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Bruck-Reilly extension of a ternary monoid

Seda OĞUZ ÜNAL*

Sivas Cumhuriyet University, Education Faculty,
Department of Math and Science Education, Campus, Sivas

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Abstract

In this study, Bruck-Reilly extension of a ternary monoid is defined. Additionally, some results about this construction are given which belongs to one of the classes of ternary semigroups; regular, inverse, orthodox and strongly regular.

Keywords: *Bruck-Reilly extension, ternary semigroup, regular.*

Bir üçlü monoidin Bruck-Reilly genişlemesi

Öz

Bu çalışmada bir üçlü monoidin Bruck-Reilly genişlemesi tanımlanmıştır. Ayrıca; regüler, tersinir, orthodox ve strongly regüler üçlü yarıgrup sınıflarından birine ait olan bu yapı ile ilgili bazı sonuçlar verilmiştir.

Anahtar kelimeler: *Bruck-Reilly genişlemesi, üçlü yarıgrup, regüler.*

1. Introduction

The Bruck-Reilly extension is a fundamental construction in semigroup theory. This extension is a generalization of constructions, which are obtained by Bruck [1], Reilly [2] and Munn [3]. Many research papers have been published about Bruck-Reilly extension and its generalization see for example [4-15].

Ternary algebraic operations were introduced by A. Cayley [16] in the nineteenth century.Ternary algebraic structures are used in many sciences for various purposes. The

* Seda OĞUZ ÜNAL, soguz@cumhuriyet.edu.tr, https://orcid.org/0000-0003-1338-1466

concept of ternary algebraic system was firstly considered by Lehmer [17] in 1932. He described some ternary operations defined on groups. S. Banach (cf. Los [18]) investigated the notion of ternary semigroups. Ternary semigroups were considered by many authors, for instance, ideal theory is considered by Sioson [19], who introduced the notion of regular ternary semigroups. Santiago [20] studied regular ternary semigroups, Sheeja and Bala [21] studied orthodox ternary semigroups and the regularity on ternary semigroup was defined and considered in [22]. Green's relations are defined on ternary semigroups in [23].

In this study, we consider Bruck-Reilly extension of a ternary monoid that belongs to the classes of ternary semigroups such as regular, inverse, orthodox and strongly regular. It is interesting to note that many known results of the theory of semigroup can also be obtained for ternary semigroups. All informations about semigroup theory can be found in [24, 25].

Now we remind some definitions related to ternary semigroups.

Definition 1.1: A nonempty set T together with a ternary multiplication $[\,]:(a,b,c)\to[abc]$ defined on T is called a ternary semigroup if $[\,]$ satisfying the associative law of the first kind

$$[[abc]\,uv] = [a[bcu]v] = \big[ab[cuv]\big]$$

for any $a,b,c,u,v \in T$.

Definition 1.2: Let T be a ternary semigroup. An element 1_T of T is called,

- a left identity if $1_T1_Tx = x$ for all $x \in T$;
- a right identity if $x1_T1_T = x$ for all $x \in T$;
- a lateral identity if $1_Tx1_T = x$ for all $x \in T$;
- a two-sided identity if $1_T1_Tx = x1_T1_T = x$ for all $x \in T$;
- an identity if $1_T1_Tx = 1_Tx1_T = x1_T1_T = x$ for all $x \in T$.

Definition 1.3: A pair (a,b) of elements in a ternary semigroup T is said to be an idempotent pair if $ab(abx) = abx$ and $(xab)ab = xab$ for all $x \in T$.

Definition 1.4: Two idempotent pairs (a,b) and (c,d) of a ternary semigroup S are said to be equivalent, in notation we write $(a,b) \backsim (c,d)$, if $abx = cdx$ and $xab = xcd$ for all $x \in T$.

Definition 1.5: Let T be a ternary semigroup. T is called

- regular if for each $a \in T$ there exist elements $x,y \in T$ such that $axaya = a$.
- ternary inverse if each element $x \in T$ has a unique inverse element denoted x^{-1}; that is $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.
- orthodox if it is a regular ternary semigroup in which for any two idempotent pairs (a,b) and (c,d) the pair $([abc],d)$ is also an idempotent pair.

- strongly regular if any two idempotent pairs commute. (a, b) and (c, d) commute if $[abcdt] = [cdabt]$ and $[tabcd] = [tcdab]$ for all $t \in T$.)

2. Bruck-Reilly Extension of ternary monoids

Definiton 2.1: Let T be a monoid and $\theta: T \to T$ an endomorphism. The set $\mathbb{N}^0 \times T \times \mathbb{N}^0$, where $\mathbb{N}^0$ signs the non-negative integers set, together with the binary multiplication

$$(m, a, n)(p, b, q) = (m - n + t, (a\theta^{t-n})(b\theta^{t-p}), q - p + t),$$

where $t = \max(n, p)$ and θ^0 is the identity map on T. This construction is a monoid and it is denoted by $BR(T, \theta)$. We call it the Bruck–Reilly extension of T determined by θ.

