Araştırma Makalesi / Research Article

An Efficient Nonlinear Technique For Solving Fourth-order Fractional Integro-differential equations

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Abstract
In this study univariate Padé approximation is applied to power series solutions of Fourth-order Fractional Integro-differential equations. The fractional derivatives are described in the Caputo sense. Power series solutions of the Fractional Integro-differential equations are converted into rational power series solutions by applying univariate Padé approximation. Then the numerical results were compared to show the effectiveness of univariate Padé approximation.

Keywords: Padé Approximation, Homotopy perturbation method, Integro-differential equations.

Dördüncü Mertebeden Kesirli İntegro-Diferensiyel Denklemlerin Etkili Bir Non-lineer Teknikle Çözümü

Öz
Bu çalışmada, tek değişkenli Padé yaklaşımı dördüncü mertebeden kesirli integro-diferensiyel denklemlere uygulandı. Kesirli türevle*r Caputo tanımına göre tanımlandı. Kesirli integro-diferensiyel denklemlerin seri çözümleri, Padé yaklaşımı yardımıyla rasyonel kuvvet serilerine dönüştürüldü. Sonra Padé yaklaşımının etkinliğini göstermek için nümerik sonuçlar karşılaştırıldı.

Anahtar Kelimeler: Padé Yaklaşımı, Homotopy Pertübasyon Yöntemi, İntegro-diferensiyel Denklemleri.

1. Introduction

During the past decades, many researchers were attracted by the topic of fractional calculus due to its applications in many area [1-3]. Some approximations and numerical methods were used for the solutions of the fractional differential equations [4-6]. As it is known, a combination of differential and Volterra–Fredholm integral equations are called as Integro-differential equations. Integro-differential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. Many applications of these equations can be seen in different areas [7-13].

Many theoretical and practical studies are done in literature based on the multivariate Padé approximation. Recently, the Padé approximation has become increasingly hot, which has led a large number of researchers to carry out relevant theoretical and numerical studies. In practice, the calculation principle of the Padé approximation is relatively simple, the calculation accuracy is high, the calculation amount is small, and it does not rely on the variational principle [14-17]. More details about the theory of univariate and multivarite Padé approximation can be found in [14]. In this study, univariate Padé approximation was applied to the solutions of Fourth-order Fractional Integro-differential equations in the form [18].

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$$D^{\alpha} y(x) = f(x) + \gamma y(x) + \int_0^x \left[g(t) y(t) + h(t) F(y(t)) \right] dt \qquad 0 < x < b, \quad 3 < \alpha < 4 \tag{1}$$

$$y(0) = \gamma_0, \qquad y''(0) = \gamma_{2,} \tag{2}$$

$$y(b) = \beta_0, \qquad y''(b) = \beta_{2,} \tag{3}$$

where D^{α} is the fractional derivative in the Caputo sense and $F(y(x))$ is any nonlinear function, $\gamma, \gamma_0, \gamma_2, \beta_0$ and β_2 are real constants and f, g and h are given and can be approximated by Taylor polynomials [18]. Basic definitions and properties of fractional calculus theory which are further used in this paper can be found in [18].

The paper is organized as follows. We begin by introducing necessary homotopy pertubation method (HPM) equations that constructed by Nawaz [18] in section 2. Then the univariate Padé approximation is presented which are required for establishing our results. In Section 3 the applications of the Padé approximation are presented to construct approximate solutions to linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations. In Section 3 two examples present to demonstrate the efficiency of the method. Concluding remarks are given in the last section.

2. Materials and Methods

2.1. Homotopy perturbation method (HPM)

Nawaz [18] constructed following homotopy equation (4) and (5) for equation (1) with boundary conditions (2)–(3), by using basic concepts of HPM [18]:

$$(1-p) D^{\alpha} y(x) + p\left(D^{\alpha} y(x) - f(x) - \gamma y(x) - \int_0^x \left[g(t) y(t) + h(t) F(y(t)) \right] dt \right) = 0 \tag{4}$$

or

$$(D^{\alpha} y(x) = p\left(f(x) + \gamma y(x) + \int_0^x \left[g(t) y(t) + h(t) F(y(t)) \right] dt \right) \tag{5}$$

where $p \in [0,1]$ is an embedding parameter [18]. More details can be found in [18].

2.2. Univariate Padé approximation

Consider a formal power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots \tag{6}$$

with $(c_0 \neq 0)$ [14]. The Padé approximation problem of order (m,n) or $[m,n]$ for f consists in finding polynomials

$$p(x) = \sum_{i=0}^{m} a_i x^i, \quad q(x) = \sum_{i=0}^{n} b_i x^i \tag{7}$$

such that in the power series $(fq - p)(x)$ [14]. To find the coefficients we get following linear systems of equations

$$\begin{cases} c_0 b_0 = 0 \\ c_1 b_0 + c_0 b_1 = a_1 \\ \vdots \\ c_m b_0 + c_{m-1} b_1 + \cdots + c_{m-n} b_n = a_m \end{cases} \tag{8}$$

$$\begin{cases} c_{m+1}b_0 + c_m b_1 + \ldots + c_{m-n+1}b_n = a_m \\ \vdots \\ c_{m+n}b_0 + c_{m-n+1}b_1 + \ldots + c_m b_n = 0 \end{cases} \tag{9}$$

with $c_i = 0$ for $i < 0$ [14].

In general a solution for the coefficients a_i is known after substitution of a solution for the b_i in the left hand side of (8). So the crucial point is to solve the homogeneous system of n equations (9) in the $n+1$ unknowns b_i. This system has at least one nontrivial solution because one of the unknowns can be chosen freely [14

In short, by solving the equations (8) and (9) the coefficients a_i and b_i are found. Then the Padé equations (7) are found. After finding these polynomials we get The Padé approximation of order (m,n) or $[m,n]$ for f. More details about definitions and theorems for Padé approximation can be found in [14].

3. Results and Discussion

In this section univariate Padé series solutions of fourth-order linear and nonlinear fractional integro-differential equations shall be illustrated by two examples. The full Homotopy perturbation method solutions of examples by Nawaz can be seen in [18].

Example 1.

Consider the following linear fourth-order fractional integro-differential equation [18]:

$$D^{\alpha} y(x) = x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)dt \qquad 0 < x < 1, \quad 3 < \alpha \le 4 \tag{10}$$

subject to the following boundary conditions:

$$y(0) = 1, \qquad y''(0) = 2. \tag{11}$$

$$y(1) = 1 + e, \qquad y''(1) = 3e. \tag{12}$$

The exact solution of (10) is given as $y(x) = 1 + xe^x$ for $\alpha = 4$ in [18].

According to the Homotopy Pertubation method Nawaz [18] obtained following homotopy:

$$D^{\alpha} y(x) = p\left(x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)dt \right) \tag{13}$$

Nawaz obtained following solution (14) by applying HPM on (13) in [18]:

$$\begin{aligned} y_1(x) &= 1 + Ax + x^2 + \frac{Bx^3}{6} + \frac{4x^{\alpha}}{\Gamma(\alpha+1)} + (4+A)\frac{x^{\alpha+1}}{\Gamma(\alpha+2)} \\ &+ (7-A)\frac{x^{\alpha+2}}{\Gamma(\alpha+3)} + (4+B)\frac{x^{\alpha+3}}{\Gamma(\alpha+4)} + (4-B)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)} \\ &+ \frac{4x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{x^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{2x^{2\alpha+4}}{\Gamma(2\alpha+5)} - \frac{4x^{2\alpha+5}}{\Gamma(2\alpha+6)} \end{aligned} \tag{14}$$

Table 1. Values of A and B for different values of α for equation (14) [18]

	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4$
A	1.10186984200028	1.09179499393439	1.05222793297923	0.99906052231083
B	−0.29428456125416	0.96679305229906	2.07168041387465	3.00628128299199

HPM solution (14) for $\alpha = 4$ is obtained as:

$$y_1(x)_{\alpha=4} = 1 + 0.9990605223x + x^2 + 0.5010468805x^3 + 0.1666666667x^4$$
$$+0.04165883768x^5 + 0.008334638165x^6 + 0.001390135175x^7$$
$$+0.0001238521507x^8 + 0.2755731922.10^{-6}x^{10} \tag{15}$$
$$+0.250521083910^{-7}x^{11} - 0.41753513981.10^{-8}x^{12}$$

Equation (15) was put into Padé series and following equations, respectively, $r_{6,6}(x)$, $r_{5,5}(x)$, $r_{4,4}(x)$, $r_{3,3}(x)$ were obtained for different values of m and n by applying Padé approximation using equations (8) and (9) :

$$r_{6,6}(x) = (1 + 1.067184792x + 1.55025727x^2$$
$$+0.541829923x^3 + 0.1712272163x^4 + 0.03239991461x^5$$
$$+0.001680525583x^6)/(1 + 0.06812426968x - 0.01303454103x^2 \tag{16}$$
$$-0.01431893132x^3 - 0.002232883194x^4 + 0.002467664877x^5$$
$$-0.0003776744646x^6)$$

$$r_{5,5}(x) = (0.9999999996 + 0.9821637950x$$
$$+0.9922657685x^2 + 0.4502613670x^3 + 0.1381120667x^4$$
$$+0.01278787559x^5)/(0.9999999996 - 0.01689672688x \tag{17}$$
$$+0.009146621640x^2 - 0.04302681492x^3 + 0.01375122298x^4$$
$$-0.001349216249x^5)$$

$$r_{4,4}(x) = (1 - 0.3473258101x + 0.2490539622x^2$$
$$-0.3793574163x^3 - 0.02859963464x^4)/(1 - 1.346386333x$$
$$+0.5941753945x^2 - 0.1276351444x^3 + 0.01267620996x^4)$$
(18)

$$r_{3,3}(x) = \frac{1 - 0.4137039739x + 0.5569093547x^2 + 0.04216740745x^3}{1 - 0.5853565487x + 0.1417159738x^2 - 0.01510575936x^3} \tag{19}$$

Table 2. Numerical Values for exact solution and padé approximations of equation (14) for $\alpha = 4$

	$\alpha = 4$				
x	$r_{6,6}(x)$	$r_{5,5}(x)$	$r_{4,4}(x)$	$r_{3,3}(x)$	$y(x) = 1 + xe^x$
0.1	1.110424190	1.110424191	1.110424191	1.110424191	1.110517092
0.2	1.244101028	1.244101029	1.244101029	1.244101030	1.244280552
0.3	1.404704041	1.404704042	1.244101029	1.404704079	1.404957642
0.4	1.596420961	1.596420960	1.596420901	1.596421290	1.596729879
0.5	1.824021196	1.824021198	1.824020671	1.824023019	1.824360636
0.6	2.092931688	2.092931680	2.092928360	2.092939194	2.093271280
0.7	2.409321908	2.409321859	2.409305363	2.409347245	2.409626895
0.8	2.780199088	2.780198871	2.780128937	2.780272852	2.780432742
0.9	3.213514523	3.213513724	3.213246540	3.213706334	3.213642800
1.0	3.718282084	3.718279530	3.717312570	3.718738299	3.718281828

