_4 - 3c_5) + (4c_3 - 4c_4 + c_5)i$ |
| 9 | $(4c_3 - 4c_4 + c_5) + (4c_2 - 4c_3 + c_4)i$ |
| 10 | $(4c_2 - 4c_3 + c_4) + (4c_1 - 4c_2 + c_3)i$ |
| 11 | $(4c_1 - 4c_2 + c_3) + (4c_0 - 4c_1 + c_2)i$ |

In particular, $GV_n(0, 1, 1+i, 2+i, 4+2i, 8+4i)$ $(resp.GV_n(6-i, 1+6i, 3+i, 7+3i, 15+7i, 31+15i))$ is called the sequence of Gaussian Hexanacci ($resp.$Gaussian Hexanacci-Lucas) numbers and denoted by $GH_n$ $(resp.GE_n)$. More precisely, we define formally them as follows.

Gaussian Hexanacci numbers are defined by

$$GH_n = GH_{n-1} + GH_{n-2} + GH_{n-3} + GH_{n-4} + GH_{n-5},$$

with the initial conditions

$$GH_0 = 0, GH_1 = 1,\ GH_2 = 1 + i,\ GH_3 = 2 + i,\ GH_4 = 4 + 2i,\ GH_5 = 8 + 4i,$$

and Gaussian Hexanacci-Lucas numbers are defined by

$$GE_n = GE_{n-1} + GE_{n-2} + GE_{n-3} + GE_{n-4} + GE_{n-5},$$

with the initial conditions

$$GE_0 = 6 - i,\ GE_1 = 1 + 6i,\ GE_2 = 3 + i,\ GE_3 = 7 + 3i,\ GE_4 = 15 + 7i,\ GE_5 = 31 + 15i.$$

Note that for $n \geq 0$

$$GH_n = M_n + iM_{n-1},\ GE_n = R_n + iR_{n-1},$$

and

$$GH_{-n} = M_{-n} + iM_{-n-1}, GE_{-n} = R_{-n} + iR_{-n-1}.$$

Next, we present the first few values of the Gaussian Hexanacci and Hexanacci-Lucas numbers with positive and negative subscripts in the following Table 4:

Table 4. A few Gaussian Hexanacci and Hexanacci-Lucas Numbers

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $GH_n$ | $0$ | $1$ | $1+i$ | $2+i$ | $4+2i$ | $8+4i$ | $16+8i$ | $32+16i$ | $63+32i$ |
| $GH_{-n}$ | $0$ | $0$ | $0$ | $0$ | $i$ | $1-i$ | $-1$ | $0$ | $0$ |
| $GE_n$ | $6-i$ | $1+6i$ | $3+i$ | $7+3i$ | $15+7i$ | $31+15i$ | $63+31i$ | $120+63i$ | $239+120i$ |
| $GE_{-n}$ | $6-i$ | $-1-i$ | $-1-i$ | $-1-i$ | $-1-i$ | $-1+11i$ | $11-8i$ | $-8-i$ | $-1-i$ |

The following Theorem presents the generating function $f_{GV_n}(x) = \sum\limits_{n=0}^{\infty} GVx^n$ of Gaussian generalized Hexanacci numbers.

**Theorem 3.1.** *The generating function of Gaussian generalized Hexanacci numbers is given as*

$$f_{GV_n}(x) = \frac{\begin{array}{c} GV_0 + (GV_1 - GV_0)x + (GV_2 - GV_1 - GV_0)x^2 + (GV_3 - GV_2 - GV_1 - GV_0)x^3 \\ +(GV_4 - GV_3 - GV_2 - GV_1 - GV_0)x^4 + (GV_5 - GV_4 - GV_3 - GV_2 - GV_1 - GV_0)x^5 \end{array}}{1 - x - x^2 - x^3 - x^4 - x^5 - x^6}. \tag{3.2}$$

