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Grüss Type Integral Inequalities For Generalized $\eta-$ Conformable Fractional Integrals

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Abstract. Our aim in this paper is to establish new $\eta-$conformable fractional integral. For this purpose new inequalities are obtained by using generalized $\eta-$conformable fractional integral with the help of Grüss type integrals. The inequalities that exist in the literature are obtained in case of some special choices, which shows that the inequality we achieve is a more general inequality.

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1. Introduction

If f and g are two continous functions on $[a,b]$ satisfying $m \leq f(t) \leq M$ and $p \leq g(t) \leq P$ for all $t \in [a,b]$, $m, M, p, P \in \mathbb{R}$, then

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt \right| \leq \frac{1}{4}(M-m)(P-p). \tag{1.1}$$

Inequality (1.1) is well-known in literature as Grüss inequality.

In addition to the numerous generalizations of the Grüss inequality, there are also applications in the field. Some of these are statistics, coding theory, can be counted as numerical analysis. Recently, many researchers have had numerous generalizations, variants, and extensions of grüss inequality in the literature, to name a few, see [1–10].

We given two theorems involving generalizations of Grüss type integral inequalities. We used a new fractional integral operator. Also, some special cases of the main results are mentioned.

Definition 1.1 ([4]). Let $f \in L_1[0,\infty)$, the Riemann- Liouville fractional integral of order $\alpha \geq 0$ is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\, dx,$$
$$I^0 f(t) = f(t),$$

where Γ is the gamma function.

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Definition 1.2. A function f is said to be invex with respect to arbitrary bifunction $\eta(.,.)$ if

$$f(u+\tau\eta(u,v)) \leq (1-t)f(u)+tf(v).$$

The function f is said to be preincave if and only if f is invex. For $\eta(u,v)=u-v$, the invex functions becomes convex functions in the classical sense.

Definition 1.3 ([3]). Let f be defined on $[a,b]$ and $\alpha \in \mathbb{C}$, $\mathbb{R}(\alpha)>0$, $\rho>0$, then the mixed left conformable fractional of f is defined by

$$ {}_a^bJ^{\alpha,\rho}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b f(s)\left(\frac{(b-s)^\rho-(b-x)^\rho}{\rho}\right)^{\alpha-1}(b-s)^{\rho-1}dsa$$

and the mixed right conformable fractional integral of f is defined by

$$ {}_a^bJ^{\alpha,\rho}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(s)\left(\frac{(s-a)^\rho-(x-a)^\rho}{\rho}\right)^{\alpha-1}(s-a)^{\rho-1}ds.$$

We introduced a new definition of the fractional integral called generalized mixed $\eta-$conformable fractional integral below.

Definition 1.4. Let f be defined on $[a,b]$ and $\alpha \in \mathbb{C}$, $\mathbb{R}(\alpha)>0$, $\rho>0,\eta$ be defined on $[a,b]\times[a,b]$, then the generalized mixed left $\eta-$conformable fractional of f is defined by

$$ {}_a^bJ_\eta^{\alpha,\rho}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a+\eta(x,a)}^b f(s)\left(\frac{(\eta(b,s))^\rho-(a-x+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}(\eta(b,s))^{\rho-1}ds$$

and the generalized mixed right $\eta-$conformable fractional integral of f is defined by

$$ {}_a^bJ_\eta^{\alpha,\rho}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^{a+\eta(x,a)} f(s)\left(\frac{(\eta(s,a))^\rho-(x-b+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}(\eta(s,a))^{\rho-1}ds,$$

where Γ is the gamma function.

2. Main Results

In this section, we have obtained new results for Grüss type inequalities by using generalized mixed $\eta-$conformable fractional integrals.

Theorem 2.1. *Let f be an intagrable function on $[0,\infty)$, $t>0$, $\alpha,\beta,\rho>0$. Suppose that there exist two integrable function φ_1,φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t)\leq f(t)\leq\varphi_2(t) \quad \forall t\in[0,\infty). \tag{2.1}$$

Then, we obtain the following inequality for generalized mixed $\eta-$ conformable fractional integrals;

$$ {}_a^b\mathrm{J}_\eta^{\beta,\rho}\varphi_1(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t)+{}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_2(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}f(t)\geq{}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_2(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}\varphi_1(t)+{}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}f(t).$$

Proof. From (2.1) for all $x\geq0$, $y\geq0$, we have

$$(\varphi_2(x)-f(x))(f(y)-\varphi_1(y))\geq0,$$
$$\Rightarrow\varphi_2(x)f(y)+\varphi_1(y)f(x)\geq\varphi_1(y)\varphi_2(x)+f(x)f(y). \tag{2.2}$$

If we multiply both sides of (2.2) by $\frac{1}{\Gamma(\alpha)}\left(\frac{(\eta(b,x))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}$, and integrate with respect to x on $(a,a+\eta(t,a))$, we get

$$\begin{aligned}
&\frac{f(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-x))^\rho-(\eta(a-t+\eta(b,a)))^\rho}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}\varphi_2(x)dx\\
&\quad+\frac{\varphi_1(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-x))^\rho-(\eta(a-t+\eta(b,a)))^\rho}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}f(x)dx\\
&\quad\geq\frac{\varphi_1(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-x))^\rho-(\eta(a-t+\eta(b,a)))^\rho}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}\varphi_2(x)dx\\
&\qquad+\frac{f(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-x))^\rho-(\eta(a-t+\eta(b,a)))^\rho}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}f(x)dx.
\end{aligned}$$