Table 3. Error values for padé approximations of equation (14) for $\alpha = 4$

x	$\alpha = 4$			
	$\lvert y(x) - r_{6,6}(x)\rvert$	$\lvert y(x) - r_{5,5}(x)\rvert$	$\lvert y(x) - r_{4,4}(x)\rvert$	$\lvert y(x) - r_{3,3}(x)\rvert$
0.1	0.0000929018	0.0000929008	0.0000929008	0.0000929008
0.2	0.0001795236	0.0001795226	0.0001795226	0.0001795216
0.3	0.0002536014	0.0002536004	0.0002536044	0.0002535634
0.4	0.0003089182	0.0003089182	0.0003089782	0.0003085892
0.5	0.0003394395	0.0003394375	0.0003399645	0.0003376165
0.6	0.000339592	0.000339600	0.000342920	0.000332086
0.7	0.000304987	0.000305036	0.000321532	0.000279650
0.8	0.000233654	0.000233871	0.000303805	0.000159890
0.9	0.000128277	0.000129076	0.000396260	0.000063534
1.0	2.56×10^{-7}	2.298×10^{-6}	0.000969258	0.000456471

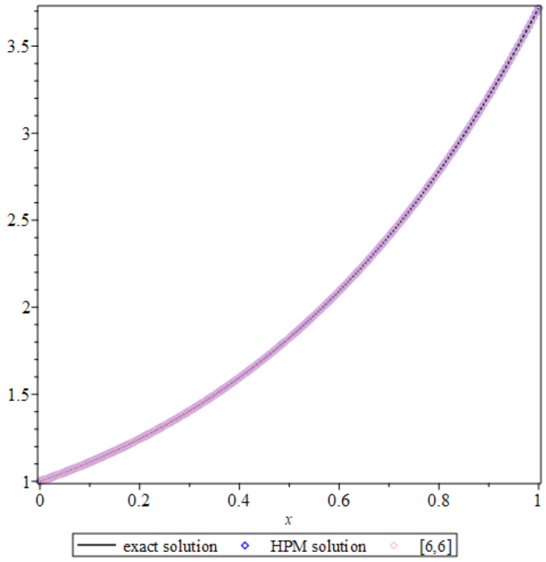

Figure 1.

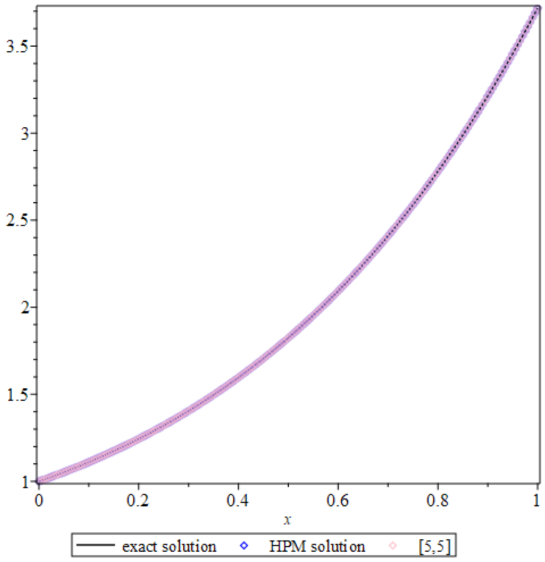

Figure 2.

Figure 1. $y(x) = 1 + xe^{x}$ (Exact solution), $y_1(x)$ (HPM solution), $r_{6,6}(x)$.

Figure 2. $y(x) = 1 + xe^{x}$ (Exact solution), $y_1(x)$ (HPM solution), $r_{5,5}(x)$.

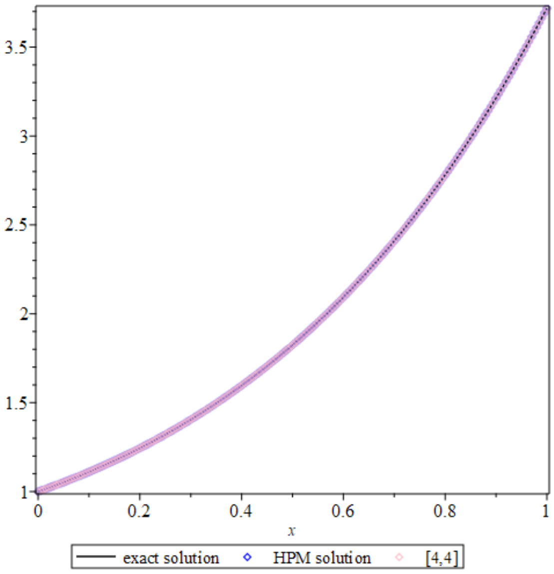

Figure 3.

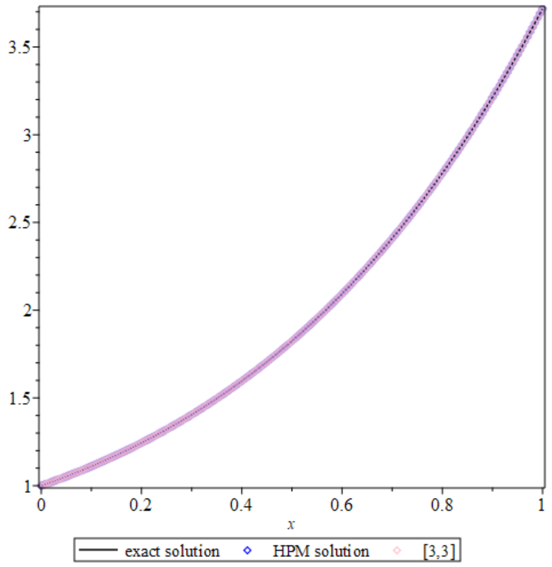

Figure 4.

Figure 3. $y(x)=1+xe^{x}$ (Exact solution), $y_1(x)$ (HPM solution), $r_{4,4}(x)$.

Figure 4. $y(x)=1+xe^{x}$ (Exact solution), $y_1(x)$ (HPM solution), $r_{3,3}(x)$.

HPM solution (14) for $\alpha=3.75$ is obtained as:

$$\begin{aligned}
y_1(x)_{\alpha=3.75} &= 1+1.052227933x+x^2+0.3452800690x^3\\
&+0.2411642464x^{3.75}+0.06412719693x^{4.75}+0.001985622146x^{6.75}\\
&+0.00008137013071x^{7.75}+0.0002850138176x+0.01312942763x^{5.75}\\
&+0.8403768765.10^{-7}x^{10.50}-0.1461525002.10^{-7}x^{11.50}-0.2338440004.10^{-8}x^{12.50}\\
&+0.8823957200.10^{-6}x^{9.50}
\end{aligned} \tag{20}$$

Equation (20) was put into Padé series and following equations, respectively, $r_{15,15}(x)$, $r_{18,18}(x)$, $r_{20,20}$ were obtained for different values of m and n by applying Padé approximation using equations (8) and (9)

$$\begin{aligned}
r_{15,15}(x) &= 1+0.2851336739x^{1/4}+0.1139716724\sqrt{x}+0.04738762916x^{3/4}\\
&+0.8112266571x+0.1942311684x^{5/4}+0.07297851730x^{3/2}\\
&+0.02914795875x^{7/4}+0.7571584845x^{2}+0.1813837208x^{9/4}\\
&+0.06884972230x^{5/2}+0.02783143989x^{11/4}+0.1191052063x^{3}\\
&+0.002021703304x^{13/4}-0.002614105724x^{7/2}+0.2381289871x^{15/4})/(1\\
&+0.2851336739x^{1/4}+0.1139716724\sqrt{x}+0.047387622916x^{3/4}\\
&-0.2410012761x-0.1057944479x^{5/4}-0.04694565998x^{3/2}\\
&-0.2071462833x^{7/4}+0.1074675887x^{2}+0.007570920095x^{9/4}\\
&+0.004275584651x^{5/2}+0.002240321285x^{11/4}+0.00351873405x^{3}\\
&+0.001398843018x^{13/4}+0.0004805177468x^{7/2}-0.001039963651x^{15/4})
\end{aligned} \tag{21}$$

$$
\begin{aligned}
r_{18,18}(x) = (&1 + 0.8485572144x^{1/4} + 0.5282597480\sqrt{x} + 2.181821340x^{3/4} \\
&+ 1.431881288x + 0.8921999198x^{5/4} + 0.4833774903x^{3/2} + 1.739490796x^{7/4} \\
&+ 1.174093423x^{2} + 0.7624491887x^{9/4} + 0.4228081410x^{5/2} + 1.625739170x^{11/4} \\
&+ 0.5090971001x^{3} + 0.2151591341x^{13/4} + 0.08604054946x^{7/2} + 04757821763x^{15/4} \\
&+ 0.134911668x^{4} + 0.05549905147x^{17/4} + 0.4767630077x^{9/2}) / (1 + 0.8485572144x^{1/4} \\
&+ 0.5282597480\sqrt{x} + 2.181821340x^{3/4} + 0.3796533547x - 0.0006756839084x^{5/4} \\
&- 0.07247217240x^{3/2} - 0.5562825620x^{7/4} - 0.2253884425x^{2} - 0.08539705218x^{9/4} \\
&- 0.02919436297x^{5/2} + 0.02925388047x^{11/4} + 0.02132369126x^{3} + 0.01270208818x^{13/4} \\
&+ 0.006834283848x^{7/2} + 0.006779318949x^{15/4} + 0.002134329333x^{4} + 0.0003665478079x^{17/4} \\
&- 0.002387956079x^{9/2})
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
r_{20,20}(x) = (&1 - 0.4325094246x^{1/4} + 0.04125808286\sqrt{x} + 0.01581636875x^{3/4} \\
&+ 0.9993919516x - 0.3811418570x^{5/4} + 0.03745701208x^{3/2} \\
&+ 0.01282848367x^{7/4} + 0.9070031282x^{2} - 0.3599513890x^{9/4} \\
&+ 0.03355620190x^{5/2} + 0.01142649967x^{11/4} + 0.2575481631x^{3} \\
&- 0.8027046362x^{13/4} + 0.007198456294x^{7/2} + 0.2414347479x^{15/4} \\
&- 0.1543286011x^{4} + 0.03086733759x^{17/4} + 0.0007466854551x^{9/2} \\
&+ 0.04902968304x^{19/4} - 0.01762806717x^{5}) / (1 - 0.4325094246x^{1/4} \\
&+ 0.04125808286\sqrt{x} + 0.01581636875x^{3/4} - 0.05283598095x \\
&+ 0.07395664083x^{5/4} - 0.005955895167x^{3/2} - 0.003813941327x^{7/4} \\
&- 0.03740137653x^{2} - 0.005261207760x^{9/4} - 0.001434921695x^{5/2} \\
&- 0.0003767334895x^{11/4} + 0.004458848127x^{3} + 0.0006457692279x^{13/4} \\
&+ 0.0004186224502x^{7/2} - 0.0009802246004x^{15/4} + 0.0009300715263x^{4} \\
&- 0.00003667958584x^{17/4} - 0.00001676984274x^{9/2} + 0.0003696667503x^{19/4} \\
&- 0.0002516931782x^{5})
\end{aligned} \tag{23}
$$