*Proof.* Using (3.1) and some calculation, we obtain

$$\begin{aligned} & f_{GV_n}(x) - xf_{GV_n}(x) - x^2 f_{GV_n}(x) - x^3 f_{GV_n}(x) - x^4 f_{GV_n}(x) - x^5 f_{GV_n}(x) - x^6 f_{GV_n}(x) \\ = \ & GV_0 + (GV_1 - GV_0)x + (GV_2 - GV_1 - GV_0)x^2 + (GV_3 - GV_2 - GV_1 - GV_0)x^3 \\ & +(GV_4 - GV_3 - GV_2 - GV_1 - GV_0)x^4 + (GV_5 - GV_4 - GV_3 - GV_2 - GV_1 - GV_0)x^5, \end{aligned}$$

which gives (3.2). □

The previous Theorem gives the following results as particular examples: the generating function of Gaussian Hexanacci numbers is

$$f_{GH_n}(x) = \frac{x + ix^2}{1 - x - x^2 - x^3 - x^4 - x^5 - x^6},$$

and the generating function of Gaussian Hexanacci-Lucas numbers is

$$f_{GE_n}(x) = \frac{(6-i) - (5-7i)\,x - (4+4i)\,x^2 - (3+3i)\,x^3 - (2+2i)\,x^4 - (1+i)\,x^5}{1 - x - x^2 - x^3 - x^4 - x^5 - x^6}.$$

We now present the Binet's formula for the Gaussian generalized Hexanacci numbers.

**Theorem 3.2.** *Binet's formula for the Gaussian generalized Hexanacci numbers*

$$GV_n = \sum_{p=1}^{6} A_p x_p^{n-10} + i \sum_{p=1}^{6} A_p x_p^{n-11},$$

*where $A_1, A_2, A_3, A_4, A_5$ and $A_6$ are as in Corollary 2.5.*

*Proof.* The proof follows from Corollary 2.5 and $GV_n = V_n + iV_{n-1}$. □

The previous Theorem gives the following results as particular examples: The Binet's formula for the Gaussian Hexanacci numbers are

$$GH_n = \sum_{p=1}^{6} \frac{x_p^{n+4}}{\prod\limits_{\substack{q=1 \\ p\neq q}}^{6} (x_p - x_q)} + i \sum_{p=1}^{6} \frac{x_p^{n+3}}{\prod\limits_{\substack{q=1 \\ p\neq q}}^{6} (x_p - x_q)},$$

or

$$GH_n = \sum_{p=1}^{6} \frac{x_p - 1}{7x_p - 12} x_p^{n-1} + i \sum_{p=1}^{6} \frac{x_p - 1}{7x_p - 12} x_p^{n-2},$$

and the Binet's formula for the Gaussian Hexanacci-Lucas numbers are

$$GE_n = \sum_{p=1}^{6} x_p^n + i \sum_{p=1}^{6} x_p^{n-1}.$$

Following the method as staded in the proof of Theorem 2.6, we obtain the following three results.

**Theorem 3.3.** *For $n \geq 0$ we have the following formulas:*

**(a):** *(Sum of the Gaussian generalized Hexanacci numbers)*

$$\sum_{k=0}^{n} GV_k = \frac{1}{5}(GV_{n+5} - GV_{n+3} - 2GV_{n+2} - 3GV_{n+1} + GV_n - GV_5 + GV_3 + 2GV_2 + 3GV_1 + 4GV_0),$$

**(b):** $\sum_{k=0}^{n} GV_{2k+1} = \frac{1}{5}(3GV_{2n+2} + 2GV_{2n} - GV_{2n-1} + GV_{2n-2} - 2GV_{2n-3} + 2GV_5 - 5GV_4 + 3GV_3 - 4GV_2 + 4GV_1 - 3GV_0)$,

**(c):** $\sum_{k=0}^{n} GV_{2k} = \frac{1}{5}(-2GV_{2n+2} + 5GV_{2n+1} + 2GV_{2n} + 4GV_{2n-1} + GV_{2n-2} + 3GV_{2n-3} - 3GV_5 + 5GV_4 - 2GV_3 + 6GV_2 - GV_1 + 7GV_0)$.