Using the definition of generalized mixed $\eta-$ conformable fractional integrals, we write

$$f(y){}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)+\varphi_1(y){}_a^bJ_\eta^{\alpha,\rho}f(t)\geq\varphi_1(y){}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)+f(y){}_a^bJ_\eta^{\alpha,\rho}f(t). \tag{2.3}$$

If we multiply both sides of (2.3) by $\frac{1}{\Gamma(\beta)}\left(\frac{(\eta(b,y))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,y))^{\rho-1}$,and integrate with respect to y on $(a, a+\eta(t,a))$, we get

$$\begin{aligned}
&{}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-y))^{\rho}-(\eta(a-t+\eta(b,a)))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,x))^{\rho-1}f(y)dy\\
&\quad+{}_a^bJ_\eta^{\alpha,\rho}f(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-y))^{\rho}-(\eta(a-t+\eta(b,a)))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,x))^{\rho-1}\varphi_1(y)dy\\
&\quad\geq{}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-y))^{\rho}-(\eta(a-t+\eta(b,a)))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,x))^{\rho-1}\varphi_1(y)dy\\
&\quad+{}_a^bJ_\eta^{\alpha,\rho}f(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b-y))^{\rho}-(\eta(a-t+\eta(b,a)))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,x))^{\rho-1}f(y)dy.
\end{aligned}$$

Using the definition of generalized mixed $\eta-$ conformable fractional integrals, we write

$${}_a^bJ_\eta^{\beta,\rho}\varphi_1(t){}_a^bJ_\eta^{\alpha,\rho}f(t)+{}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t){}_a^bJ_\eta^{\beta,\rho}f(t)\geq{}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t){}_a^bJ_\eta^{\beta,\rho}\varphi_1(t)+{}_a^bJ_\eta^{\alpha,\rho}f(t){}_a^bJ_\eta^{\beta,\rho}f(t).$$

This proves the theorem. □

Remark 2.2. If we choose $\eta(a,b)=a-b$ and $a\to 0$ in Theorem 2.1, then the following inequality holds;

$${}_0^bJ^{\beta,\rho}\varphi_1(t){}_0^bJ^{\alpha,\rho}f(t)+{}_0^bJ^{\alpha,\rho}\varphi_2(t){}_0^bJ^{\beta,\rho}f(t)\geq{}_0^bJ^{\alpha,\rho}\varphi_2(t){}_0^bJ^{\beta,\rho}\varphi_1(t)+{}_0^bJ^{\alpha,\rho}f(t){}_0^bJ^{\beta,\rho}f(t),$$

which was proved by Çelik et al. in [3].

Remark 2.3. If we choose $\eta(a,b)=a-b, a\to 0$ and $\rho=1$ in Theorem 2.1, then the following inequality holds;

$$J^{\beta}\varphi_1(t)J^{\alpha}f(t)+J^{\alpha}\varphi_2(t)J^{\beta}f(t)\geq J^{\alpha}\varphi_2(t)J^{\beta}\varphi_1(t)+J^{\alpha}f(t)J^{\beta}f(t).$$

which was proved by Tariboon et al. in [11]

Theorem 2.4. *Let f and g be two integrable functions on $[0,\infty)$, Then, $t>0$, $\alpha,\beta,\rho>0$. Suppose that (2.1) holds and moreover, assume that there exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t)\leq g(t)\leq\psi_2(t),\quad\forall t\in[0,\infty). \tag{2.4}$$

Then, the following inequality for generalized mixed $\eta-$ conformable fractional integrals:

a. $${}_a^b\mathrm{J}_\eta^{\beta,\rho}\psi_1(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t)+{}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_2(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}g(t)\geq{}_a^b\mathrm{J}_\eta^{\beta,\rho}\psi_1(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_2(t)+{}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}g(t).$$

b. $${}_a^b\mathrm{J}_\eta^{\beta,\rho}\varphi_1(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}g(t)+{}_a^b\mathrm{J}_\eta^{\alpha,\rho}\psi_2(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}f(t)\geq{}_a^b\mathrm{J}_\eta^{\beta,\rho}\varphi_1(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}\psi_2(t)+{}_a^b\mathrm{J}_\eta^{\beta,\rho}f(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}g(t).$$

c. $${}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_2(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}\psi_2(t)+{}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}g(t)\geq{}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_2(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}g(t)+{}_a^b\mathrm{J}_\eta^{\beta,\rho}\psi_2(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t).$$

d. $${}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_1(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}\psi_1(t)+{}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}g(t)\geq{}_a^b\mathrm{J}_\eta^{\alpha,\rho}\varphi_1(t){}_a^b\mathrm{J}_\eta^{\beta,\rho}g(t)+{}_a^b\mathrm{J}_\eta^{\beta,\rho}\psi_1(t){}_a^b\mathrm{J}_\eta^{\alpha,\rho}f(t).$$