Table 4. Numerical Values of equation (20) and its padé approximations

	$\alpha = 3.75$			
x	$r_{15,15}(x)$	$r_{18,18}(x)$	$r_{20,20}(x)$	$y_1(x)_{\alpha=3.75}$
0.1	1.115612122	1.115612123	1.115612123	1.115612122
0.2	1.253816808	1.253816809	1.253816809	1.253816810
0.3	1.417854532	1.417854536	1.417854534	1.417854534
0.4	1.611650010	1.611650047	1.611650029	1.611650029
0.5	1.839846025	1.839846228	1.839846113	1.839846120
0.6	2.107862057	2.107862888	2.107862329	2.107862361
0.7	2.421969134	2.421971911	2.421969739	2.421969889
0.8	2.789378640	2.789386580	2.789379509	2.789380129
0.9	3.218344559	3.218364771	3.218344693	3.218346864
1.0	3.718279437	3.718326368	3.718275141	3.718281828

Table 5. Error values for padé approximations of equation (20) for $\alpha = 3.75$

x	$\alpha = 3.75$		
	$\lvert y_1(x)_{\alpha=3.75} - r_{15,15} \rvert$	$\lvert y_1(x)_{\alpha=3.75} - r_{18,18} \rvert$	$\lvert y_1(x)_{\alpha=3.75} - r_{20,20} \rvert$
0.1	0	1×10^{-9}	1×10^{-9}
0.2	$2.\times10^{-9}$	1×10^{-9}	1×10^{-9}
0.3	$2.\times10^{-9}$	2×10^{-9}	0
0.4	1.9×10^{-8}	1.8×10^{-8}	0
0.5	9.5×10^{-8}	1.08×10^{-7}	7×10^{-9}
0.6	3.04×10^{-7}	5.27×10^{-7}	3.2×10^{-8}
0.7	2.022×10^{-6}	2.022×10^{-6}	1.50×10^{-7}
0.8	1.489×10^{-6}	6.451×10^{-6}	6.20×10^{-7}
0.9	2.305×10^{-6}	0.000017907	2.171×10^{-6}
1.0	2.391×10^{-6}	0.000044540	6.687×10^{-6}

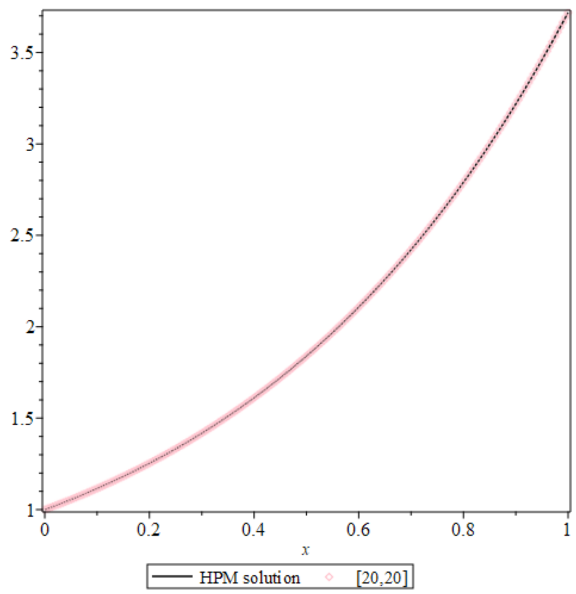

Figure 5.

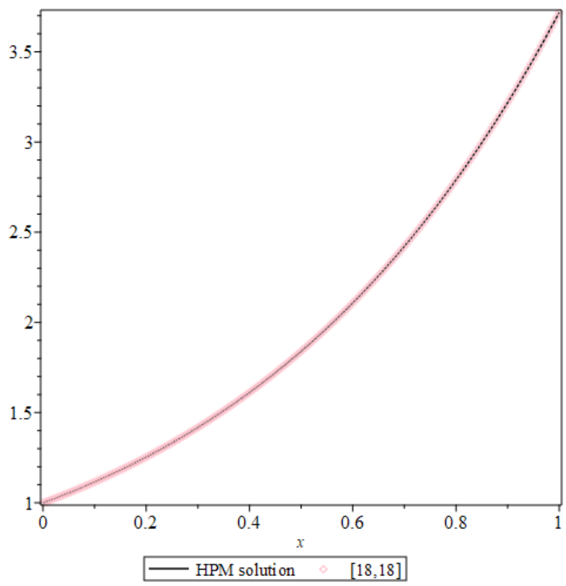

Figure 6.

Figure 5. $y_1(x)_{\alpha=3.75}$ (HPM solution), $r_{20,20}(x)$.

Figure 6. $y_1(x)_{\alpha=3.75}$ (HPM solution), $r_{18,18}(x)$.

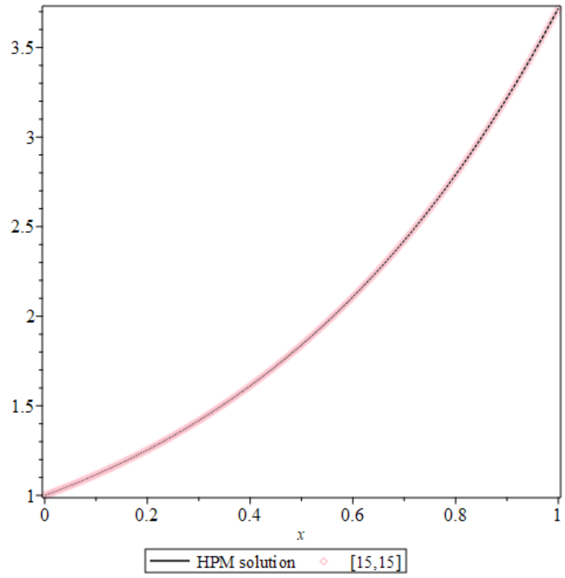

Figure 7. $y_1(x)_{\alpha=3.75}$ (HPM solution), $r_{15,15}(x)$.

HPM solution (14) for $\alpha = 3.50$ is obtained as:

$$\begin{aligned} y_1(x)_{\alpha=3.50} &= 1+1.091794994x+x^2+0.1611321754x^3+0.3438869842x^{3.5}+0.09727789028x^{4.5} \\ &+0.02052277578x^{5.5}+0.002654258716x^{6.5}+0.0002161264730x^{7.5}+0.0007936507936x^{7.0} \\ &+0.2755731922.10^{-5}x^{9.0}+0.1922275573.10^{-6}x^{10.0}-0.5010421678.10^{-7}x^{11.0}-0.8350702796.10^{-8}x^{12.0} \end{aligned} \tag{24}$$

Equation (24) was put into Padé series and following equations, respectively, $r_{8,8}(x)$, $r_{10,10}(x)$, $r_{12,12}(x)$ were obtained for different values of m and n by applying Padé approximation using equations (8) and (9):

$$\begin{aligned} r_{8,8}(x) = (&1+0.2835525045\sqrt{x}+0.08224512078x+0.2119091973x^{3/2} \\ &+0.7163321022x^2++0.1875802077x^{5/2}-0.09162252208x^3+0.3039322961x^{7/2} \\ &+0.06939585279x^4)/(1+0.2835525045\sqrt{x}-0.2693437860x-0.09767200760x^{3/2} \\ &+0.01040029944x^2+0.01066551221x^{5/2}+0.005234093640x^3+0.0003833347164x^{7/2} \\ &-0.0008290693842x^4) \end{aligned} \tag{25}$$

$$\begin{aligned} r_{10,10}(x) = (&1-0.3054195159\sqrt{x}+0.9347649705x-0.3270464958x^{3/2} \\ &+0.8128516542x^2-0.2840601115x^{5/2}-0.006110844493x^3+0.3145224527x^{7/2} \\ &-0.1387501437x^4+0.05629854454x^{9/2}-0.02353116120x^5)/(1-0.3054195159\sqrt{x} \\ &-0.1570300234x+0.006409002742\,x^{3/2}-0.01570375229x^2+0.01436208733x^{5/2} \\ &+0.006932281634x^3-0.002241078214x^{7/2}-0.0002826362218x^4+0.00007324951870x^{9/2} \\ &-0.0001178887241x^5) \end{aligned} \tag{26}$$

$$\begin{aligned}
r_{12,12}(x) = {} & (0.9999999997 - 2.188941160\sqrt{x} + 1.436483957x - 2.166426874x^{3/2} \\
& + 1.189627699x^2 - 1.924661063x^{5/2} + 0.3304087829x^3 + 0.2360963679x^{7/2} \\
& - 0.8516004838x^4 + 0.2699054008x^{9/2} - 0.1374299967x^5 - 0.008971297337x^{11/2} \\
& - 0.01138337076x^6) / (0.9999999997 - 2.188941160\sqrt{x} + 0.3446889634x \\
& + 0.2234481266x^{3/2} - 0.1867019855x^2 + 0.02032054977x^{5/2} + 0.02842793720x^3 \\
& - 0.0007157663508x^{7/2} + 0.001271913680x^4 - 0.001450299947x^{9/2} \\
& - 0.001068230175x^5 + 0.0002045966444x^{11/2} + 0.0001289622666x^6)
\end{aligned} \tag{27}$$

Table 6. Numerical Values of equation (24) and its padé approximations

	$\alpha = 3.50$			
x	$r_{8,8}(x)$	$r_{10,10}(x)$	$r_{12,12}(x)$	$y_1(x)_{\alpha=3.5}$
0.1	1.119452522	1.119452520	1.119452521	1.119452520
0.2	1.260951009	1.260951006	1.260951007	1.260951013
0.3	1.427434668	1.427434668	1.427434670	1.427434796
0.4	1.622665407	1.622665405	1.622665403	1.622666350
0.5	1.851219467	1.851219446	1.851219443	1.851223956
0.6	2.118526691	2.118526625	2.118526604	2.118542772
0.7	2.430933149	2.430933014	2.430932911	2.430980469
0.8	2.795780103	2.795780057	2.795779617	2.795900712
0.9	3.221496607	3.221497578	3.221495977	3.221772101
1.0	3.717704529	3.717709848	3.717704665	3.718281829

Table 7. Error values for padé approximations of equation (24) for $\alpha = 3.50$