*Proof.* (a), (b) and (c) can be proved exactly as in the proof of Theorem 2.6. □

As special cases of the above Theorem, we have the following two Corollaries. First one present summation formulas of Gaussian Hexanacci numbers.

**Corollary 3.4.** *For $n \geq 0$ we have the following formulas:*

**(a):** *(Sum of the Gaussian Hexanacci numbers)*

$$\sum_{k=0}^{n} GH_k = \frac{1}{5}(GH_{n+5} - GH_{n+3} - 2GH_{n+2} - 3GH_{n+1} + GH_n - 1),$$

**(b):** $\sum_{k=0}^{n} GH_{2k+1} = \frac{1}{5}(3GH_{2n+2} + 2GH_{2n} - GH_{2n-1} + GH_{2n-2} - 2GH_{2n-3} + 2)$,

**(c):** $\sum_{k=0}^{n} GH_{2k} = \frac{1}{5}(-2GH_{2n+2} + 5GH_{2n+1} + 2GH_{2n} + 4GH_{2n-1} + GH_{2n-2} + 3GH_{2n-3} - 3)$.

**Corollary 3.5.** *For $n \geq 0$ we have the following formulas:*

**(a):** *(Sum of the Gaussian Hexanacci-Lucas numbers)*

$$\sum_{k=0}^{n} GE_k = \frac{1}{5}(GE_{n+5} - GE_{n+3} - 2GE_{n+2} - 3GE_{n+1} + GE_n + 9),$$

**(b):** $\sum_{k=0}^{n} GE_{2k+1} = \frac{1}{5}(3GE_{2n+2} + 2GE_{2n} - GE_{2n-1} + GE_{2n-2} - 2GE_{2n-3} - 18)$,

**(c):** $\sum_{k=0}^{n} GE_{2k} = \frac{1}{5}(-2GE_{2n+2} + 5GE_{2n+1} + 2GE_{2n} + 4GE_{2n-1} + GE_{2n-2} + 3GE_{2n-3} + 27)$.

## 4. Basic Relations and Simson Formulas

In this section, we obtain some identities of Hexanacci numbers and Hexanacci-Lucas numbers and some identities of Gaussian Hexanacci numbers and Gaussian Hexanacci-Lucas numbers. Moreover, we present Simson formulas of these numbers.

Some basic relaton between $\{H_n\}$ and $\{E_n\}$ ($resp.\{GH_n\}$ and $\{GE_n\}$ ) are demonstrated in the following two theorems.

**Theorem 4.1.** *The following equalities are true:*

$$\begin{aligned} E_n &= -H_{n+4} + H_{n+2} + 9H_{n+1} - 2H_n - H_{n-1}, \\ E_{2n+1} &= -235H_{n+4} - 686H_{n+3} + 2393H_{n+2} - 80H_{n+1} - 75H_n - 58H_{n-1}, \\ E_{2n} &= -113H_{n+4} - 352H_{n+3} + 1195H_{n+2} - 33H_{n+1} - 42H_n - 33H_{n-1}, \end{aligned} \quad (4.1)$$

*and*

$$205\,937H_n = 2401E_{n+4} + 1715E_{n+3} + 539E_{n+2} - 1477E_{n+1} - 4933E_n + 18\,562E_{n-1}.$$