Definition 2.2: Let T be a ternary monoid with a ternary endomorphism θ defined on T. On the set $\mathbb{N}^0$x T x $\mathbb{N}^0$ ($\mathbb{N}^0$ is non-negative integers set) we define a ternary operation by

$$(m, a, n)(p, b, q)(r, c, s) = (m - n - q + p + v, (a\theta^{u-n})(b\theta^{u-p})\theta^{v-q+p-u}c\theta^{v-r}, s - r + v),$$

where $u = \max\{n, p\}, v = \max\{q - p + u, r\}$ and θ^0 is the identity map on T, forms a ternary monoid. Then this construction is called the Bruck-Reilly extension of T determined by θ and we denote it by $BR(T, \theta)$ ([1, 10, 17]).

Through this paper T will denote a ternary monoid and θ will denote an identity homomorphism. In order to provide ease of operation we select θ homomorphism as an identity.

Theorem 2.1 $BR(T, \theta)$ is regular if and only if T is regular.

Proof. Let $BR(T, \theta)$ be regular. Then for any $(m, a, n) \in BR(T, \theta)$ there exist elements $(n, b, m), (n, c, m) \in BR(T, \theta)$ such that

$$((m, a, n)(n, b, m)(m, a, n))\ (n, c, m)(m, a, n) = (m, a, n).$$

By considering this equation we get

$$\big((m, a, n)(n, b, m)(m, a, n)\big)(n, c, m)(m, a, n) = ((m - n - m + n + m, aba, n - m + m)(n, c, m))(m, a, n)$$
$$= ((m, aba, n)(n, c, m))(m, a, n)$$
$$= (m - n - m + n + m, (aba)ca, n - m + m)$$
$$= (m, (aba)ca, n) = (m, a, n)$$

Since $(aba)ca = a$ we get T is regular.

Conversely, let T be regular. Then we know that for any $a \in T$ there exist $b, c \in T$ such that $abaca = a$. Now we need to show that for any $(m, a, n) \in BR(T, \theta)$ there exist $(x, b, y), (z, c, t) \in BR(T, \theta)$ such that

$$((m, a, n)(x, b, y)(m, a, n))\ (z, c, t)(m, a, n) = (m, a, n).$$

Here we take $x = z = n$ and $y = t = m$, then we have

$((m, a, n)(n, b, m)(m, a, n))(n, c, m)(m, a, n) = (m, a, n)$.

Hence $BR(T, \theta)$ is regular.

Theorem 2.2 $BR(T; \theta)$ is ternary inverse monoid if and only if T is ternary inverse monoid.

Proof. Let $BR(T; \theta)$ be ternary inverse monoid. Then for any $(m, a, n) \in BR(T, \theta)$ there exists a unique inverse element $(n, b, m) \in BR(T, \theta)$ such that

$(m, a, n)(n, b, m)(m, a, n) = (m, a, n)$ and $(n, b, m)(m, a, n)(n, b, m) = (n, b, m)$.

Then we get for any $a \in T$ there exists $b \in T$ such that $aba = a$ and $bab = b$. Hence T is ternary inverse monoid.

Conversely, let T be a ternary inverse monoid. Then for any $a \in T$ there exists a unique inverse $b \in T$ such that $aba = a$ and $bab = b$. Now we need to show that for any $(m, a, n) \in BR(T, \theta)$ there exist$(x, b, y) \in BR(T, \theta)$ such that

$(m, a, n)(x, b, y)(m, a, n) = (m, a, n)$ and $(x, b, y)(m, a, n)(x, b, y) = (x, b, y)$.

Here we take $x = n$ and $y = m$, then we get $BR(T; \theta)$ is ternary inverse monoid.

Theorem 2.3 If $BR(T; \theta)$ is orthodox then T is orthodox.

Proof. Let θ be an identity homomorphism and $BR(T, \theta)$ be orthodox. Then $BR(T, \theta)$ is regular and for any two idempotent pairs $\big((m, a, n), (p, b, q)\big), ((r, c, s), (t, d, k))$ of $BR(T, \theta)$ the pair $([(m, a, n)(p, b, q)(r, c, s)], (t, d, k))$ is also an idempotent pair in $BR(T, \theta)$. By Theorem 2.1 T is regular. Now we need to show that for any $((a, b), (c, d))$ idempotent pairs of elements of T the pair $([abc], d)$ is also an idempotent pair in T.
For any idempotent pairs $\big((m, a, n), (p, b, q)\big), ((r, c, s), (t, d, k))$ of elements in $BR(T, \theta)$ we have the following

$(m, a, n)(p, b, q)\ ((m, a, n)(p, b, q)(x, y, z)) = ((m, a, n)(p, b, q)(x, y, z))$
$((x, y, z)(m, a, n)(p, b, q))\ (m, a, n)(p, b, q) = ((x, y, z)(m, a, n)(p, b, q))$

and

$(r, c, s)(t, d, k)\ ((r, c, s)(t, d, k)(x, y, z)) = (r, c, s)(t, d, k)(x, y, z)$
$((x, y, z)(r, c, s)(t, d, k))\ (r, c, s)(t, d, k) = (x, y, z)(r, c, s)(t, d, k)$

for any $(x, y, z) \in BR(T, \theta)$. From idempotent pair definition we have $m = n$, $q = p$, $k = t$, $r = s$, $(ab[aby] = aby)$, $([yab]ab) = yab$, $(cd[cdy]) = cdy$, $([ycd]cd) = ycd$.
On the other hand we know that $([(m, a, n)(p, b, q)(r, c, s)], (t, d, k))$ is also an idempotent pair. Then we have the following equalities:

$[(m, a, n)(p, b, q)(r, c, s)](t, d, k)([(m, a, n)(p, b, q)(r, c, s)])(t, d, k)(x, y, z)) =$
$[(m, a, n)(p, b, q)(r, c, s)](t, d, k)(x, y, z)$

$[(x,y,z)[(m,a,n)(p,b,q)(r,c,s)](t,d,k)] = (x,y,z)[(m,a,n)(p,b,q)(r,c,s)](t,d,k)$.

From these equalities we get $[abc]d([abc]dy) = [abc]dy$ and $(y[abc]d)[abc]d = y[abc]d$. Hence $([abc],d)$ is an idempotent pair. Therefore T is orthodox.

Theorem 2.4 If $BR(T,\theta)$ is strongly regular then T is strongly regular.

Proof. Let θ be an identity homomorphism and $BR(T,\theta)$ be strongly regular. Then two idempotent pairs $((m,a,n),(p,b,q))$ and $((r,c,s),(t,d,k))$ of $BR(T,\theta)$ commute i.e.

$$[(m,a,n)(p,b,q)(r,c,s)(t,d,k)(x,y,z)] = [(r,c,s)(t,d,k)(m,a,n)(p,b,q)(x,y,z)]$$

and

$$[(x,y,z)(m,a,n)(p,b,q)(r,c,s)(t,d,k)] = [(x,y,z)(r,c,s)(t,d,k)(m,a,n)(p,b,q)]$$

for all $(x,y,z) \in BR(T,\theta)$.

Now we write the equivalents of $[(m,a,n)(p,b,q)(r,c,s)(t,d,k)(x,y,z)]$, $[(r,c,s)(t,d,k)(m,a,n)(p,b,q)(x,y,z)]$, $[(x,y,z)(m,a,n)(p,b,q)(r,c,s)(t,d,k)]$ and $[(x,y,z)(r,c,s)(t,d,k)(m,a,n)(p,b,q)]$ respectively, we obtain

$$\begin{aligned}
&[(m,a,n)(p,b,q)(r,c,s)(t,d,k)(x,y,z)] \\
&= (m-n-q+p+\max\{q-p+\max\{n,p\},r\}, abc, s-r+\max\{q-p+\\
&\max\{n,p\},r\})\ (t,d,k)(x,y,z) \\
&= (m-n-q+p-s+r+\max\{k-t+\max\{s-r+\max\{q-p+\\
&\max\{n,p\},r\},t\}, [abc]dy, z-x+\max\{k-t+\max\{s-r+\max\{q-p+\\
&\max\{n,p\},r\},t\})
\end{aligned}$$

$$\begin{aligned}
&[(r,c,s)(t,d,k)(m,a,n)(p,b,q)(x,y,z)] \\
&= (r-s-k+t+\max\{k-t+\max\{s,t\},m\}, cda, n-m+\max\{k-t+\\
&\max\{s,t\},m\})(p,b,q)(x,y,z) \\
&= (r-s-k+t-n+m+\\
&\max\{q-p+\max\{n-m+\max\{k-t+\max\{s,t\},m\},p\}\}, [cda]by,\\
&z-x+\max\{q-p+\max\{n-m+\max\{k-t+\max\{s,t\},m\},p\}\})
\end{aligned}$$

$$\begin{aligned}
&[(x,y,z)(m,a,n)(p,b,q)(r,c,s)(t,d,k)] \\
&= (x-z-n+m+\max\{n-m+\max\{z,m\},p\}, yab, q-p+\max\{n-m+\\
&\max\{z,m\},p\})(r,c,s)(t,d,k) \\
&= (x-z-n+m-q+p+\max\{s-r+\max\{q-p+\max\{n-m+\\
&\max\{z,m\},p\},r\}\}, [yab]cd, k-t+\max\{s-r+\max\{q-p+\max\{n-m+\\
&\max\{z,m\},p\},r\}\})
\end{aligned}$$

$$\begin{aligned}
&[(x,y,z)(r,c,s)(t,d,k)(m,a,n)(p,b,q)] = \\
&= (x-z-s+r+\max\{s-r+\max\{z,r\},t\}, ycd, k-t+\max\{s-r+\\
&\max\{z,r\},t\})(m,a,n)(p,b,q) \\
&= (x-z-s+r-k+t+\max\{n-m+\max\{k-t+\max\{s-r+\\
&\max\{z,r\},t\},m\}\}, [ycd]ab, q-p+\max\{n-m+\max\{k-t+\max\{s-r+\\
&\max\{z,r\},t\},m\}\}).
\end{aligned}$$

Since $[\mathrm{abc}]\mathrm{dy} = [cda]by$ and $[yab]cd = [ycd]ab$ for all $y \in T$ then T is strongly regular.

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