	$\alpha = 3.50$		
x	$\lvert y_1(x)_{\alpha=3.5} - r_{8,8}\rvert$	$\lvert y_1(x)_{\alpha=3.5} - r_{10,10}\rvert$	$\lvert y_1(x)_{\alpha=3.5} - r_{12,12}\rvert$
0.1	$2.\times10^{-9}$	0	$1.\times10^{-9}$
0.2	$4.\times10^{-9}$	$7.\times10^{-9}$	$6.\times10^{-9}$
0.3	1.28×10^{-7}	1.28×10^{-7}	1.26×10^{-7}
0.4	9.43×10^{-7}	9.45×10^{-7}	9.47×10^{-7}
0.5	4.489×10^{-6}	4.510×10^{-6}	4.513×10^{-6}
0.6	0.000016081	0.000016147	0.000016168
0.7	0.000047320	0.000047455	0.000047558
0.8	0.000120609	0.000120655	0.000121095
0.9	0.000275494	0.000274523	0.000276124
1.0	0.000577300	0.000571981	0.000577164

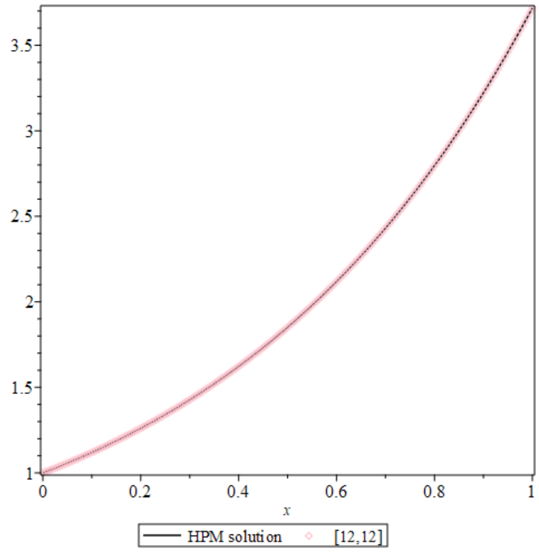

Figure 8.

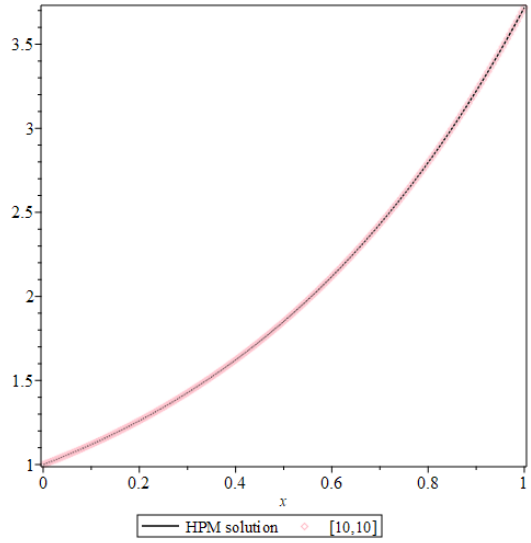

Figure 9.

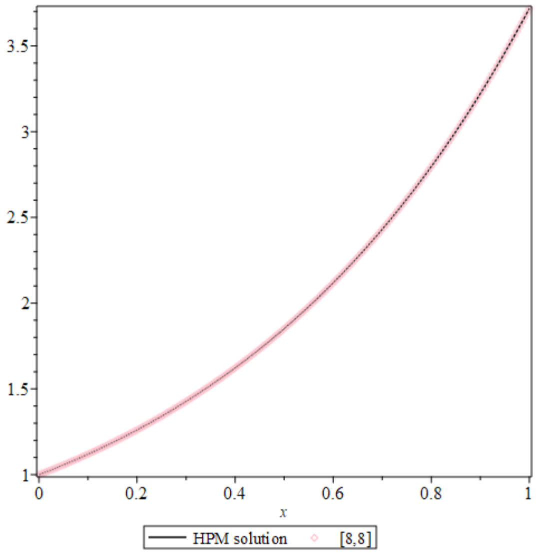

Figure 10.

Figure 8. $y_1(x)_{\alpha=3.5}$ (HPM solution), $r_{12,12}(x)$.

Figure 9. $y_1(x)_{\alpha=3.5}$ (HPM solution), $r_{10,10}(x)$.

Figure 10. $y_1(x)_{\alpha=3.5}$ (HPM solution), $r_{8,8}(x)$.

HPM solution (14) for $\alpha = 3.25$ is obtained as:

$$\begin{aligned} y_1(x)_{\alpha=3.25} = {} & 1 + 1.101869842x + x^2 - 0.04904742688x^3 + 0.4827952800x^{3.25} + 0.1448916870x^{4.25} \\ & + 0.03190576360x^{5.25} + 0.00320735358x^{6.25} + 0.0005126578201x^{7.25} + 0.002137603632x^{6.5} \\ & + 0.8382759340.10^{-5}x^{8.5} + 0.8823957200.10^{-6}x^{9.5} + 0.1680753753.10^{-6}x^{10.5} - 0.2923050004.10^{-7}x^{11.5} \end{aligned} \tag{28}$$

Equation (28) was put into Padé series and following equations, respectively, $r_{10,10}(x)$, $r_{15,15}(x)$, $r_{22,22}(x)$ were obtained for different values of m and n by applying Padé approximation using equations (8) and (9):

$$r_{10,10}(x) = (0.9999999999 - 0.5060267858x^{1/4} - 0.0008190350517\sqrt{x} - 0.0004133682437x^{3/4} + 1.148193838x - 0.8643176139x^{5/4} - 0.0006185576570x^{3/2} - 0.00008032639278x^{7/4} + 1.053514659x^{2} - 1.026318098x^{9/4} + 0.2209203695x^{5/2}) / (0.9999999999 - 0.5060267858x^{1/4} - 0.0008190350517\sqrt{x} - 0.0004133682437x^{3/4} + 0.04632399572x - 0.3067419597x^{5/4} + 0.0002839123659x^{3/2} + 0.0003751516086x^{7/4} + 0.002471645030x^{2} - 0.1823015973x^{9/4} + 0.2214265701x^{5/2}) \quad (29)$$

$$r_{15,15}(x) = (1 - 4.963699699x^{1/4} - 0.7033630287x^{3/4} + 0.097201755491x - 5.985197421x^{5/4} + 17.18436284x^{3/2} - 0.4339815735x^{7/4} + 0.2339900549x^{2} - 5.180097105x^{9/4} + 15.02183239x^{5/2} - 0.3881288340x^{11/4} - 0.7239863084x^{3} + 0.5645537719x^{13/4} - 8.619883215x^{7/2} + 10.34062473x^{15/4}) / (1 - 4.963699699x^{1/4} + 20.970262232\sqrt{x} - 0.7033630287x^{3/4} - 1.004668087x - 0.5158464198x^{5/4} - 5.922136796x^{3/2} + 0.3410329359x^{7/4} + 0.3410035214x^{2} + 0.3519982059x^{9/4} + 0.5769940001x^{5/2} - 0.06053971248x^{11/4} - 0.04601229086x^{3} - 0.03370799392x^{13/4} + 0.09146948587x^{7/2} - 0.09254313671x^{15/4}) \quad (30)$$

$$r_{22,22}(x) = (0.9999999997 - 0.1125883731x^{1/4} - 0.006063564488\sqrt{x} - 0.03611626203x^{3/4} + 1.075623477x - 0.1871666827x^{5/4} - 0.006555198037x^{3/2} - 0.03498711183x^{7/4} - 0.9558080376x^{2} - 0.1859749748x^{9/4} - 0.006330511891x^{5/2} - 0.03147808000x^{11/4} - 0.09297976073x^{3} + 0.4227405134x^{13/4} - 0.05349050669x^{7/2} + 0.003366660477x^{15/4} - 0.03152385778x^{4} + 0.1339000391x^{17/4} - 0.04617763939x^{9/2} - 0.001320221677x^{19/4} - 0.002192485028x^{5} + 0.02309692696x^{21/4} - 0.01369317035x^{11/2}) / (0.9999999997 - 0.1125883731x^{1/4} - 0.006063564488\sqrt{x} - 0.03611626203x^{3/4} - 0.02624646449x - 0.6310894978x^{5/4} + 0.0001260608082x^{3/2} + 0.0048008308101x^{7/4} - 0.01527188472x^{2} - 0.003848753232x^{9/4} - 0.0004058500055x^{5/2} - 0.0006599476604x^{11/4} - 0.0008583401711x^{3} + 0.001772838272x^{13/4} + 0.0008903592430x^{7/2} + 0.0004415793950x^{15/4} + 0.0008432502832x^{4} + 0.0004799575988x^{17/4} + 0.00003515598350x^{9/2} - 0.00009330337098x^{19/4} - 0.0001008258828x^{5} - 0.0001232232683x^{21/4} - 0.00004783271947x^{11/2}) \quad (31)$$

Table 8. Numerical Values of equation (28) and its padé approximations

	$\alpha = 3.25$			
x	$r_{10,10}(x)$	$r_{15,15}(x)$	$r_{22,22}(x)$	$y_1(x)_{\alpha=3.25}$
0.1	1.120417823	1.120417763	1.120417762	1.120417763
0.2	1.262730346	1.262726566	1.262726568	1.262726568
0.3	1.429850771	1.429812595	1.429812625	1.429812625
0.4	1.625597615	1.625408087	1.625408213	1.625408215
0.5	1.854711288	1.854074014	1.854074097	1.854074097
0.6	2.122902481	2.121242645	2.121240998	2.121240995
0.7	2.436853952	2.433288385	2.433277674	2.433277653
0.8	2.804082549	2.797619921	2.797577223	2.797577117
0.9	3.232508324	3.222792691	3.222658734	3.222658299
1.0	3.729479822	3.718643358	3.718283348	3.718281830

Table 9. Error values for padé approximations of equation (28) for $\alpha = 3.25$

x	$\alpha = 3.25$		
	$\lvert y_1(x)_{\alpha=3.25} - r_{10,10} \rvert$	$\lvert y_1(x)_{\alpha=3.25} - r_{15,15} \rvert$	$\lvert y_1(x)_{\alpha=3.25} - r_{22,22} \rvert$
0.1	6.0×10^{-8}	0	1×10^{-9}
0.2	3.778×10^{-6}	2×10^{-9}	0
0.3	0.000038146	3×10^{-8}	0
0.4	0.000189400	1.28×10^{-7}	2×10^{-9}
0.5	0.000637191	8.3×10^{-8}	0
0.6	0.001661486	1.650×10^{-6}	3×10^{-9}
0.7	0.003576299	0.000010732	2.1×10^{-8}
0.8	0.006505432	0.000042804	1.06×10^{-7}
0.9	0.009850025	0.000134392	4.35×10^{-7}
1.0	0.011197992	0.000361528	1.518×10^{-6}

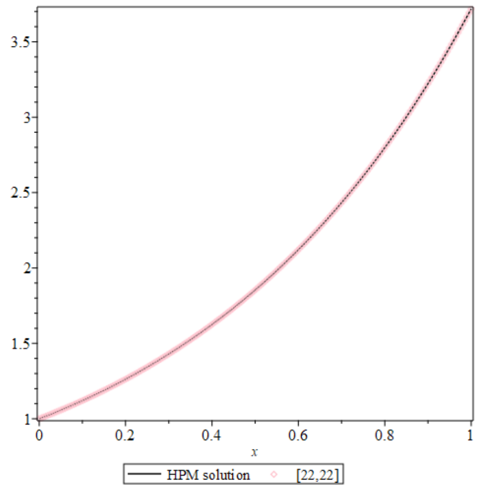

Figure 11.