Proof. To prove (a), from (2.1) and (2.4) for $\forall t\in[0,\infty)$, we have

$$(\varphi_2(x)-f(x))(g(y)-\psi_1(y))\geq 0,$$

then

$$\varphi_2(x)g(y)+\psi_1(y)f(x)\geq\psi_1(y)\varphi_2(x)+f(x)g(y).$$

If we multiply both sides of (1.10) by $\frac{1}{\Gamma(\alpha)}\left(\frac{(\eta(b,x))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}$and integrate with respect to x on $(a,a+\eta(t,a))$, we get

$$\begin{aligned}
&\frac{g(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,x))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}\varphi_2(x)dx\\
&+\frac{\psi_1(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,x))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}f(x)dx\\
&\geq\frac{\psi_1(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,x))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}\varphi_2(x)dx\\
&+\frac{g(y)}{\Gamma(\alpha)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,x))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}f(x)dx.
\end{aligned}$$

Using the definition of generalized mixed $\eta-$ conformable fractional integrals, we can write;

$$g(y)_a^bJ_{\eta}^{\alpha,\rho}\varphi_2(t)+\psi_1(y)_a^bJ_{\eta}^{\alpha,\rho}f(t)\geq\psi_1(y)_a^bJ_{\eta}^{\alpha,\rho}\varphi_2(t)+g(y)_a^bJ_{\eta}^{\alpha,\rho}f(t). \tag{2.5}$$

If we multiply both sides of (2.5) by $\frac{1}{\Gamma(\beta)}\left(\frac{(\eta(b,y))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,y))^{\rho-1}$ and integrate with respect to y on $(a,a+\eta(t,a))$, we get

$$\begin{aligned}
&{}_a^bJ_{\eta}^{\alpha,\rho}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,y))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,y))^{\rho-1}g(y)dy\\
&+_a^bJ_{\eta}^{\alpha,\rho}f(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,y))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,y))^{\rho-1}\psi_1(y)dy\\
&\geq_a^bJ_{\eta}^{\alpha,\rho}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,y))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,y))^{\rho-1}\psi_1(y)dy\\
&+_0^bJ_{\eta}^{\alpha,\rho}f(t)\frac{1}{\Gamma(\beta)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,y))^{\rho}-(a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\beta-1}(\eta(b,y))^{\rho-1}g(y)dy.
\end{aligned}$$

Using the definition of generalized mixed $\eta-$ conformable fractional integrals, we can write;

$${}_a^bJ_{\eta}^{\beta,\rho}\psi_1(t)_a^bJ_{\eta}^{\alpha,\rho}f(t)+_a^bJ_{\eta}^{\alpha,\rho}\varphi_2(t)_a^bJ_{\eta}^{\beta,\rho}g(t)\geq_a^bJ_{\eta}^{\beta,\rho}\psi_1(t)_a^bJ_{\eta}^{\alpha,\rho}\varphi_2(t)+_a^bJ_{\eta}^{\alpha,\rho}f(t)_a^bJ_{\eta}^{\beta,\rho}g(t).$$

This proves (a). To prove (b)-(d), we use the following inequalities:

b. $(\psi_2(x)-g(x))(f(y)-\varphi_1(y))\geq 0,$

c. $(\varphi_2(x)-f(x))(g(y)-\psi_2(y))\geq 0,$

d. $(\varphi_1(x)-f(x))(g(y)-\psi_1(y))\geq 0.$

□

The following inequalities are the special case of Theorem 2.4.

Remark 2.5. If we choose $\eta(a,b)=a-b$ and $a\to 0$ in Theorem 2.4, then the following inequality holds;

$$\begin{aligned}
&(a)\ {}_0^bJ^{\beta,\rho}\psi_1(t)_0^bJ^{\alpha,\rho}f(t)+_0^bJ^{\alpha,\rho}\varphi_2(t)_0^bJ^{\beta,\rho}g(t)\geq_0^bJ^{\beta,\rho}\psi_1(t)_0^bJ^{\alpha,\rho}\varphi_2(t)+_0^bJ^{\alpha,\rho}f(t)_0^bJ^{\beta,\rho}g(t),\\
&(b)_0^bJ^{\beta,\rho}\varphi_1(t)_0^bJ^{\alpha,\rho}g(t)+_0^bJ^{\alpha,\rho}\psi_2(t)_0^bJ^{\beta,\rho}f(t)\geq_0^bJ^{\beta,\rho}\varphi_1(t)_0^bJ^{\alpha,\rho}\psi_2(t)+_0^bJ^{\beta,\rho}f(t)_0^bJ^{\alpha,\rho}g(t),\\
&(c)\ {}_0^bJ^{\alpha,\rho}\varphi_2(t)_0^bJ^{\beta,\rho}\psi_2(t)+_0^bJ^{\alpha,\rho}f(t)_0^bJ^{\beta,\rho}g(t)\geq_0^bJ^{\alpha,\rho}\varphi_2(t)_0^bJ^{\beta,\rho}g(t)+_0^bJ^{\beta,\rho}\psi_2(t)_0^bJ^{\alpha,\rho}f(t),\\
&(d)\ {}_0^bJ^{\alpha,\rho}\varphi_1(t)_0^bJ^{\beta,\rho}\psi_1(t)+_0^bJ^{\alpha,\rho}f(t)_0^bJ^{\beta,\rho}g(t)\geq_0^bJ^{\alpha,\rho}\varphi_1(t)_0^bJ^{\beta,\rho}g(t)+_0^bJ^{\beta,\rho}\psi_1(t)_0^bJ^{\alpha,\rho}f(t),
\end{aligned}$$

which was proved by Çelik et al.in [3].