*Proof.* The last six identities. For example, to show (4.1), writing

$$E_n = z_1H_{n+4} + z_2H_{n+3} + z_3H_{n+2} + z_4H_{n+1} + z_5H_n + z_6H_{n-1},$$

and solving the system of equations

$$\begin{aligned} E_0 &= z_1H_4 + z_2H_3 + z_3H_2 + z_4H_1 + z_5H_0 + z_6H_{-1}, \\ E_1 &= z_1H_5 + z_2H_4 + z_3H_3 + z_4H_2 + z_5H_1 + z_6H_0, \\ E_2 &= z_1H_6 + z_2H_5 + z_3H_4 + z_4H_3 + z_5H_2 + z_6H_1, \\ E_3 &= z_1H_7 + z_2H_6 + z_3H_5 + z_4H_4 + z_5H_3 + z_6H_2, \\ E_4 &= z_1H_8 + z_2H_7 + z_3H_6 + z_4H_5 + z_5H_4 + z_6H_3, \\ E_5 &= z_1H_9 + z_2H_8 + z_3H_7 + z_4H_6 + z_5H_5 + z_6H_4, \end{aligned}$$

we find that $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, $z_4 = 9$, $z_5 = -2$, $z_6 = -1$. The other equalities can be proved similarly. □

**Theorem 4.2.** *The following equalities are true:*

$$\begin{aligned} GE_n &= -GH_{n+4} + GH_{n+2} + 9GH_{n+1} - 2GH_n - GH_{n-1}, \\ GE_{2n+1} &= -235GH_{n+4} - 686GH_{n+3} + 2393GH_{n+2} - 80GH_{n+1} - 75GH_n - 58GH_{n-1}, \\ GE_{2n} &= -113GH_{n+4} - 352GH_{n+3} + 1195GH_{n+2} - 33GH_{n+1} - 42GH_n - 33GH_{n-1} \end{aligned} \quad (4.2)$$

*and*

$$205\,937GH_n = 2401GE_{n+4} + 1715GE_{n+3} + 539GE_{n+2} - 1477GE_{n+1} - 4933GE_n + 18\,562GE_{n-1}.$$

*Proof.* The last four identities. For example, to show (4.2), writing

$$GE_n = z_1GH_{n+4} + z_2GH_{n+3} + z_3GH_{n+2} + z_4GH_{n+1} + z_5GH_n + z_6GH_{n-1},$$

and solving the system of equations

$$\begin{aligned} GE_0 &= z_1GH_4 + z_2GH_3 + z_3GH_2 + z_4GH_1 + z_5GH_0 + z_6GH_{-1}, \\ GE_1 &= z_1GH_5 + z_2GH_4 + z_3GH_3 + z_4GH_2 + z_5GH_1 + z_6GH_0, \\ GE_2 &= z_1GH_6 + z_2GH_5 + z_3GH_4 + z_4GH_3 + z_5GH_2 + z_6GH_1, \\ GE_3 &= z_1GH_7 + z_2GH_6 + z_3GH_5 + z_4GH_4 + z_5GH_3 + z_6GH_2, \\ GE_4 &= z_1GH_8 + z_2GH_7 + z_3GH_6 + z_4GH_5 + z_5GH_4 + z_6GH_3, \\ GE_5 &= z_1GH_9 + z_2GH_8 + z_3GH_7 + z_4GH_6 + z_5GH_5 + z_6GH_4, \end{aligned}$$

we find that $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, $z_4 = 9$, $z_5 = -2$, $z_6 = -1$. The other equalities can be proved similarly.

Using the relations $GH_n = H_n + iH_{n-1}$, $GE_n = E_n + iE_{n-1}$ and the identity $E_n = -H_{n+4} + H_{n+2} + 9H_{n+1} - 2H_n - H_{n-1}$ (see Theorem 4.1) we obtain the identity (4.2). In fact, note that

$$\begin{aligned} GE_n &= E_n + iE_{n-1} \\ &= (-H_{n+4} + H_{n+2} + 9H_{n+1} - 2H_n - H_{n-1}) + i(-H_{n+3} + H_{n+1} + 9H_n - 2H_{n-1} - H_{n-2}) \\ &= -(H_{n+4} + iH_{n+3}) + (H_{n+2} + iH_{n+1}) + 9(H_{n+1} + iH_n) - 2(H_n + iH_{n-1}) - (H_{n-1} + iH_{n-2}) \\ &= -GH_{n+4} + GH_{n+2} + 9GH_{n+1} - 2GH_n - GH_{n-1}. \end{aligned}$$

The other equalities can be proved similarly. □

We present an identity related with Gaussian generalized Hexanacci numbers and Hexanacci numbers.