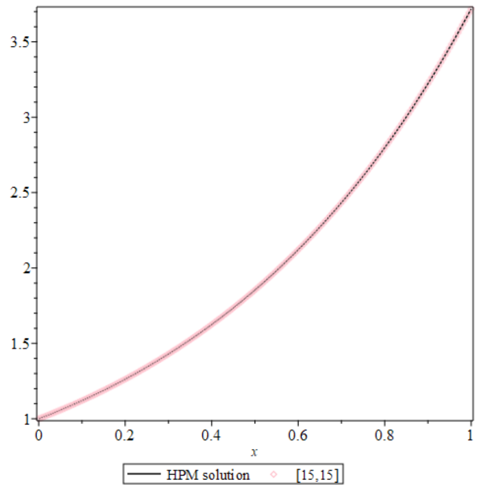

Figure 12.

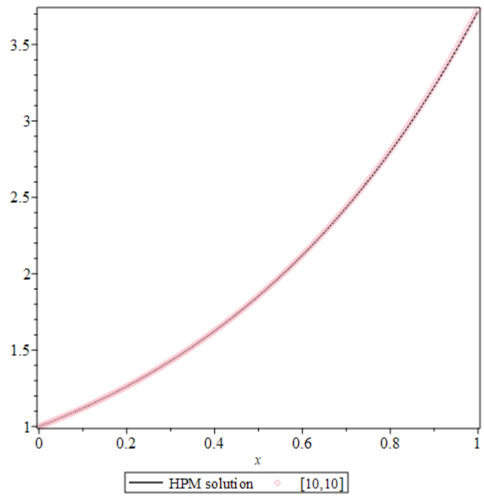

Figure 13.

Figure 11. $y_1(x)_{\alpha=3.25}$ (HPM solution), $r_{22,22}(x)$.

Figure 12. $y_1(x)_{\alpha=3.25}$ (HPM solution), $r_{15,15}(x)$.

Figure 13. $y_1(x)_{\alpha=3.25}$ (HPM solution), $r_{10,10}(x)$.

Example 2.

Consider the following nonlinear fourth-order fractional integro-differential equation:

$$D^{\alpha} y(x)=1+\int_0^x e^{-t} y^2(t)dt \qquad 0<x<1, \quad 3<\alpha\le 4 \tag{32}$$

subject to the following boundary conditions:

$$y(0)=1, \qquad y''(0)=1, \tag{33}$$

$$y(1)=e, \qquad y''(1)=e. \tag{34}$$

The exact solution of (32) is given as $y(x)=e^x$ for $\alpha=4$ in [18].

According to the Homotopy Pertubation method Nawaz [18] obtained following homotopy:

$$D^{\alpha} y(x)=p\left(1+\int_0^x e^{-t} y^2(t)dt\right) \tag{35}$$

Nawaz obtained following solution (36) by applying HPM on (35) in [18]:

$$\begin{aligned} y_2(x)=&1+Ax+\frac{x^2}{2}+\frac{Bx^3}{6}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}+(2A-1)\frac{x^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{3x^{\alpha+3}}{\Gamma(\alpha+4)}-(2B+7)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)}\\ &+\frac{12x^{2\alpha}}{\Gamma(\alpha+6)}-\frac{8Ax^{\alpha+5}}{\Gamma(\alpha+6)}-\frac{6Bx^{\alpha+5}}{\Gamma(\alpha+6)}-\frac{20x^{2\alpha+4}}{\Gamma(\alpha+7)}+\frac{20Bx^{2\alpha+4}}{\Gamma(\alpha+7)}-\frac{40Bx^{\alpha+6}}{\Gamma(\alpha+8)}+\frac{2x^{2\alpha+1}}{\Gamma(2\alpha+2)}\end{aligned} \quad (36$$

Table 10. Values of A and B for different values of α for equation (14) [18]

	$\alpha=3.25$	$\alpha=3.5$	$\alpha=3.75$	$\alpha=4$
A	1.00646865931986	1.01085715673040	1.00647005332874	0.99746675420551
B	0.34838722251386	0.59592879361901	0.59592879361901	1.01767767908914

HPM solution (36) for $\alpha=4$ is obtained as:

$$\begin{aligned} y_2\ x\ _{\alpha=4}=&1+0.9974667542x+0.5000000000x^2+0.1696129465x^3+0.04166666667x^4\\ &+0.008333333333x^5+0.001381852095x^6+0.0005952380952x^7-0.0002240911547x^8\\ &-2.364420965\times10^{-7}x^9+9.742988865\times10^{-8}x^{10}-1.019798861\times10^{-6}x^{11}\end{aligned} \tag{37}$$

Equation (37) was put into Padé series and following equations, respectively, $r_{2,2}(x)$, $r_{4,4}(x)$, $r_{5,5}(x)$, $r_{6,6}(x)$, were obtained for different values of m and n by applying Padé approximation using equations (8) and (9):

$$r_{2,2}(x)=\frac{(1+0.4623626828x+0.06443930170x^2)}{(1-0.5351040718x+0.09818782314x^2)} \tag{38}$$

$$\begin{aligned} r_{4,4}(x)=&(0.9999999996+1.938451454x+0.8336731784x^2\\ &+0.1801187584x^3+0.02516736845x^4)/(0.9999999996\\ &+0.9409846998x-0.6049277756x^2+0.1434088068x^3\\ &-0.01668411508x^4)\end{aligned} \tag{39}$$

$$\begin{aligned} r_{5,5}(x) &= (0.9999999998+1.383647533x+0.9841959081x^2+0.4786366029x^3 \\ &+0.1347491064x^4+0.01875726599x^5)/(0.9999999998+0.3861807790x \\ &+0.09899341999x^2+0.01719062162x^3-0.03906260359x^4+0.007910838822x^5) \end{aligned} \tag{40}$$

$$\begin{aligned} r_{6,6}(x) &= (1+1.758575892x+1.469417086x^2+0.7826093180x^3+0.2862358740x^4 \\ &+0.06323598953x^5+0.006188328148x^6)/(1+0.7611091378x+0.2102360243x^2 \\ &+0.02273835783x^3-0.01232352438x^4-0.01154584914x^5+0.003525708542x^6) \end{aligned} \tag{41}$$

Table 11. Numerical Values for exact solution and padé approximations of equation (36) for $\alpha=4$

	$\alpha=4$				
x	$r_{6,6}(x)$	$r_{5,5}(x)$	$r_{4,4}(x)$	$r_{2,2}(x)$	$y(x)=e^x$
0.1	1.104920538	1.104920540	1.104920540	1.104920510	1.105170918
0.2	1.220919683	1.220919682	1.220919684	1.220918643	1.221402758
0.3	1.349178450	1.349178448	1.349178455	1.349169722	1.349858808
0.4	1.491000418	1.491000420	1.491000508	1.490959762	1.491824698
0.5	1.647824945	1.647824948	1.647825564	1.647687631	1.648721271
0.6	1.821241812	1.821241843	1.821244804	1.820863442	1.822118800
0.7	2.013007344	2.013007501	2.013018650	2.012101404	2.013752707
0.8	2.225061816	2.225062430	2.225097517	2.223104923	2.225540928
0.9	2.459547883	2.459603111	2.459646288	2.455641034	2.459603111
1.0	2.718829673	2.718835589	2.719073522	2.711500699	2.718281828

Table 12. Error values for padé approximations of equation (36) for $\alpha=4$

	$\alpha=4$			
x	$\lvert y(x)-r_{6,6}(x)\rvert$	$\lvert y(x)-r_{5,5}(x)\rvert$	$\lvert y(x)-r_{4,4}(x)\rvert$	$\lvert y(x)-r_{2,2}(x)\rvert$
0.1	0.000250380	0.000250378	0.000250378	0.000250408
0.2	0.000483075	0.000483076	0.000483074	0.000484115
0.3	0.000680358	0.000680360	0.000680353	0.000689086
0.4	0.000824280	0.000824278	0.000824190	0.000864936
0.5	0.000896326	0.000896323	0.000895707	0.001033640
0.6	0.000876988	0.000876957	0.000873996	0.001255358
0.7	0.000745363	0.000745206	0.000734057	0.001651303
0.8	0.000479112	0.000478498	0.000443411	0.002436005
0.9	0.000055228	0.000053182	0.000043177	0.003962077
1.0	0.000547845	0.000553761	0.000791694	0.006781129

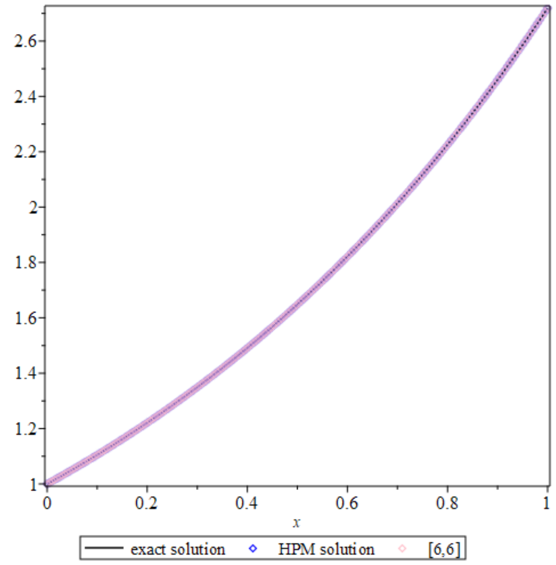

Figure 14.

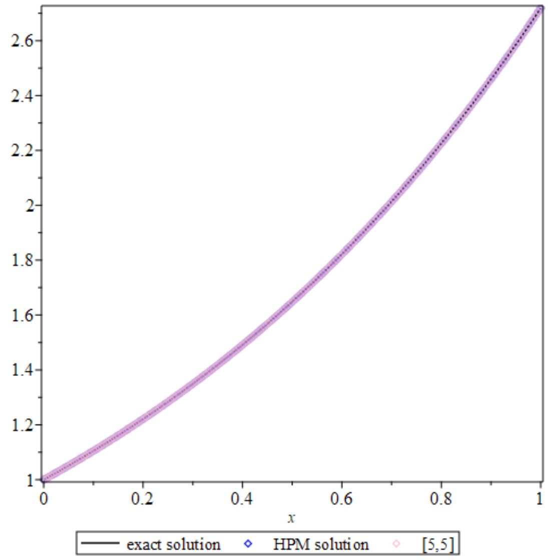

Figure 15.