Remark 2.6. If we choose $\eta(a,b) = a - b$, $a \to 0$ and $\rho = 1$ in Theorem 2.4, then the following inequality holds;

$$(a)\ J^{\beta}\psi_1(t)J^{\alpha}f(t) + J^{\alpha}\varphi_2(t)J^{\beta}g(t) \geq J^{\beta}\psi_1(t)J^{\alpha}\varphi_2(t) + J^{\alpha}f(t)J^{\beta}g(t).$$
$$(b)J^{\beta}\varphi_1(t)J^{\alpha}g(t) + J^{\alpha}\psi_2(t)J^{\beta}f(t) \geq J^{\beta}\varphi_1(t)J^{\alpha}\psi_2(t) + J^{\beta}f(t)J^{\alpha}g(t).$$
$$(c)\ J^{\alpha}\varphi_2(t)J^{\beta}\psi_2(t) + J^{\alpha}f(t)J^{\beta}g(t) \geq J^{\alpha}\varphi_2(t)J^{\beta}g(t) + J^{\beta}\psi_2(t)J^{\alpha}f(t).$$
$$(d)\ J^{\alpha}\varphi_1(t)J^{\beta}\psi_1(t) + J^{\alpha}f(t)J^{\beta}g(t) \geq J^{\alpha}\varphi_1(t)J^{\beta}g(t) + J^{\beta}\psi_1(t)J^{\alpha}f(t).$$

which is proved by Tariboon et al. in [11].

Lemma 2.7. *Let f be a integrable on $[0,\infty)$, invex function, φ_1, φ_2 be two integrable functions on $[0,\infty)$ and $t > 0$, $\alpha,\rho > 0$. Suppose that the condition (2.1) holds. Then, we have following equality for generalized mixed $\eta-$conformable fractional integrals;*

$$
\begin{aligned}
&{}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f^2(t) \times \left[\frac{(\eta(b,a+\eta(t,a))^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(b,a)^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\right] - ({}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f(t))^2 \\
&= ({}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_2(t) - {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f(t))({}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f(t) - {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_1(t)) - ({}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_2(t) - {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f(t))({}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f(t) - {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_1(t)) \\
&\times \left[\frac{(\eta(b,a+\eta(t,a))^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(b,a)^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\right] \\
&+ {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_1(t)f(t) \times \left[\frac{(\eta(b,a+\eta(t,a))^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(b,a)^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\right] \\
&- {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_1(t){}_{0}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f(t) + {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_2(t)f(t) \\
&\times \left[\frac{(\eta(b,a+\eta(t,a))^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(b,a)^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\right] \\
&- {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_2(t){}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho} f(t) + {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_1(t)\varphi_2(t) - {}_{a}^{b}\mathrm{J}_{\eta}^{\alpha,\rho}\varphi_1(t)\varphi_2(t) \\
&\times \left[\frac{(\eta(b,a+\eta(t,a))^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(b,a)^{\rho} - (a-t+\eta(b,a))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\right].
\end{aligned}
$$

Proof. For any $x, y > 0$, we have

$$
\begin{aligned}
&(\varphi_2(y) - f(y))(f(x) - \varphi_1(x)) + (\varphi_2(x) - f(x))(f(y) - \varphi_1(y)) \\
&-(\varphi_2(x) - f(x))(f(x) - \varphi_1(x)) - (\varphi_2(y) - f(y))(f(y) - \varphi_1(y)) \\
&\qquad = f^2(x) + f^2(y) - 2f(x)f(y) + \varphi_2(y)f(x) + \varphi_1(x)f(y) \\
&\qquad -\varphi_1(x)\varphi_2(y) + \varphi_2(x)f(y) + \varphi_1(y)f(x) - \varphi_1(y)\varphi_2(x) \\
&\qquad -\varphi_2(x)f(x) + \varphi_1(x)\varphi_2(x) - \varphi_1(x)f(x) - \varphi_2(y)f(y) \\
&\qquad +\varphi_1(y)\varphi_2(y) - \varphi_1(y)f(y).
\end{aligned}
\tag{2.6}
$$

If we multiply both sides of (2.6) by $\frac{1}{\Gamma(\alpha)}\left(\frac{(\eta(b,x))^{\rho} - (a-t+\eta(b,a))^{\rho}}{\rho}\right)^{\alpha-1}(\eta(b,x))^{\rho-1}$ and integrate with respect to x on $(a, a+\eta(t,a))$, we obtain

$$(\varphi_2(y) - f(y))({}_a^bJ_\eta^{\alpha,\rho} f(t) -{}_a^b J_\eta^{\alpha,\rho}\varphi_1(t)) + ({}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t) -{}_a^b J_\eta^{\alpha,\rho} f(t))(f(y) - \varphi_1(y))$$

$$-{}_a^b J_\eta^{\alpha,\rho}(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) - (\varphi_2(y) - f(y))(f(y) - \varphi_1(y))$$