**Theorem 4.3.** *For $n \geq 0$ and $m \geq 0$, the following identity holds:*

$$\begin{aligned} GV_{m+n} &= H_{m-5}GV_n + (H_{m-5}+H_{m-6})GV_{n+1} + (H_{m-5}+H_{m-6}+H_{m-7})GV_{n+2} \\ &\quad +(H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8})GV_{n+3} + (H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8}+H_{m-9})GV_{n+4} + H_{m-4}GV_{n+5}. \end{aligned} \tag{4.3}$$

*Proof.* The identity (4.3) can be proved by induction on $m$ as in Theorem 2.3. □

In particular, taking $GV_{n+m} = GH_{n+m}$ $(n \geq 0,\ m \geq 0)$, we have,

$$\begin{aligned} GH_{m+n} &= H_{m-5}GH_n + (H_{m-5}+H_{m-6})GH_{n+1} + (H_{m-5}+H_{m-6}+H_{m-7})GH_{n+2} \\ &\quad +(H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8})GH_{n+3} + (H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8}+H_{m-9})GH_{n+4} + H_{m-4}GH_{n+5}. \end{aligned}$$

and similarly

$$\begin{aligned} GE_{m+n} &= H_{m-5}GE_n + (H_{m-5}+H_{m-6})GE_{n+1} + (H_{m-5}+H_{m-6}+H_{m-7})GE_{n+2} \\ &\quad +(H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8})GE_{n+3} + (H_{m-5}+H_{m-6}+H_{m-7}+H_{m-8}+H_{m-9})GE_{n+4} + H_{m-4}GE_{n+5}. \end{aligned}$$

One of the oldest and best known identities for the Fibonacci sequence $\{F_n\}$ is

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$

which was derived first by R. Simson in 1753 [19]. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n,$$

and called as Simson or Cassini formula (Identity). The following Theorem gives a generalization of this result to generalized Hexanacci numbers.

**Theorem 4.4.** *(Simson's formula of generalized Hexanacci numbers) For all integers $n$, we have*

$$\begin{vmatrix} V_{n+5} & V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} & V_{n-5} \end{vmatrix} = (-1)^n \begin{vmatrix} V_5 & V_4 & V_3 & V_2 & V_1 & V_0 \\ V_4 & V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_3 & V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} & V_{-5} \end{vmatrix}. \tag{4.4}$$

*Proof.* (4.4) is given in Soykan [24]. □

**Corollary 4.5.** *For all integers $n$, we have*

**(a):** (Simson's formula of Hexanacci numbers)

$$\begin{vmatrix} H_{n+5} & H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} & H_{n-5} \end{vmatrix} = (-1)^n,$$

**(b):** (Simpson's formula of Hexanacci-Lucas numbers)

$$\begin{vmatrix} E_{n+5} & E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} & E_{n-5} \end{vmatrix} = -205\,937(-1)^n = 205\,937(-1)^{n+1}.$$

**(c):** (Simpson's formula of Gaussian Hexanacci numbers)

$$\begin{vmatrix} GH_{n+5} & GH_{n+4} & GH_{n+3} & GH_{n+2} & GH_{n+1} & GH_{n} \\ GH_{n+4} & GH_{n+3} & GH_{n+2} & GH_{n+1} & GH_{n} & GH_{n-1} \\ GH_{n+3} & GH_{n+2} & GH_{n+1} & GH_{n} & GH_{n-1} & GH_{n-2} \\ GH_{n+2} & GH_{n+1} & GH_{n} & GH_{n-1} & GH_{n-2} & GH_{n-3} \\ GH_{n+1} & GH_{n} & GH_{n-1} & GH_{n-2} & GH_{n-3} & GH_{n-4} \\ GH_{n} & GH_{n-1} & GH_{n-2} & GH_{n-3} & GH_{n-4} & GH_{n-5} \end{vmatrix} = (2-i)(-1)^n,$$