Figure 14. $y(x)=e^{x}$ (Exact Solution), $y_2(x)_{\alpha=4}$ (HPM Solution), $r_{6,6}(x)$.

Figure 15. $y(x)=e^{x}$ (Exact Solution), $y_2(x)_{\alpha=4}$ (HPM Solution), $r_{5,5}(x)$.

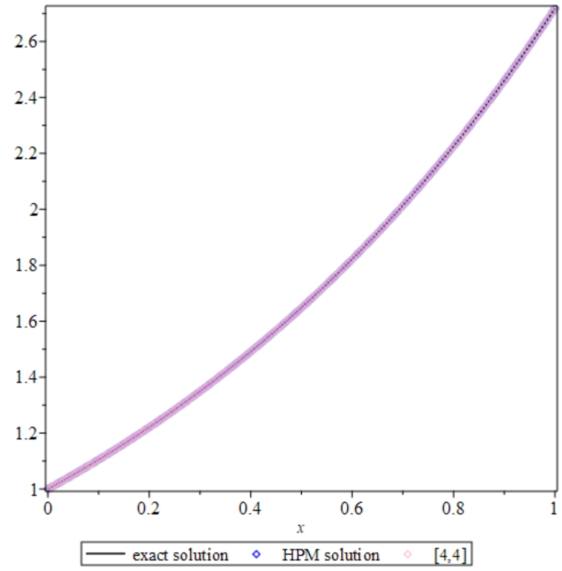

Figure 16.

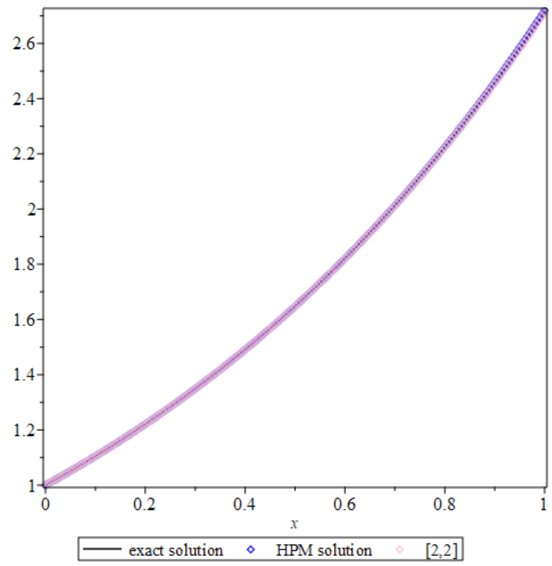

Figure 17.

Figure 16. $y(x)=e^{x}$ (Exact Solution), $y_2(x)_{\alpha=4}$ (HPM Solution), $r_{4,4}(x)$.

Figure 17. $y(x)=e^{x}$ (Exact Solution), $y_2(x)_{\alpha=4}$ (HPM Solution), $r_{2,2}(x)$.

HPM solution (36) for $\alpha=3.75$ is obtained as:

$$\begin{aligned}
y_2(x)_{\alpha=3.75} &= 1.+1.006470053x+0.5000000000x^{2}+0.09932146560x^{3}\\
&+0.06029106159x^{3.75}+0.01269285507x^{4.75}+0.002236017733x^{5.75}\\
&+0.0009810902472x^{6.75}-0.0003456753369x^{7.75}\\
&+1.797209347\times10^{-6}x^{8.75}-3.997249276\times10^{-6}x^{9.75}\\
&-1.096779260\times10^{-6}x^{10.75}+0.00001676551868x^{8.50}
\end{aligned} \tag{42}$$

Equation (42) was put into Padé series and following equations, respectively, $r_{12,12}(x)$, $r_{15,15}(x)$, $r_{21,21}$ were obtained for different values of m and n by applying Padé approximation using equations (8) and (9):

$$\begin{aligned}
r_{12,12}(x) = {} & (0.9999999999 - 0.4998656165x^{1/4} - 0.1543483031\sqrt{x} \\
& -0.03842260783x^{3/4} + 1.291176132x - 0.4000733738x^{5/4} \\
& -0.1197986446x^{3/2} - 0.02658187017x^{7/4} + 0.7918534690x^{2} \\
& -0.1480936423x^{9/4} - 0.04382419089x^{5/2} - 0.06916138576x^{11/4} \\
& +0.2462265934x^{3}) / (0.9999999999 - 0.4998656165x^{1/4} \\
& -0.1543483031\sqrt{x} - 0.03842260783x^{3/4} + 0.2847060794x \\
& +0.1030263998x^{5/4} + 0.03554830018x^{3/2} + 0.01208933398x^{7/4} \\
& +0.005305326228x^{2} - 0.001853819941x^{9/4} - 0.002428338916x^{5/2} \\
& -0.06211763447x^{11/4} - 0.0007875639496x^{3}
\end{aligned} \tag{43}$$

$$\begin{aligned}
r_{15,15}(x) = {} & (1.000000000 + 0.4030869103x^{1/4} + 0.1024948641\sqrt{x} - 0.01074528209x^{3/4} \\
& +0.7655942410x + 0.3721445846x^{5/4} + 0.1265136466x^{3/2} + 0.01276299228x^{7/4} \\
& +0.2770567424x^{2} + 0.1764605694x^{9/4} + 0.07780359138x^{5/2} + 0.01902455166x^{11/4} \\
& -0.01155813334x^{3} + 0.02485742532x^{13/4} + 0.02175087116x^{7/2} \\
& +0.07084391716x^{15/4}) / (1.000000000 + 0.4030869103x^{1/4} + 0.1024948641\sqrt{x} \\
& -0.01074528209x^{3/4} - 0.2408758121x - 0.03355031953x^{5/4} \\
& +0.02335563536x^{3/2} + 0.02357779692x^{7/4} + 0.01949103390x^{2} \\
& +0.008684505890x^{9/4} + 0.003049411793x^{5/2} + 0.0006668461939x^{11/4} \\
& -0.01005883480x^{3} - 0.007143292714x^{13/4} - 0.003176028294x^{7/2} - 0.0008399664623x^{15/4})
\end{aligned} \tag{44}$$

$$\begin{aligned}
r_{21,21}(x) = {} & (1.000000000 - 1.158398906x^{1/4} + 0.3364640838x^{1/2} - 0.08611768495x^{3/4} \\
& +0.9472603216x - 1.260489798x^{5/4} + 0.3877969178x^{3/2} - 0.1082642056x^{7/4} \\
& +0.4160796197x^{2} - 0.6592257893x^{9/4} + 0.2236365778x^{5/2} - 0.06539043178x^{11/4} \\
& +0.03973008852x^{3} - 0.1315134578x^{13/4} + 0.06660885270x^{7/2} \\
& +0.04187365896x^{15/4} - 0.08347642648x^{4} + 0.02773853132x^{17/4} \\
& +0.005797006079x^{9/2} + 0.007502722245x^{19/4} - 0.01710418722x^{5} \\
& +0.01000050629x^{21/4}) / (1.000000000 - 1.158398906x^{1/4} + 0.3364640838x^{1/2} \\
& -0.08611768495x^{3/4} - 0.05920973144x - 0.09459598977x^{5/4} + 0.04915589367x^{3/2} \\
& -0.02158933471x^{7/4} - 0.02432755886x^{2} + 0.01518169463x^{9/4} + 0.005930601033x^{5/2} \\
& -0.0006025704559x^{11/4} - 0.005501551911x^{3} + 0.01555849322x^{13/4} + 0.002643827608x^{7/2} \\
& +0.001537068522x^{15/4} + 0.009946397292x^{4} - 0.006401837938x^{17/4} + 0.0004806634939x^{9/2} \\
& -0.0007217591044x^{19/4} - 0.001541231401x^{5} - 0.00007770117320x^{21/4})
\end{aligned} \tag{45}$$

Table 13. Numerical Values of equation (42) and its padé approximations.

	$\alpha=3.75$			
x	$r_{12,12}(x)$	$r_{15,15}(x)$	$r_{21,21}(x)$	$y_2(x)_{\alpha=3.75}$
0.1	1.105757279	1.105757278	1.105757278	1.105757277
0.2	1.222239265	1.222239141	1.222239139	1.222239139
0.3	1.350328524	1.350326760	1.350326701	1.350326704
0.4	1.491074036	1.491062618	1.491062077	1.491062076
0.5	1.645703533	1.645655243	1.645652185	1.645652171
0.6	1.815647202	1.815491046	1.815478523	1.815478365
0.7	2.002571450	2.002151634	2.002110838	2.002109417
0.8	2.208423663	2.207436620	2.207329371	2.207316696
0.9	2.435489914	2.433392811	2.433371328	2.433090997
1.0	2.686468944	2.682351045	2.681373791	2.681660338

Table 14. Error values for padé approximations of equation (42) for $\alpha=3.75$

	$\alpha=3.75$		
x	$\lvert y_2(x)_{\alpha=3.75}-r_{12,12}\rvert$	$\lvert y_2(x)_{\alpha=3.75}-r_{15,15}\rvert$	$\lvert y_2(x)_{\alpha=3.75}-r_{21,21}\rvert$
0.1	$2.\times10^{-9}$	1×10^{-9}	$1.\times10^{-9}$
0.2	1.26×10^{-7}	2×10^{-9}	0
0.3	1.820×10^{-6}	5.6×10^{-8}	$3.\times10^{-9}$
0.4	0.000011960	5.42×10^{-7}	$1.\times10^{-9}$
0.5	0.000051362	3.072×10^{-6}	1.4×10^{-8}
0.6	0.000168837	0.000012681	1.58×10^{-7}
0.7	0.000462033	0.000042217	1.421×10^{-6}
0.8	0.001106967	0.000119924	0.000012675
0.9	0.002398917	0.000301814	0.000280331
1.0	0.004808606	0.000690707	0.000286547

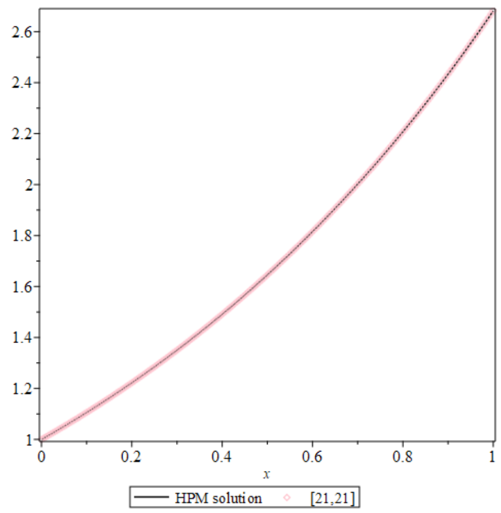

Figure 18.