$$\times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right]$$

$$={}_a^b J_\eta^{\alpha,\rho} f^2(t) + f^2(y) \times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right]$$

$$- 2f(y){}_a^bJ_\eta^{\alpha,\rho} f(t) + \varphi_2(y){}_a^bJ_\eta^{\alpha,\rho} f(t) + f(y){}_a^bJ_\eta^{\alpha,\rho}\varphi_1(t) - \varphi_2(y){}_a^bJ_\eta^{\alpha,\rho}\varphi_1(t) + f(y){}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)$$

$$+ \varphi_1(y){}_a^bJ_\eta^{\alpha,\rho} f(t) - \varphi_1(y){}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t) -{}_a^b J_\eta^{\alpha,\rho}\varphi_2(t)f(t) +{}_a^b J_\eta^{\alpha,\rho}\varphi_1(t)\varphi_2(t) -{}_a^b J_\eta^{\alpha,\rho}\varphi_1(t)f(t) - \varphi_2(y)f(y)$$

$$\times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right] + \varphi_1(y)\varphi_2(y) \tag{2.7}$$

$$\times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right] - \varphi_1(y)f(y)$$

$$\times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right].$$

If we multiply both sides of (2.7) by $\frac{1}{\Gamma(\alpha)}\left(\frac{(\eta(b,y))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}(\eta(b, y))^{\rho-1}$ and integrate with respect to y on $(a, a+\eta(t, a))$, we have

$$({}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t) -{}_a^b J_\eta^{\alpha,\rho} f(t))({}_a^bJ_\eta^{\alpha,\rho} f(t) -{}_a^b J_\eta^{\alpha,\rho}\varphi_1(t)) + ({}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t) -{}_a^b J_\eta^{\alpha,\rho} f(t))({}_a^bJ_\eta^{\alpha,\rho} f(t) -{}_a^b J_\eta^{\alpha,\rho}\varphi_1(t))$$

$$- ({}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t) -{}_a^b J_\eta^{\alpha,\rho} f(t))({}_a^bJ_\eta^{\alpha,\rho} f(t) -{}_a^b J_\eta^{\alpha,\rho}\varphi_1(t))$$

$$\times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right]$$

$$- ({}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t) -{}_a^b J_\eta^{\alpha,\rho} f(t))({}_a^bJ_\eta^{\alpha,\rho} f(t) -{}_a^b J_\eta^{\alpha,\rho}\varphi_1(t))$$

$$\times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right]$$

$$={}_a^b J_\eta^{\alpha,\rho} f^2(t) \times\left[\frac{(\eta(b, a + \eta(t, a))^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha} - \frac{(\eta(b, a)^\rho - (a - t + \eta(b, a))^\rho)^\alpha}{\Gamma(\alpha + 1)\rho^\alpha}\right]$$

$$+{}_0^b J_\eta^{\alpha,\rho} f^2(t) \times \left[\frac{(\eta(b,a+\eta(t,a))^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta(b,a)^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right]$$

$$-2{}_a^b J_\eta^{\alpha,\rho} f(t) {}_a^b J_\eta^{\alpha,\rho} f(t) + {}_a^b J_\eta^{\alpha,\rho} \varphi_2(t) {}_a^b J_\eta^{\alpha,\rho} f(t) + {}_a^b J_\eta^{\alpha,\rho} \varphi_1(t) {}_a^b J_\eta^{\alpha,\rho} f(t)$$

$$-{}_a^b J_\eta^{\alpha,\rho} \varphi_1(t) {}_a^b J_\eta^{\alpha,\rho} \varphi_2(t) + {}_a^b J_\eta^{\alpha,\rho} \varphi_2(t) {}_a^b J_\eta^{\alpha,\rho} f(t) + {}_a^b J_\eta^{\alpha,\rho} \varphi_1(t) {}_a^b J_\eta^{\alpha,\rho} f(t) - {}_a^b J_\eta^{\alpha,\rho} \varphi_1(t) {}_a^b J_\eta^{\alpha,\rho} \varphi_2(t) - {}_a^b J_\eta^{\alpha,\rho} \varphi_2(t) f(t)$$

$$\times \left[\frac{(\eta(b,a+\eta(t,a))^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta(b,a)^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right] + {}_a^b J_\eta^{\alpha,\rho} \varphi_1(t)\varphi_2(t)$$

$$\times \left[\frac{(\eta(b,a+\eta(t,a))^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta(b,a)^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right] - {}_a^b J_\eta^{\alpha,\rho} \varphi_1(t) f(t)$$

$$\times \left[\frac{(\eta(b,a+\eta(t,a))^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta(b,a)^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right] - {}_a^b J_\eta^{\alpha,\rho} \varphi_2(t) f(t)$$

$$\times \left[\frac{(\eta(b,a+\eta(t,a))^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta(b,a)^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right] + {}_0^b J_\eta^{\alpha,\rho} \varphi_1(t)\varphi_2(t)$$
$$\times \left[\frac{(\eta(b,a+\eta(t,a))^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta(b,a)^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right] - {}_0^b J_\eta^{\alpha,\rho} \varphi_1(t) f(t)$$
$$\times \left[\frac{(\eta(b,a+\eta(t,a))^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta(b,a)^\rho - (a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right].$$