**(d):** (Simpson's formula of Gaussian Hexanacci-Lucas numbers)

$$\begin{vmatrix} GE_{n+5} & GE_{n+4} & GE_{n+3} & GE_{n+2} & GE_{n+1} & GE_{n} \\ GE_{n+4} & GE_{n+3} & GE_{n+2} & GE_{n+1} & GE_{n} & GE_{n-1} \\ GE_{n+3} & GE_{n+2} & GE_{n+1} & GE_{n} & GE_{n-1} & GE_{n-2} \\ GE_{n+2} & GE_{n+1} & GE_{n} & GE_{n-1} & GE_{n-2} & GE_{n-3} \\ GE_{n+1} & GE_{n} & GE_{n-1} & GE_{n-2} & GE_{n-3} & GE_{n-4} \\ GE_{n} & GE_{n-1} & GE_{n-2} & GE_{n-3} & GE_{n-4} & GE_{n-5} \end{vmatrix} = -205\,937(2-i)(-1)^n.$$

## 5. Matrix Formulations of $V_n$ and $GV_n$

In this section, we present some matrix formulation of generalized Hexanacci numbers and Gaussian generalized Hexanacci numbers.

We define the square matrix $A$ of order 6 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

such that $\det A = 1$. Induction proof may be used to establish

$$A^n = \begin{pmatrix} H_{n+1} & a_{12} & H_n + H_{n-1} + H_{n-2} + H_{n-3} & H_n + H_{n-1} + H_{n-2} & H_n + H_{n-1} & H_n \\ H_n & a_{22} & H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} & H_{n-1} + H_{n-2} + H_{n-3} & H_{n-1} + H_{n-2} & H_{n-1} \\ H_{n-1} & a_{32} & H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} & H_{n-2} + H_{n-3} + H_{n-4} & H_{n-2} + H_{n-3} & H_{n-2} \\ H_{n-2} & a_{42} & H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6} & H_{n-3} + H_{n-4} + H_{n-5} & H_{n-3} + H_{n-4} & H_{n-3} \\ H_{n-3} & a_{52} & H_{n-4} + H_{n-5} + H_{n-6} + H_{n-7} & H_{n-4} + H_{n-5} + H_{n-6} & H_{n-4} + H_{n-5} & H_{n-4} \\ H_{n-4} & a_{62} & H_{n-5} + H_{n-6} + H_{n-7} + H_{n-8} & H_{n-5} + H_{n-6} + H_{n-7} & H_{n-5} + H_{n-6} & H_{n-5} \end{pmatrix},$$

where

$$\begin{aligned} a_{12} &= H_n + H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4}, \\ a_{22} &= H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5}, \\ a_{32} &= H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6}, \\ a_{42} &= H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6} + H_{n-7}, \\ a_{52} &= H_{n-4} + H_{n-5} + H_{n-6} + H_{n-7} + H_{n-8}, \\ a_{62} &= H_{n-5} + H_{n-6} + H_{n-7} + H_{n-8} + H_{n-9}. \end{aligned}$$

Matrix formulation of $H_n$ and $E_n$ can be given as

$$\begin{pmatrix} H_{n+5} \\ H_{n+4} \\ H_{n+3} \\ H_{n+2} \\ H_{n+1} \\ H_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} H_5 \\ H_4 \\ H_3 \\ H_2 \\ H_1 \\ H_0 \end{pmatrix}$$

and

$$\begin{pmatrix} E_{n+5} \\ E_{n+4} \\ E_{n+3} \\ E_{n+2} \\ E_{n+1} \\ E_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} E_5 \\ E_4 \\ E_3 \\ E_2 \\ E_1 \\ E_0 \end{pmatrix},$$

which are proved by induction on $n$. Similarly, matrix formulation of $V_n$ can be given as

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_5 \\ V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