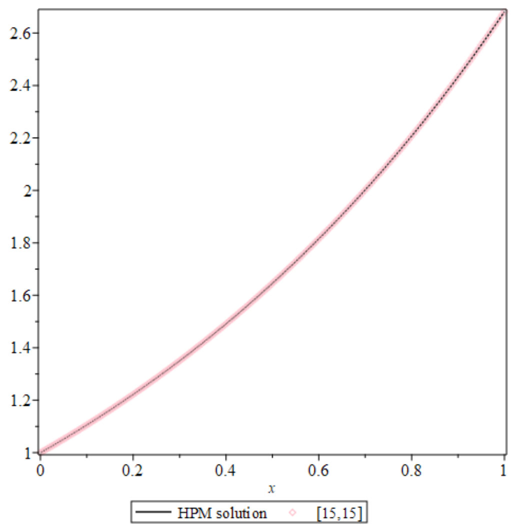

Figure 19.

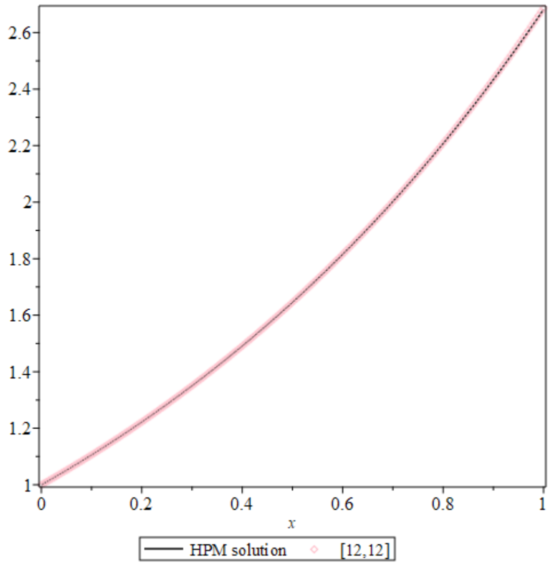

Figure 20.

Figure 18. $y_2(x)_{\alpha=3.75}$ (HPM solution), $r_{21,21}(x)$.

Figure 19. $y_2(x)_{\alpha=3.75}$ (HPM solution), $r_{15,15}(x)$.

Figure 20. $y_2(x)_{\alpha=3.75}$ (HPM solution), $r_{12,12}(x)$.

HPM solution (36) for $\alpha = 3.50$ is obtained as:

$$\begin{aligned} y_2(x)_{\alpha=3.50} &= 1.+1.010857157x+0.5000000000x^2+0.09932146560x^3 \\ &+0.08597174604x^{3.5}+0.01910483246x^{4.5}+0.003549032871x^{5.5} \\ &+0.001603202724x^{6.5}-0.0005836981511x^{7.5}+2.829767920\times10^{-6}x^{8.5} \\ &-7.131014064\times10^{-6}x^{9.5}-2.003219112\times10^{-6}x^{10.5}+0.00004960317460x^{8.0} \end{aligned} \tag{46}$$

Equation (46) was put into Padé series and following equations, respectively, $r_{7,7}(x)$, $r_{8,8}(x)$, $r_{10,10}(x)$ were obtained for different values of m and n by applying Padé approximation using equations (8) and (9):

$$\begin{aligned} r_{7,7}(x) &= (1.000000000+0.2126062027\sqrt{x}+0.8252329868x \\ &+0.2026831312x^{3/2}+0.3332301421x^2+0.09962505282x^{5/2} \\ &+0.01743951061x^3+0.1019969367x^{7/2})/(1.000000000+0.2126062027\sqrt{x} \\ &-0.1856241703x-0.01223137047x^{3/2}+0.02086966324x^2 \\ &+0.005686119835x^{5/2}-0.01016611830x^3-0.004723338690x^{7/2}) \end{aligned} \tag{47}$$

$$\begin{aligned} r_{8,8}(x) &= (1.000000000-0.1639253396\sqrt{x}+1.004377559x-0.1576658605x^{3/2} \\ &+0.5043928389x^2-0.1089945598x^{5/2}+0.08835048971x^3+0.03642187344x^{7/2} \\ &-0.01762723677x^4)/(1.000000000-0.1639253396\sqrt{x}-0.006479598170x \\ &+0.008039242267x^{3/2}+0.01094278700x^2-0.03515841557x^{5/2} \\ &-0.01879277138x^3-0.001748052755x^{7/2}+0.01063468802x^4) \end{aligned} \tag{48}$$

$$
\begin{aligned}
r_{10,10}(x) &= (1.000000000 + 0.5084255416\sqrt{x} + 1.273941216x \\
&+0.5221862006x^{3/2} + 0.7545614502x^2 + 0.2329494241x^{5/2} \\
&+0.1824810720x^3 + 0.08935668578x^{7/2} + 0.03056014946x^4 \\
&+0.01213168829x^{9/2} - 0.003043783177x^5)/(1.000000000 \\
&+0.5084255416\sqrt{x} + 0.2630840591x + 0.008240602845x^{3/2} \\
&-0.01137895403x^2 - 0.02959341910x^{5/2} - 0.03687992599x^3 \\
&-0.02131821215x^{7/2} + 0.003689837731x^4 \\
&+0.005936968039x^{9/2} + 0.002374611522x^5)
\end{aligned}
\tag{49}
$$

Table 15. Numerical Values of equation (46) and its padé approximations

	$\alpha = 3.50$			
x	$r_{7,7}(x)$	$r_{8,8}(x)$	$r_{10,10}(x)$	$y_2(x)_{\alpha=3.5}$
0.1	1.106212839	1.106212839	1.106212841	1.106212840
0.2	1.223287811	1.223287806	1.223287805	1.223287806
0.3	1.352300397	1.352300274	1.352300277	1.352300278
0.4	1.494516261	1.494515164	1.494515214	1.494515219
0.5	1.651385685	1.651379728	1.651380068	1.651380086
0.6	1.824551802	1.824527954	1.824529522	1.824529657
0.7	2.015867171	2.015789706	2.015795471	2.015796133
0.8	2.227417708	2.227201781	2.227219615	2.227222275
0.9	2.461554571	2.461018677	2.461067162	2.461076224
1.0	2.720935053	2.719720795	2.719839903	2.719867038

Table 16. Error values for padé approximations of equation (46) for $\alpha = 3.50$

	$\alpha = 3.50$		
x	$\lvert y_2(x)_{\alpha=3.5} - r_{7,7}\rvert$	$\lvert y_2(x)_{\alpha=3.5} - r_{8,8}\rvert$	$\lvert y_2(x)_{\alpha=3.5} - r_{10,10}\rvert$
0.1	1×10^{-9}	1×10^{-9}	1×10^{-9}
0.2	5×10^{-9}	0	1×10^{-9}
0.3	1.19×10^{-7}	4×10^{-9}	1×10^{-9}
0.4	1.042×10^{-6}	5.5×10^{-8}	5×10^{-9}
0.5	5.599×10^{-6}	3.58×10^{-7}	1.8×10^{-8}
0.6	0.000022145	1.703×10^{-6}	1.35×10^{-7}
0.7	0.000071038	6.427×10^{-6}	6.62×10^{-7}
0.8	0.000195433	0.000020494	2.660×10^{-6}
0.9	0.000478347	0.000057547	9.062×10^{-6}
1.0	0.001068015	0.000146243	0.000027135

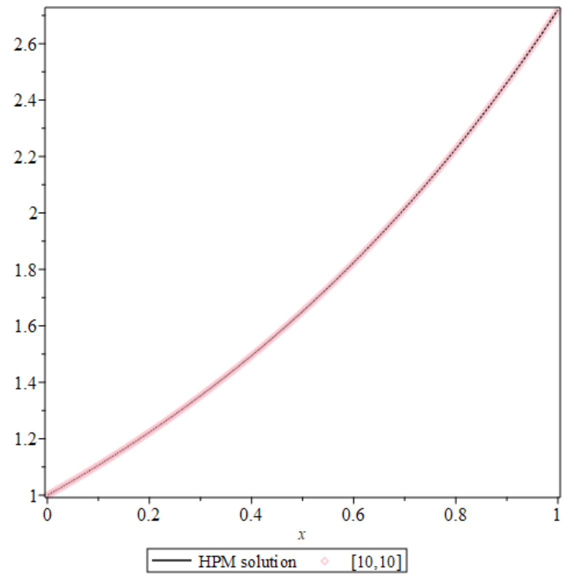

Figure 21.

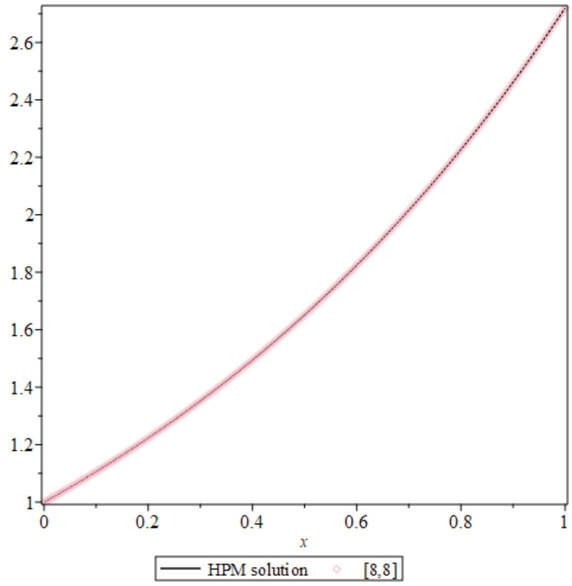

Figure 22.

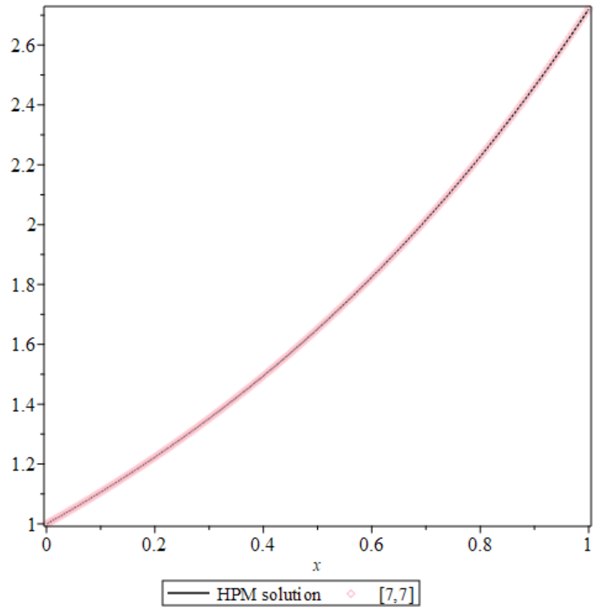

Figure 23.