This proves lemma. □

Remark 2.8. If we choose $\eta(a,b) = a - b$ and $a \to 0$, in Lemma 2.7, then the following inequality holds;

$$\begin{aligned}
{}_0^b J^{\alpha,\rho} f^2(t) \tfrac{(b^\rho-(b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - ({}_0^b J^{\alpha,\rho} f(t))^2 = {} & ({}_0^b J^{\alpha,\rho} \varphi_2(t) - {}_0^b J^{\alpha,\rho} f(t))({}_0^b J^{\alpha,\rho} f(t) - {}_0^b J^{\alpha,\rho} \varphi_1(t)) \\
& - \frac{(b^\rho - (b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} ({}_0^b J^{\alpha,\rho} \varphi_2(t) - {}_0^b J^{\alpha,\rho} f(t))({}_0^b J^{\alpha,\rho} f(t) - {}_0^b J^{\alpha,\rho} \varphi_1(t)) \\
& + {}_0^b J^{\alpha,\rho} \varphi_1(t) f(t) \tfrac{(b^\rho-(b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - {}_0^b J^{\alpha,\rho} \varphi_1(t) {}_0^b J^{\alpha,\rho} f(t) \\
& + {}_0^b J^{\alpha,\rho} \varphi_2(t) f(t) \frac{(b^\rho - (b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - {}_0^b J^{\alpha,\rho} \varphi_2(t) {}_0^b J^{\alpha,\rho} f(t) + {}_0^b J^{\alpha,\rho} \varphi_1(t)\varphi_2(t) \\
& - {}_0^b J^{\alpha,\rho} \varphi_1(t)\varphi_2(t) \frac{(b^\rho - (b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}.
\end{aligned}$$

which is proved by Çelik et al. in [3].

Remark 2.9. If we choose $\eta(a,b) = a - b$, $a \to 0$and $\rho = 1$ in Lemma 2.7 then the following inequality holds;

$$\begin{aligned}
J^\alpha f^2(t) \frac{t^\alpha}{\Gamma(\alpha+1)} - (J^\alpha f(t))^2 = {} & (J^\alpha \varphi_2(t) - J^\alpha f(t))(J^\alpha f(t) - J^\alpha \varphi_1(t)) \\
& - \frac{t^\alpha}{\Gamma(\alpha+1)} (J^\alpha \varphi_2(t) - J^\alpha f(t))(J^\alpha f(t) - J^\alpha \varphi_1(t)) \\
& + J^\alpha \varphi_1(t) f(t) \frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha \varphi_1(t) J^\alpha f(t) + {}_0^b J^\alpha \varphi_2(t) f(t) \frac{t^\alpha}{\Gamma(\alpha+1)} \\
& - J^\alpha \varphi_2(t) J^\alpha f(t) + J^\alpha \varphi_1(t)\varphi_2(t) - J^\alpha \varphi_1(t)\varphi_2(t) \frac{t^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

which is proved by Tariboon et al. in [11].

Theorem 2.10. *Let $f, g, \varphi_1, \varphi_2, \psi_1$ and ψ_2 be six integrable functions on $[0,\infty)$.Satisfying the conditions (2.1) and (2.4) on $[0,\infty)$. Then, for all $t>0$, $\alpha,\rho>0$, we have following inequality for generalized mixed $\eta-$ conformable fractional integrals*

$$\begin{aligned} &\Big| {}_a^b\mathrm{J}_\eta^{\alpha,\rho} f(t)g(t) \times \Big\{ \tfrac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \tfrac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \Big\} - {}_a^b\mathrm{J}_\eta^{\alpha,\rho} f(t) {}_0^b\mathrm{J}_\eta^{\alpha,\rho} g(t) \Big| \\ &\quad \leq \sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)}, \end{aligned} \tag{2.8}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned} T(u,v,w) &= ({}_a^b\mathrm{J}_\eta^{\alpha,\rho} w(t) -_a^b \mathrm{J}_\eta^{\alpha,\rho} u(t))({}_a^b\mathrm{J}_\eta^{\alpha,\rho} u(t) -_a^b \mathrm{J}_\eta^{\alpha,\rho} v(t)) +_a^b \mathrm{J}_\eta^{\alpha,\rho} v(t)u(t) \\ &\quad \times \left\{ \frac{(\eta\,(b, a+\eta\,(t,a))^\rho - (a-t+\eta\,(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta\,(b,a)^\rho - (a-t+\eta\,(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right\} -_a^b \mathrm{J}_\eta^{\alpha,\rho} v(t) {}_0^b\mathrm{J}_\eta^{\alpha,\rho} u(t) \\ &\quad +_a^b \mathrm{J}_\eta^{\alpha,\rho} w(t)u(t) \times \left\{ \frac{(\eta\,(b, a+\eta\,(t,a))^\rho - (a-t+\eta\,(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta\,(b,a)^\rho - (a-t+\eta\,(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right\} \\ &\quad -_a^b \mathrm{J}_\eta^{\alpha,\rho} w(t) {}_a^b\mathrm{J}_\eta^{\alpha,\rho} u(t) +_a^b \mathrm{J}_\eta^{\alpha,\rho} v(t) {}_a^b\mathrm{J}_\eta^{\alpha,\rho} w(t) \\ &\quad -_a^b \mathrm{J}_\eta^{\alpha,\rho} v(t)w(t) \times \left\{ \frac{(\eta\,(b, a+\eta\,(t,a))^\rho - (a-t+\eta\,(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \frac{(\eta\,(b,a)^\rho - (a-t+\eta\,(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right\}. \end{aligned}$$