Consider the matrices $N_H$, $M_H$ defined by

$$N_H = \begin{pmatrix} 8+4i & 4+2i & 2+i & 1+i & 1 & 0 \\ 4+2i & 2+i & 1+i & 1 & 0 & 0 \\ 2+i & 1+i & 1 & 0 & 0 & 0 \\ 1+i & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 1-i \end{pmatrix},$$

$$M_H = \begin{pmatrix} GH_{n+5} & GH_{n+4} & GH_{n+3} & GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+4} & GH_{n+3} & GH_{n+2} & GH_{n+1} & GH_n & GH_{n-1} \\ GH_{n+3} & GH_{n+2} & GH_{n+1} & GH_n & GH_{n-1} & GH_{n-2} \\ GH_{n+2} & GH_{n+1} & GH_n & GH_{n-1} & GH_{n-2} & GH_{n-3} \\ GH_{n+1} & GH_n & GH_{n-1} & GH_{n-2} & GH_{n-3} & GH_{n-4} \\ GH_n & GH_{n-1} & GH_{n-2} & GH_{n-3} & GH_{n-4} & GH_{n-5} \end{pmatrix}.$$

The next Theorem presents the relations between $A^n, N_H$ and $M_H$.

**Theorem 5.1.** *For $n \geq 4$, we have*

$$A^n N_H = M_H.$$

*Proof.* The proof requires some lengthy calculation, so we omit it. □

Consider the matrices $N_E$, $M_E$ defined by as follows:

$$N_E = \begin{pmatrix} 31+15i & 15+7i & 7+3i & 3+i & 1+6i & 6-i \\ 15+7i & 7+3i & 3+i & 1+6i & 6-i & -1-i \\ 7+3i & 3+i & 1+6i & 6-i & -1-i & -1-i \\ 3+i & 1+6i & 6-i & -1-i & -1-i & -1-i \\ 1+6i & 6-i & -1-i & -1-i & -1-i & -1-i \\ 6-i & -1-i & -1-i & -1-i & -1-i & -1+11i \end{pmatrix},$$

$$M_E = \begin{pmatrix} GE_{n+5} & GE_{n+4} & GE_{n+3} & GE_{n+2} & GE_{n+1} & GE_n \\ GE_{n+4} & GE_{n+3} & GE_{n+2} & GE_{n+1} & GE_n & GE_{n-1} \\ GE_{n+3} & GE_{n+2} & GE_{n+1} & GE_n & GE_{n-1} & GE_{n-2} \\ GE_{n+2} & GE_{n+1} & GE_n & GE_{n-1} & GE_{n-2} & GE_{n-3} \\ GE_{n+1} & GE_n & GE_{n-1} & GE_{n-2} & GE_{n-3} & GE_{n-4} \\ GE_n & GE_{n-1} & GE_{n-2} & GE_{n-3} & GE_{n-4} & GE_{n-5} \end{pmatrix}.$$

The following Theorem presents the relations between $A^n$, $N_E$ and $M_E$.

**Theorem 5.2.** *We have*

$$A^n N_E = M_E.$$

*Proof.* The proof requires some lengthy calculation, so we omit it. □

## 6. Conclusions

- In the section 1, we present some background about generalized Hexanacci numbers.
- In the section 2, we present Binet's formulas, the generating functions, and the summation formulas for generalized Hexanacci numbers.
- In the section 3, first we recall Gaussian integers and then we define Gaussian generalized Hexanacci numbers and as special cases, we investigate Gaussian Hexanacci and Gaussian Hexanacci-Lucas numbers, with their properties such as the generating functions, Binet's formulas and sums formulas of these Gaussian numbers.
- In the section 4, we obtain some identities of Hexanacci numbers and Hexanacci-Lucas numbers and some identities of Gaussian Hexanacci numbers and Gaussian Hexanacci-Lucas numbers. Furthermore, we present Simson formulas of those numbers.
- In the section 5, we give some matrix formulation of generalized Hexanacci numbers and Gaussian generalized Hexanacci numbers.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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