Figure 21. $y_2(x)_{\alpha=3.5}$ (HPM Solution), $r_{10,10}(x)$.

Figure 22. $y_2(x)_{\alpha=3.5}$ (HPM Solution), $r_{8,8}(x)$.

Figure 23. $y_2(x)_{\alpha=3.5}$ (HPM Solution), $r_{7,7}(x)$.

HPM solution (36) for $\alpha = 3.25$ is obtained as:

$$\begin{aligned} y_2(x)_{\alpha=3.25} &= 1.+1.006468659x+0.5000000000x^2+0.05806453708x^3 \\ &+0.1206988200x^{3.25}+0.02839972235x^{4.25}+0.005479454970x^{5.25} \\ &+0.002596546043x^{6.25}-0.0009188519189x^{7.25}+0.00002688509381x^{8.25} \\ &-0.00002038734503x^{9.25}-2.126862577\times10^{-6}x^{10.25}+0.0001425069088x^{7.50} \end{aligned} \tag{50}$$

Equation (50) was put into Padé series and following equations, respectively, $r_{14,14}(x)$, $r_{16,16}(x)$, $r_{20,20}(x)$ were obtained for different values of m and n by applying Padé approximation using equations (8) and (9):

$$\begin{aligned}
r_{14,14}(x) &= (0.9999999999 + 4.263879226x^{1/4} - 0.3885403001\sqrt{x} - 0.1579395441x^{3/4} \\
&+ 1.562612515x + 3.198410715x^{5/4} - 0.3162362639x^{3/2} - 0.02314731625x^{7/4} \\
&+ 1.087772286x^{2} + 1.077692304x^{9/4} - 0.1132748338x^{5/2} + 0.04595884244x^{11/4} \\
&+ 0.3372792344x^{3} - 0.1867173627x^{13/4} + 0.5388464882x^{7/2}) / (0.9999999999 \\
&+ 4.263879226x^{1/4} - 0.3885403001\sqrt{x} - 0.1579395441x^{3/4} + 0.5561438555x \\
&- 1.093050092x^{5/4} + 0.07481737085x^{3/2} + 0.1358138849x^{7/4} + 0.02803092528x^{2} \\
&+ 0.04587335198x^{9/4} + 0.005693977274x^{5/2} - 0.01176380418x^{11/4} - 0.02706947829x^{3} \\
&- 0.05464140122x^{13/4} + 0.003622214619x^{7/2})
\end{aligned} \tag{51}$$

$$\begin{aligned}
r_{16,16}(x) &= (0.9999999999 + 0.1625183353x^{1/4} + 0.07575788017\sqrt{x} - 0.1274966670x^{3/4} \\
&+ 0.7950946696x + 0.2330041062x^{5/4} + 0.1805863678x^{3/2} - 0.1677934518x^{7/4} \\
&+ 0.3160963058x^{2} + 0.1486207748x^{9/4} + 0.1065972410x^{5/2} - 0.1412472664x^{11/4} \\
&- 0.03459603471x^{3} + 0.1566928373x^{13/4} + 0.03875285169x^{7/2} - 0.04961972605x^{15/4} \\
&- 0.01472945396x^{4}) / (0.9999999999 + 0.1625183353x^{1/4} + 0.07575788017\sqrt{x} \\
&- 0.1274966670x^{3/4} - 0.2113739892x + 0.06943449513x^{5/4} + 0.1043384357x^{3/2} \\
&- 0.03947205233x^{7/4} + 0.02883760131x^{2} - 0.002522036074x^{9/4} - 0.03629506454x^{5/2} \\
&- 0.03777154940x^{11/4} - 0.01599771909x^{3} - 0.005621431918x^{13/4} - 0.0009011387704x^{7/2} \\
&+ 0.006391328990x^{15/4} + 0.01461497835x^{4})
\end{aligned} \tag{52}$$

$$
\begin{aligned}
r_{20,20}(x) &= (1.000000000 + 1.177320138x^{1/4} + 0.8756705052\sqrt{x} \\
&-0.1170237480x^{3/4} + 1.309196257x + 1.289318324x^{5/4} \\
&+0.9862598369x^{3/2} - 0.1872455303x^{7/4} + 0.7671040230x^{2} \\
&+0.6342508801x^{9/4} + 0.5162736177x^{5/2} - 0.1501256840x^{11/4} \\
&+0.1067802041x^{3} + 0.1411506487x^{13/4} + 0.2010131998x^{7/2} \\
&+0.03396821765x^{15/4} - 0.06923044055x^{4} + 0.01470345623x^{17/4} \\
&+0.02919531710x^{9/2} + 0.01517209895x^{19/4} - 0.03157746416x^{5}) \\
&/(1.000000000 + 1.177320138x^{1/4} + 0.8756705052\sqrt{x} \\
&-0.1170237480x^{3/4} + 0.3027275978x + 0.1043825033x^{5/4} \\
&+0.1049249179x^{3/2} - 0.06946479555x^{7/4} - 0.03758181674x^{2} \\
&-0.05946690684x^{9/4} - 0.02716527662x^{5/2} - 0.02169967037x^{11/4} \\
&-0.06482321121x^{3} - 0.04024839378x^{13/4} - 0.01705481355x^{7/2} \\
&-0.008356813365x^{15/4} + 0.01134988874x^{4} + 0.01394614867x^{17/4} \\
&+0.007816263918x^{9/2} + 0.0009331330703x^{19/4} + 0.003300765725x^{5})
\end{aligned}
\tag{53}
$$

Table 17. Numerical Values of equation (50) and its padé approximations

	$\alpha = 3.25$			
x	$r_{14,14}(x)$	$r_{16,16}(x)$	$r_{20,20}(x)$	$y_2(x)_{\alpha=3.25}$
0.1	1.105774433	1.105774433	1.105774432	1.105774434
0.2	1.222435653	1.222435640	1.222435642	1.222435641
0.3	1.351101759	1.351101526	1.351101544	1.351101544
0.4	1.493078835	1.493076879	1.493077066	1.493077070
0.5	1.649853823	1.649843598	1.649844736	1.649844784
0.6	1.823104509	1.823064829	1.823069799	1.823070101
0.7	2.014719014	2.014593489	2.014610773	2.014612209
0.8	2.226823522	2.226481229	2.226532308	2.226537855
0.9	2.461818715	2.460984945	2.461118222	2.461136504
1.0	2.722426248	2.720566966	2.720882567	2.720935765

Table 18. Error values for padé approximations of equation (50) for $\alpha = 3.25$

	$a = 3.25$		
x	$\lvert y_2(x)_{a=3.25} - r_{14,14}\rvert$	$\lvert y_2(x)_{a=3.25} - r_{16,16}\rvert$	$\lvert y_2(x)_{a=3.25} - r_{20,20}\rvert$
0.1	1×10^{-9}	$1.\times10^{-9}$	2×10^{-9}
0.2	1.2×10^{-8}	1×10^{-9}	1×10^{-9}
0.3	2.15×10^{-7}	1.8×10^{-8}	0
0.4	1.765×10^{-6}	1.91×10^{-7}	$4.\times10^{-9}$
0.5	9.039×10^{-6}	1.186×10^{-6}	4.8×10^{-8}
0.6	0.000034408	5.272×10^{-6}	3.02×10^{-7}
0.7	0.000106805	0.000018720	1.436×10^{-6}
0.8	0.000285667	0.000056626	5.547×10^{-6}
0.9	0.000682211	0.000151559	0.000018282
1.0	0.001490483	0.000368799	0.000053198

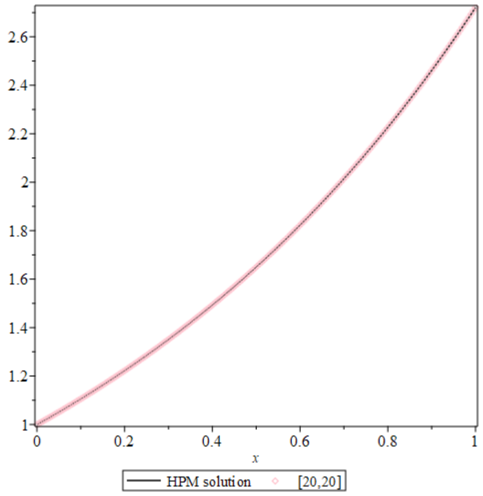

Figure 24.

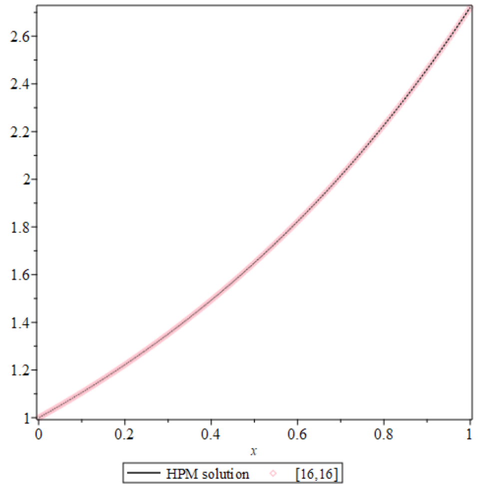

Figure 25.

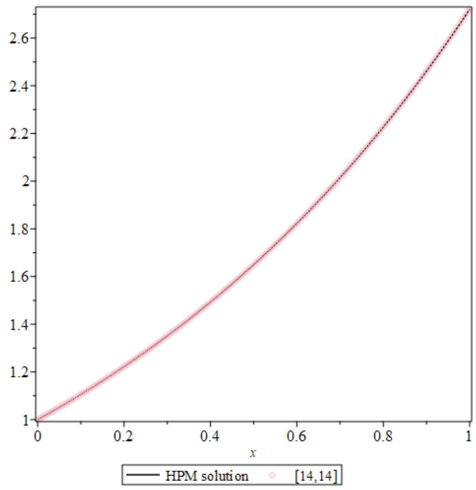

Figure 26.

Figure 24. $y_2(x)_{\alpha=3.25}$ (HPM Solution), $r_{20,20}(x)$.

Figure 25. $y_2(x)_{\alpha=3.25}$ (HPM Solution), $r_{16,16}(x)$.

Figure 26. $y_2(x)_{\alpha=3.25}$ (HPM Solution), $r_{14,14}(x)$

4. Conclusion

In this Paper, Padé approximation has been successfully applied to linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations. It gives a simple and a powerful mathematical tool for nonlinear problems. Numerical results show the accuracy and efficency of Padé approximation. Comparisons of HPM and Padé approximation have been shown by numerical results for different values of α show the efficiency of the Padé approximation. It provides also more realistic rational series solutions that converge very rapidly in real physical problems.

Author's Contributions

All contributions belong to the author in this paper.

Statement of Conflicts of Interest

No potential conflict of interest was reported by the author.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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