Proof. Let f and g be two integrable functions defined $[0,\infty)$ satisfying (2.1) and (2.4). We define

$$H(x,y) = (f(x)-f(y))(g(x)-g(y)), \quad x,y \in (0,t), \ t>0. \tag{2.9}$$

Multiplying both sides of (2.9) by

$$\frac{1}{\Gamma^2(\alpha)} \left(\frac{(\eta\,(b,x))^\rho - (a-t+\eta\,(b,a))^\rho}{\rho} \right)^{\alpha-1} \left(\frac{(\eta\,(b,y))^\rho - (a-t+\eta\,(b,a))^\rho}{\rho} \right)^{\alpha-1} \times (\eta\,(b,x))^{\rho-1} (\eta\,(b,y))^{\rho-1},$$

where $x,y \in (0, a+\eta\,(t,a))$ and integrating the resulting identity with respect to x and y, from a to $a+\eta\,(t,a)$, we can state that

$$\begin{aligned} &\tfrac{1}{2\Gamma^2(\alpha)} \int_a^{a+\eta(t,a)} \int_a^{a+\eta(t,a)} \left(\tfrac{(\eta(b,x))^\rho-(a-t+\eta(b,a))^\rho}{\rho} \right)^{\alpha-1} (\eta\,(b,x))^{\rho-1} \\ &\times \left(\tfrac{(\eta(b,y))^\rho-(a-t+\eta(b,a))^\rho}{\rho} \right)^{\alpha-1} (\eta\,(b,y))^{\rho-1} H(x,y)dxdy \\ &=_a^b J_\eta^{\alpha,\rho} f(t)g(t) \times \left[\tfrac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} - \tfrac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha} \right] \\ &-_a^b J_\eta^{\alpha,\rho} f(t) {}_a^b J_\eta^{\alpha,\rho} g(t). \end{aligned} \tag{2.10}$$

Applying the Cauchy-Schwarz inequality to (2.10), we have

$$\begin{aligned}
&\left[\frac{1}{2\Gamma^2(\alpha)}\int_a^{a+\eta(t,a)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,x))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}\left(\frac{(\eta(b,y))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}\right.\\
&\qquad\left.(f(x)-f(y))(g(x)-g(y))\,(\eta(b,y))^{\rho-1}\,(\eta(b,x))^{\rho-1}\,dxdy\right]^2\\
&\leq\left[\frac{1}{2\Gamma^2(\alpha)}\int_a^{a+\eta(t,a)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,x))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}\left(\frac{(\eta(b,y))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}\right.\\
&\qquad\left.(f(x)-f(y))^2\,(\eta(b,y))^{\rho-1}\,(\eta(b,x))^{\rho-1}\,dxdy\right]\\
&\times\left[\frac{1}{2\Gamma^2(\alpha)}\int_a^{a+\eta(t,a)}\int_a^{a+\eta(t,a)}\left(\frac{(\eta(b,x))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}\left(\frac{(\eta(b,y))^\rho-(a-t+\eta(b,a))^\rho}{\rho}\right)^{\alpha-1}\right.\\
&\qquad\left.(g(x)-g(y)^2\,(\eta(b,y))^{\rho-1}\,(\eta(b,x))^{\rho-1}\,dxdy\right].
\end{aligned}\tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\begin{aligned}
&\left[{}_a^bJ_\eta^{\alpha,\rho}f(t)g(t)\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}-{}_a^bJ_\eta^{\alpha,\rho}f(t){}_a^bJ_\eta^{\alpha,\rho}g(t)\right]^2\\
&\leq\left[{}_a^bJ_\eta^{\alpha,\rho}f^2(t)\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}-({}_a^bJ_\eta^{\alpha,\rho}f(t))^2\right]\\
&\times\left[{}_a^bJ_\eta^{\alpha,\rho}g^2(t)\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}-({}_a^bJ_\eta^{\alpha,\rho}g(t))^2\right].
\end{aligned}$$

Since $(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\geq 0$ and $(\psi_2(t)-g(t))(g(t)-\psi_1(y))\geq 0$, for $t\in[0,\infty)$, we have

$$\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}\times{}_a^bJ_\eta^{\alpha,\rho}(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\geq 0,$$

$$\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,0)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}\times{}_0^bJ_\eta^{\alpha,\rho}(\psi_2(t)-g(t))(g(t)-\psi_1(y))\geq 0.$$

Thus, from Lemma 2.7, we obtain

$$\begin{aligned}
&{}_a^bJ_\eta^{\alpha,\rho}f^2(t)\times\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}-({}_a^bJ_\eta^{\alpha,\rho}f(t))^2\\
&\leq({}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)-{}_a^bJ_\eta^{\alpha,\rho}f(t))({}_a^bJ_\eta^{\alpha,\rho}f(t)-{}_a^bJ_\eta^{\alpha,\rho}\varphi_1(t))+{}_a^bJ_\eta^{\alpha,\rho}\varphi_1(t)f(t)\\
&\times\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}-{}_a^bJ_\eta^{\alpha,\rho}\varphi_1(t){}_a^bJ_\eta^{\alpha,\rho}f(t)\\
&+{}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)\times\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}\\
&-{}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t){}_a^bJ_\eta^{\alpha,\rho}f(t)+{}_a^bJ_\eta^{\alpha,\rho}\varphi_1(t){}_a^bJ_\eta^{\alpha,\rho}\varphi_2(t)-{}_a^bJ_\eta^{\alpha,\rho}\varphi_1(t)\varphi_2(t)\\
&\times\left\{\frac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\frac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}\\
&=T(f,\varphi_1,\varphi_2)
\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}
&{}_a^bJ_\eta^{\alpha,\rho} g^2(t)\times\left\{\tfrac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\tfrac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}-({}_a^bJ_\eta^{\alpha,\rho}g(t))^2\\
&\le ({}_a^bJ_\eta^{\alpha,\rho}\psi_2(t)-{}_a^bJ_\eta^{\alpha,\rho}g(t))({}_a^bJ_\eta^{\alpha,\rho}g(t)-{}_a^bJ_\eta^{\alpha,\rho}\psi_1(t))+{}_a^bJ_\eta^{\alpha,\rho}\psi_1(t)g(t)\\
&\times\left\{\tfrac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\tfrac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}-{}_a^bJ_\eta^{\alpha,\rho}\psi_1(t){}_a^bJ_\eta^{\alpha,\rho}g(t)\\
&+{}_a^bJ_\eta^{\alpha,\rho}\psi_2(t)\times\left\{\tfrac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\tfrac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\} \qquad (2.13)\\
&-{}_a^bJ_\eta^{\alpha,\rho}\psi_2(t){}_a^bJ_\eta^{\alpha,\rho}g(t)\ +{}_a^bJ_\eta^{\alpha,\rho}\psi_1(t){}_a^bJ_\eta^{\alpha,\rho}\psi_2(t)-{}_a^bJ_\eta^{\alpha,\rho}\psi_1(t)\psi_2(t)\\
&\times\left\{\tfrac{(\eta(b,a+\eta(t,a))^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-\tfrac{(\eta(b,a)^\rho-(a-t+\eta(b,a))^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\right\}\\
&=T(g,\psi_1,\psi_2).
\end{aligned}$$

From (2.11), (2.12), and (2.13), we get (2.8). □

Remark 2.11. If we choose $\eta(a,b)=a-b$ and $a\to 0$ in Theorem 2.10 then the following inequality holds;

$$\left|{}_0^bJ^{\alpha,\rho}f(t)g(t)\tfrac{(b^\rho-(b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}-{}_0^bJ^{\alpha,\rho}f(t){}_0^bJ^{\alpha,\rho}g(t)\right|\le\sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)}\,,$$

where $T(u,v,w)$ is defined by

$$\begin{aligned}
T(u,v,w)&=({}_0^bJ^{\alpha,\rho}w(t)-{}_0^bJ^{\alpha,\rho}u(t))({}_0^bJ^{\alpha,\rho}u(t)-{}_0^bJ^{\alpha,\rho}v(t))+{}_0^bJ^{\alpha,\rho}v(t)u(t)\tfrac{(b^\rho-(b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\\
&-{}_0^bJ^{\alpha,\rho}v(t){}_0^bJ^{\alpha,\rho}u(t)+{}_0^bJ^{\alpha,\rho}w(t)u(t)\tfrac{(b^\rho-(b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}\\
&-{}_0^bJ^{\alpha,\rho}w(t){}_0^bJ^{\alpha,\rho}u(t)+{}_0^bJ^{\alpha,\rho}v(t){}_0^bJ^{\alpha,\rho}w(t)-{}_0^bJ^{\alpha,\rho}v(t)w(t)\tfrac{(b^\rho-(b-t)^\rho)^\alpha}{\Gamma(\alpha+1)\rho^\alpha}.
\end{aligned}$$

which is proved by Çelik et al. in [3].

Remark 2.12. If we choose in Theorem 2.10 $\eta(a,b)=a-b$, $a\to 0$ and $\rho=1$ then the following inequality holds;

$$\left|J^\alpha f(t)g(t)\tfrac{t^\alpha}{\Gamma(\alpha+1)}-J^\alpha f(t)J^\alpha g(t)\right|\le\sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)}\,,$$

where $T(u,v,w)$ is defined by

$$\begin{aligned}
T(u,v,w)&=(J^\alpha w(t)-J^\alpha u(t))(J^\alpha u(t)-J^\alpha v(t))+J^\alpha v(t)u(t)\tfrac{t^\alpha}{\Gamma(\alpha+1)}-J^\alpha v(t)J^\alpha u(t)+J^\alpha w(t)u(t)\tfrac{t^\alpha}{\Gamma(\alpha+1)}\\
&-J^\alpha w(t)J^\alpha u(t)+J^\alpha v(t)J^\alpha w(t)-J^\alpha v(t)w(t)\tfrac{t^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

which is proved by Tariboon et al. in [11].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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